TEMPERLEY-LIEB OPERATORS AND CRITICAL A–D–E MODELS

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A review is given of recent work connecting exactly solvable lattice models with Temperley-Lieb and Hecke algebras, Dynkin diagrams of simply-laced Lie algebras, conformal and modular invariance and Virasoro characters.

1. Introduction

These notes are a summary of lectures presented during 1989 as part of the special year on mathematical physics at the Centre for Mathematical Analysis, Australian National University. The intent of these lectures was to introduce to an audience well versed in statistical mechanics some of the notions, ideas and results flowing from the recent advances connecting two-dimensional critical lattice models with conformal field theory. The exposition is for the most part elementary and designed to whet appetites for further research. There are some very good reviews elsewhere [1,2,3,4,5,6] which cover much the same area. Indeed, some have appeared since these lectures were given. I refer the interested reader to these for more details of some of the topics discussed here. Of course, ultimately, there is no substitute for the original papers.

2. Solvable A–D–E Lattice Models

One of the overriding goals of modern statistical mechanics is to understand how symmetry determines the various observed universality classes of critical systems and to classify them accordingly. In two dimensions, the application of conformal field theory has ushered in the first major step in this direction. Many of the theoretical advances have been either motivated by, or corroborated by, direct calculations on exactly solvable two-dimensional lattice models [7]. So recent developments on solvable models and conformal theory have been closely linked. It is known that Yang-Baxter equations give a sufficient condition for solvability of lattice models. These same equations arise, albeit in different guises, in other branches of physics as well. An important open problem is to classify all solutions to the Yang-Baxter equations. This is a difficult problem in general but large classes of solutions are known. We will start by considering a particularly interesting class of solutions leading to the critical A–D–E models. The rest of the review will then focus on the application of conformal theory as it pertains to this class of critical lattice models.

2.1 Temperley-Lieb interaction models

The Temperley-Lieb interaction (TLI) models [8,9] are a large class of exactly solvable interaction-round-a-face or IRF models [7]. These models were studied by Owczarek and Baxter [9] in 1987 and will be used in this review to construct solvable lattice models corresponding to arbitrary graphs. In particular, we show that these models include Pasquier's critical A–D–E models [10,11], which were also introduced in 1987.

The face weights of the TLI models we consider are given by

$$W\begin{pmatrix} d & c \\ a & b \end{pmatrix} = \rho[\delta(a,c) + x h_{ab} h_{cd} \delta(b,d)]. \tag{2.1}$$

Here a, b, c, d are discrete L-state spins around an elementary square of the lattice, δ is the Kronecker delta function, ρ is a normalization factor and x is a free parameter. The face edges (a,b), (b,c), (c,d) and (d,a) are restricted to certain allowed pairs. The interactions h_{ab} are in general not symmetric $h_{ab} \neq h_{ba}$ and should be thought of as three-spin interactions, that is, $h_{ab} = h_{abd}\delta(b,d)$. The TLI face weights can be used to construct two diagonal transfer matrices V_1 and V_2 corresponding to the black and white faces of the square lattices. Explicitly, the elements of V_1 are given by

$$V_1(\sigma, \sigma') = \prod_{i=0}^{N} W\begin{pmatrix} \sigma_{2i-1} & \sigma'_{2i} \\ \sigma_{2i} & \sigma_{2i+1} \end{pmatrix} \delta(\sigma_{2i-1}, \sigma'_{2i-1}) \tag{2.2}$$

where $\sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_{2N+1}\}$ and $\sigma' = \{\sigma'_0, \sigma'_1, \ldots, \sigma'_{2N+1}\}$ are consecutive diagonal rows of spins. The dimension of the transfer matrix V_1 is given by the number of configurations or allowed paths of spins along a diagonal row with the end spins σ_0, σ_{2N+1} fixed. The partition function of a lattice with M diagonal rows closed on a cylinder can then be written as

$$Z = \mathrm{Tr}(V_1 V_2)^M. \tag{2.3}$$

It is convenient to factorize the transfer matrices V_1 and V_2 by introducing face transfer matrices

$$X_j = \rho(I + xU_j) \tag{2.4}$$

where I is the identity matrix and

$$U_j(\sigma, \sigma') = h_{\sigma_j, \sigma_{j+1}} h_{\sigma'_j, \sigma_{j-1}} \delta(\sigma_{j-1}, \sigma_{j+1}) \prod_{i \neq j} \delta(\sigma_i, \sigma'_i). \tag{2.5}$$

The transfer matrices V_1 and V_2 are then given by the ordered matrix products

$$\begin{aligned} V_1 &= \rho^N(1+xU_1)(1+xU_3)\dots(1+xU_{2N-1}) && (2.6)\\ V_2 &= \rho^N(1+xU_2)(1+xU_4)\dots(1+xU_{2N}). && (2.7)\end{aligned}$$

Let us consider a graph whose vertices are the L states of the spins and whose edges are the allowed nearest-neighbour pairs. The adjacency matrix of this graph is

$$A_{\sigma,\sigma'} = \begin{cases} 1, & (\sigma,\sigma') \text{ allowed} \\ 0, & \text{otherwise.} \end{cases} \tag{2.8}$$

Owczarek and Baxter [9] have shown that if

$$V_{ab} = h_{ba}^2 \tag{2.9}$$

satisfies the conditions

$$\begin{aligned} V_{ab}V_{ba} &= 1, \quad (a,b) \text{ allowed} && (2.10)\\ \sum_b A_{ab}V_{ab} &= \sqrt{q} && (2.11)\end{aligned}$$

then the matrices $\{U_j, j=1,\dots,2N\}$ satisfy the Temperley-Lieb algebra [8]

$$\begin{aligned} U_j^2 &= \sqrt{q}U_j && (2.12)\\ U_jU_{j\pm1}U_j &= U_j \\ U_iU_j &= U_jU_i, \quad |i-j|\ge 2. \end{aligned}$$

A Temperley-Lieb algebra $\{U_j\}$ gives rise to a lattice model satisfying the Yang-Baxter equations

$$\sum_g W\begin{pmatrix} a & g \\ b & c\end{pmatrix} W'\begin{pmatrix} f & e \\ a & g\end{pmatrix} W''\begin{pmatrix} e & d \\ g & c\end{pmatrix} = \sum_g W''\begin{pmatrix} f & g \\ a & b\end{pmatrix} W'\begin{pmatrix} g & d \\ b & c\end{pmatrix} W\begin{pmatrix} f & e \\ g & d\end{pmatrix}. \tag{2.13}$$

Following Baxter [12] we define face transfer matrices

$$\begin{aligned} X_j &= \rho(I+xU_j) && (2.14)\\ X_j' &= \rho'(I+x'U_j) \\ X_j'' &= \rho''(I+x''U_j). \end{aligned}$$

The Yang-Baxter equation then becomes

$$X_jX_{j+1}'X_j'' = X_{j+1}X_j'X_{j+1}'' \tag{2.15}$$

and is satisfied provided only that the parameters x, x', x'' are related by

$$x'' = \frac{x'-x}{1+\sqrt{q}x+xx'}. \tag{2.16}$$

To see this just expand the Yang-Baxter equation in the U_j and equate the coefficients of $I, U_j, U_{j+1}, U_jU_{j+1}$ and $U_{j+1}U_j$. In fact, the U_j need only satisfy the conditions

$$\begin{aligned} U_j^2 - \sqrt{q}U_j &= U_{j+1}^2 - \sqrt{q}U_{j+1} && (2.17) \\ U_jU_{j+1}U_j - U_j &= U_{j+1}U_jU_{j+1} - U_{j+1} && (2.18) \end{aligned}$$

so it is sufficient that the U_j satisfy a Hecke algebra.

The most convenient way to ensure that x, x', x'' satisfy the required identity is to set

$$\sqrt{q} = \Delta = \begin{cases} 2\cos\lambda, & q < 4 \\ 2, & q = 4 \\ 2\cosh\lambda, & q > 4 \end{cases} \tag{2.19}$$

where λ is the crossing parameter and set

$$x = \frac{s}{s_-}, \quad x' = \frac{s'}{s'_-}, \quad x'' = \frac{s''}{s''_-} \tag{2.20}$$

where

$$s = \begin{cases} \dfrac{\sin u}{\sin\lambda}, & q < 4 \\ u, & q = 4 \\ \dfrac{\sinh u}{\sinh\lambda}, & q > 4 \end{cases} \qquad s_- = \begin{cases} \dfrac{\sin(\lambda - u)}{\sin\lambda}, & q < 4 \\ 1 - u, & q = 4 \\ \dfrac{\sinh(\lambda - u)}{\sinh\lambda}, & q > 4 \end{cases} \tag{2.21}$$

and similarly for the primed and double primed variables with u replaced by u' and u''. If the spectral parameters satisfy the difference property

$$u'' = u' - u \tag{2.22}$$

then the required relation follows from the generalized trigonometric identity

$$ss'_-s''_- + s_-s'_-s'' + ss's'' + \Delta ss'_-s'' = s_-s's''_-. \tag{2.23}$$

Physically, the spectral parameter u is related to spatial anisotropy. This anisotropy acts to distort a square face into a rhombus with angle θ given by

$$\theta = \begin{cases} \dfrac{\pi u}{\lambda}, & q \neq 4 \\ \pi u, & q = 4. \end{cases} \tag{2.24}$$

The difference property of the Yang-Baxter equation then becomes

$$\theta'' = \theta' - \theta \tag{2.25}$$

which has the meaningful geometric interpretation shown in Fig. 1.

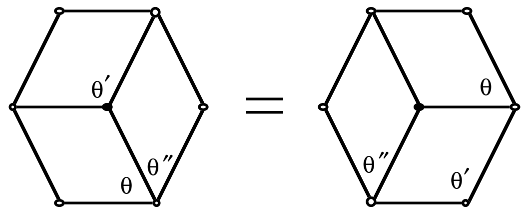

Figure 1. Diagrammatic representation of the Yang-Baxter equation showing the anisotropy angles satisfying the geometric constraint $\theta''=\theta'-\theta$. The central spin is summed out and equality holds for all values of the six perimeter spins.

2.2 Critical A–D–E models

The TLI models contain the critical A–D–E models [10,11] as a special case. To see this notice that the second Owczarek-Baxter condition (2.11) looks like an eigenvalue equation. In particular, if we set

$$V_{ab}=S_b/S_a \tag{2.26}$$

then the first condition (2.11) is automatically satisfied and the second condition (2.11) takes the form

$$\sum_b V_{ab}S_b=\sqrt{q}S_a. \tag{2.27}$$

Clearly this is satisfied if S_a is an eigenvector of the adjacency matrix A with eigenvalue $\sqrt{q}$. It follows that if S_a is any fixed eigenvector of A, then the matrices

$$U_j(\sigma,\sigma')=\sqrt{\frac{S_{\sigma_j}S_{\sigma'_j}}{S_{\sigma_{j-1}}S_{\sigma_{j-1}}}}\;\delta(\sigma_{j-1},\sigma_{j+1})\prod_{i\neq j}\delta(\sigma_i,\sigma'_i). \tag{2.28}$$

satisfy the Temperley-Lieb algebra.

Putting everything together, it has been shown that if S_a are the components of an eigenvector of the adjacency matrix A with corresponding eigenvalue $\sqrt{q}$, then the face weights

$$W\begin{pmatrix} d & c\\ a & b\end{pmatrix}=\rho\left[s_-\delta(a,c)+s\sqrt{\frac{S_aS_c}{S_bS_d}}\;\delta(b,d)\right] \tag{2.29}$$

satisfy the Yang-Baxter equations and so the corresponding lattice model is exactly solvable. It is usual, but not necessary, to take S_a to be the positive Perron-Frobenius vector of A. The face weights are then real and positive for $0\le u\le\lambda$ (or $0\le u\le 1$ in the case $q=4$).

To make the discussion more concrete at this point let us consider the q-state Potts model. The q-state Potts model has an adjacency graph consisting of a q-pointed star. The spins assume the states $\sigma=0,1,2,\ldots,q$ and the $(q+1)\times(q+1)$

adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} \tag{2.30}$$

has largest eigenvalue $\sqrt{q}$ with Perron-Frobenius vector

$$S = (\sqrt{q}, 1, 1, \dots, 1). \tag{2.31}$$

The TLI model constructed starting with the star graph is precisely the self-dual q-state Potts model. This model is critical for $q \le 4$ and exhibits first-order phase coexistence for $q > 4$. The Temperley-Lieb equivalence [8,7,12] implies that the free energy of a Temperley-Lieb interaction model is given by the free energy of the Potts model with the corresponding value of q. This equivalence can also be extended to certain correlation functions. It therefore follows that a Temperley-Lieb interaction model is critical if the corresponding Potts model is critical, that is, if the largest eigenvalue $\Lambda_{\max}$ of the adjacency matrix A satisfies

$$\Lambda_{\max} \le 2. \tag{2.32}$$

2.3 Dynkin diagrams

In statistical mechanics the object of central study is critical behaviour. It is therefore natural to ask which graphs lead via the Temperley-Lieb construction to critical lattice models. More precisely, the question is which undirected connected graphs without loops have adjacency matrices with $\Lambda_{\max} \le 2$. The answer is an old result in the spectra of graphs [13] and surprisingly has links to Lie algebras. The only connected graphs for which $\Lambda_{\max} = 2$ are the Dynkin diagrams of the simply-laced affine Lie algebras. The allowed connected graphs for which $\Lambda_{\max} < 2$ are the Dynkin diagrams of the simply-laced classical Lie algebras. These graphs are shown in Fig. 2. The Cartan matrix C of the Lie algebra is related to the adjacency matrix by $C = 2I - A$. Both the classical and affine Dynkin diagrams consist of an infinite A and D series plus a few exceptional cases. These results can be proved from the following lemmas [13] which we state without proof:

Lemma 1. *The maximal eigenvalue of an irreducible nonnegative matrix A is a strictly increasing function of its matrix elements.*

Lemma 2. *If G is a connected graph then either (i) G is an affine Dynkin diagram (ii) G is a subgraph of an affine Dynkin diagram, that is, G is a classical Dynkin diagram or (iii) G has a subgraph which is an affine Dynkin diagram.*

Another surprise is that the eigenvalues of the Dynkin diagrams can all be written in the form

$$\Lambda_j = 2\cos(m_j\pi/h) \tag{2.33}$$

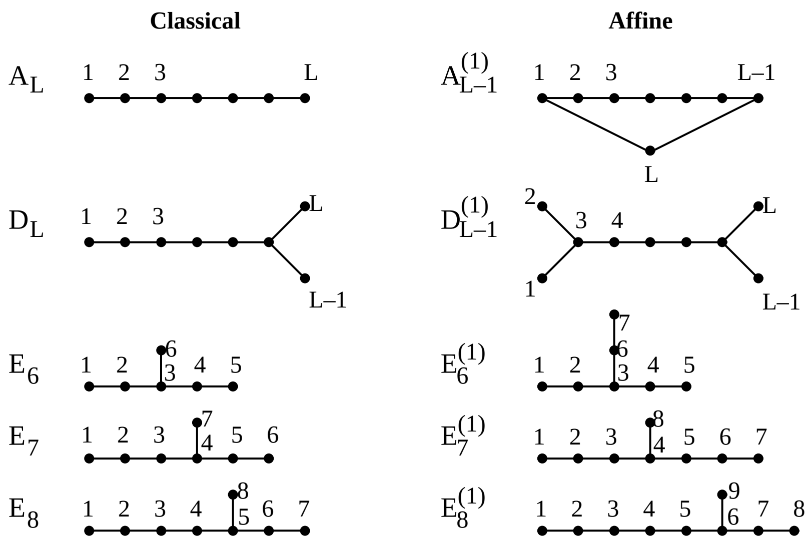

Figure 2. Dynkin diagrams of the simply-laced classical and affine Lie algebras.

Coxeter Exponents

Lie Algebra	h	m_j
A_L	L+1	1, 2, 3,…, L
D_L	2L–2	L–1, 1, 3, 5,…, 2L–3
E_6	12	1, 4, 5, 7, 8, 11
E_7	18	1, 5, 7, 9, 11, 13, 17
E_8	30	1, 7, 11, 13, 17, 19, 23, 29
$A_{L-1}^{(1)}$	L	0, 2, 4,…, 2L–2
$D_{L-1}^{(1)}$	2(L–3)	0, 2, 4,…, 2(L–3), L–3, L–3
$E_6^{(1)}$	6	0, 2, 2, 3, 4, 4, 6
$E_7^{(1)}$	12	0, 3, 4, 6, 6, 8, 9, 12
$E_8^{(1)}$	30	0, 6, 10, 12, 15, 18, 20, 24, 30

Table 1. The Coxeter number h and the Coxeter exponents m_j for simply-laced classical and affine Lie algebras.

where h and m_j are integers called the Coxeter number and Coxeter exponents respectively. These numbers are listed in Table 1. The corresponding Perron-Frobenius eigenvectors are listed in Table 2.

Algebra	Perron-Frobenius Eigenvector
A_L	$\left(\sin\frac{\pi}{L+1}, \sin\frac{2\pi}{L+1},\ldots, \sin\frac{L\pi}{L+1}\right)$
D_L	$\left(2\cos\frac{(L-2)\pi}{2L-2},\ldots, 2\cos\frac{2\pi}{2L-2}, 2\cos\frac{\pi}{2L-2}, 1, 1\right)$
E_6	$\left(\sin\frac{\pi}{12}, \sin\frac{\pi}{6}, \sin\frac{\pi}{4}, \sin\frac{\pi}{3}-\frac{\sin\pi/4}{2\cos\pi/12}, \sin\frac{5\pi}{12}-\sin\frac{\pi}{4}, \frac{\sin\pi/4}{2\cos\pi/12}\right)$
E_7	$\left(\sin\frac{\pi}{18}, \sin\frac{\pi}{9}, \sin\frac{\pi}{6}, \sin\frac{2\pi}{9}, \sin\frac{5\pi}{18}-\frac{\sin 2\pi/9}{2\cos\pi/18}, \sin\frac{\pi}{3}-\sin\frac{2\pi}{9}, \frac{\sin 2\pi/9}{2\cos\pi/18}\right)$
E_8	$\left(\sin\frac{\pi}{30}, \sin\frac{\pi}{15}, \sin\frac{\pi}{10}, \sin\frac{2\pi}{15}, \sin\frac{\pi}{6}, \sin\frac{\pi}{5}-\frac{\sin\pi/6}{2\cos\pi/30}, \sin\frac{7\pi}{30}-\sin\frac{\pi}{6}, \frac{\sin\pi/6}{2\cos\pi/30}\right)$
$A^{(1)}_{L-1}$	$(1, 1,\ldots, 1)$
$D^{(1)}_{L-1}$	$(1, 1, 2, 2,\ldots,2, 2, 1, 1)$
$E^{(1)}_6$	$(1, 2, 3, 2, 1, 2, 1)$
$E^{(1)}_7$	$(1, 2, 3, 4, 3, 2, 1, 2)$
$E^{(1)}_8$	$(1, 2, 3, 4, 5, 6, 4, 2, 3)$

Table 2. The Perron-Frobenius eigenvectors for simply-laced classical and affine Lie algebras. The order of the components is fixed by the labelling of sites in the Dynkin diagrams of Fig. 2

In the A–D–E classification, the critical $q = 2, 3, 4$ state Potts models correspond to Lie algebras A_3, D_4 and $D_4^{(1)}$ respectively. More generally, the classical A series corresponds to the ABF or RSOS models [14] and the affine A series corresponds to the cyclic solid-on-solid or CSOS models [15,16,17]. Classical D lattice models have been introduced by Pasquier [18] and affine D models by Akutsu, Kuniba and Wadati [19] and Choi, Kwon and Kim [20]. The A and D models admit off-critical elliptic extensions to the solution of the Yang-Baxter equations but no elliptic solutions are known for the exceptional cases.

3. $c < 1$ Theories

3.1 Conformal invariance

It has been known for some time that statistical mechanics models are scale invariant at criticality. The principle of scale invariance has led to the formulation of the renormalization group and implies, among other things, scaling relations among the critical exponents that characterize critical behaviour. Recently, the notion of scale invariance of critical systems has been refined by considering invariance under more general conformal [21,22] and modular [23,24] transformations. In two dimensions, the theory of conformal invariance predicts that the critical exponents fall into various universality classes classified by the central charge c of the corresponding Virasoro algebra. In particular, for unitary theories with $c < 1$, the critical exponents are quantized and a complete classification of critical behaviours can be given in terms of the unitary series of conformal field theories with central charge [25]

$$c = 1 - \frac{6}{h(h-1)} \tag{3.1}$$

where $h = 4, 5, 6, \ldots$. For these theories the scaling dimensions x and spins s of the various primary scaling operators are given by

$$x = \Delta_{r,s} + \bar{\Delta}_{r,s} \tag{3.2}$$
$$s = \Delta_{r,s} - \bar{\Delta}_{r,s} \tag{3.3}$$

in terms of the conformal weights $(\Delta_{r,s}, \bar{\Delta}_{r,s})$ of two copies of the Virasoro algebra, each given independently by the Kac formula [26]

$$\Delta_{r,s} = \frac{[hr - (h-1)s]^2 - 1}{4h(h-1)} \qquad 1 \le r \le h-2, 1 \le s \le h-1, r \ge s. \tag{3.4}$$

with $\bar{\Delta}_{r,s} = \Delta_{\bar{r},\bar{s}}$. The allowed values of the conformal weights for $h = 4, 5$ and 6 are listed in Table 3.

h = 4

s			
2		1/16	
1	0	1/2	
	1	2	r

h=5

s				
3			1/10	
2		3/80	3/5	
1	0	7/16	3/2	
	1	2	3	r

h=6

s					
4				1/8	
3			1/15	2/3	
2		1/40	21/40	13/8	
1	0	2/5	7/5	3	
	1	2	3	4	r

Table 3. Grids of conformal weights $\Delta_{r,s}$ for $h = 4, 5$ and 6.

The simplest possibility is $h = 4, c = 1/2$ with three spinless operators given by $(\Delta, \bar{\Delta}) = (0, 0)$, $(1/2, 1/2)$ and $(1/16, 1/16)$. In two dimensions, the thermal and magnetic scaling dimensions are related to the usual critical exponents by

$$2-\alpha=\frac{2}{2-x_\epsilon},\qquad \frac{2\beta}{2-\alpha}=x_\sigma. \tag{3.5}$$

Hence these three scaling fields are easily recognized as the identity (1), energy (ϵ) and magnetization (σ) operators of the Ising model which has critical exponents $\alpha=0$ and $\beta=1/8$. More generally, it was pointed by Huse [27] that the generic critical points of the ABF or A_L models, possessing Z_2 symmetry, realize the entire unitary series with $h=L+1$. More specifically, the scaling dimensions calculated from the known critical exponents of the order parameters [14] agree with allowed zero-spin values given by the Kac formula. Thus, for $h=4,5,6,\ldots$ the unitary series describes the critical point of the Ising model, the tricritical Ising model, the tetracritical Ising model and so on.

The application of conformal invariance gives a list of possible scaling dimensions that can appear in a conformal theory or critical lattice model with central charge $c<1$. It does not give the precise combinations of scaling operators allowed nor does it give any information on the degeneracy of operators. The answers to these questions are found by applying the stronger requirement of modular invariance on a torus.

3.2 Partition function on a torus

Let us consider a conformal field theory or critical lattice model on a finite $\ell\times\ell'$ periodic lattice or torus. The partition function can be written as

$$Z_{\ell,\ell'}=\exp(-\ell\ell' f)Z(q) \tag{3.6}$$

where f is the bulk free energy and $Z(q)$ is a universal term describing the leading finite-size corrections in the limit of ℓ,ℓ' large with the aspect ratio $\delta=\ell'/\ell$ fixed. The argument q is the modular parameter. For a spatially isotropic model, it is simply related to the aspect ratio δ by $q=\exp(-2\pi\delta)$. The partition function on a torus can be calculated from the eigenvalues of the row transfer matrix T of a periodic row of ℓ faces by use of the formula

$$Z_{\ell,\ell'}=\mathrm{Tr}\,T^{\ell'}=\sum_n\Lambda_n^{\ell'}=\sum_n\exp(-\ell' E_n) \tag{3.7}$$

where

$$\Lambda_n=\exp(-E_n) \tag{3.8}$$

are the eigenvalues of T and E_n are corresponding energy levels. But now conformal invariance dictates [28,29] that the leading finite-size corrections to the energy levels take the form

$$E_0 = \ell f-\frac{\pi c}{6\ell}\sin\theta \tag{3.9}$$

$$E_n-E_0 = \frac{2\pi}{\ell}\left(x_n\sin\theta+is_n\cos\theta\right) \tag{3.10}$$

where E_0 is the groundstate energy, $n = 1, 2, \ldots$, and

$$x_n = \Delta + \bar{\Delta} + k + \bar{k}, \quad s_n = \Delta - \bar{\Delta} + k - \bar{k}, \quad k, \bar{k} \in \mathrm{N}. \tag{3.11}$$

The angle θ is determined by the spatial anisotropy [30] and is given by (2.24).

The energy spectrum consists of a groundstate and an excited state for each primary operator. Above each of these is a tower of equally spaced levels or descendents. It follows that

$$Z(q) = (q\bar{q})^{-c/24} \sum_{\Delta,\bar{\Delta}} \sum_{k,\bar{k}} d_\Delta(k) d_{\bar{\Delta}}(\bar{k}) q^{\Delta+k} \bar{q}^{\bar{\Delta}+\bar{k}}. \tag{3.12}$$

The first sum is over all conformal weights allowed by the Kac formula including multiplicities. The second sum is over all nonnegative integers $k, \bar{k}$ and the factors $d_\Delta(k), d_{\bar{\Delta}}(\bar{k})$ are integers giving the degeneracy of the levels. The modular parameter is

$$q = \exp(2\pi i\tau), \qquad \tau = \frac{\ell'}{\ell} \exp[i(\pi - \theta)] \tag{3.13}$$

and $\bar{q}$ is the complex conjugate. Hence the universal finite-size partition function on a torus can be written as a sesquilinear form in Virasoro characters

$$Z(q) = \sum_{\Delta,\bar{\Delta}} \chi_\Delta(q) \mathcal{N}(\Delta, \bar{\Delta}) \chi_{\bar{\Delta}}(\bar{q}) \tag{3.14}$$

where the sum is now over distinct conformal weights and the integer $\mathcal{N}(\Delta, \bar{\Delta})$ gives the multiplicity of the primary operator $(\Delta, \bar{\Delta})$. Here

$$\chi_\Delta(q) = q^{-c/24} \sum_{k=0}^{\infty} d_\Delta(k) q^{\Delta+k} \tag{3.15}$$

is the character of the representation of the Virasoro algebra. It is given explicitly by

$$\chi_\Delta(q) = \chi_{\Delta_{r,s}}(q) = \chi_{r,s}(q) \tag{3.16}$$

$$= \frac{q^{-c/24}}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{n=-\infty}^{\infty} \left\{ q^{\frac{[2h(h-1)n+hr-(h-1)s]^2-1}{4h(h-1)}} - q^{\frac{[2h(h-1)n+hr+(h-1)s]^2-1}{4h(h-1)}} \right\}. \tag{3.17}$$

The set of numbers $\mathcal{N}(\Delta, \bar{\Delta}) \in \mathrm{N}$ is called the operator content of the theory. For the Ising model we have seen that there are three scaling operators. In this case the operator content is given by

$$\mathcal{N}(0,0) = \mathcal{N}(1/2, 1/2) = \mathcal{N}(1/16, 1/16) = 1, \qquad \mathcal{N}(\Delta, \bar{\Delta}) = 0, \Delta \neq \bar{\Delta} \tag{3.18}$$

and so

$$Z(q) = |\chi_0(q)|^2 + |\chi_{1/2}(q)|^2 + |\chi_{1/16}(q)|^2 \tag{3.19}$$

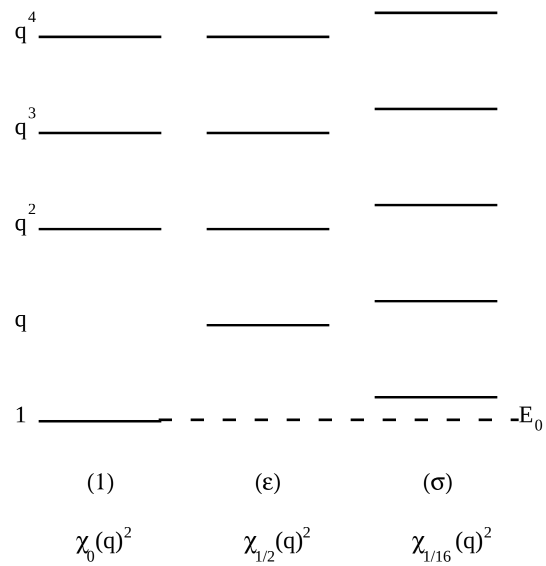

Figure 3. Equally spaced towers of levels for the isotropic Ising model above the identity, energy and magnetization operators. The degeneracies of the levels are given by the coefficients in the expansion of the squares of the Virasoro characters

where the $c = 1/2$ Virasoro characters can be written as

$$\begin{aligned} \chi_0(q) &= \tfrac{1}{2}q^{-\frac{1}{48}}\left\{\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}) + \prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}})\right\} \qquad (3.20)\\ &= q^{-\frac{1}{48}}(1+q^2+q^3+2q^4+\ldots)\\ \chi_{1/2}(q) &= \tfrac{1}{2}q^{-\frac{1}{48}}\left\{\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}) - \prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}})\right\} \qquad (3.21)\\ &= q^{-\frac{1}{48}+\frac{1}{2}}(1+q+q^2+q^3+\ldots)\\ \chi_{1/16}(q) &= q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1+q^n) \qquad (3.22)\\ &= q^{-\frac{1}{48}+\frac{1}{16}}(1+q+q^2+2q^3+\ldots). \end{aligned}$$

This situation is shown schematically, for the isotropic Ising model, in Fig. 3.

3.3 Modular invariance

The universal partition function $Z(q)$ must be invariant under transformations of the modular group generated by

$$T \;:\; \tau \mapsto 1+\tau \qquad (3.23)$$

$$S \;:\; \tau \mapsto -\frac{1}{\tau}. \qquad (3.24)$$

Here the torus is formed by identifying opposite sides of a parallelogram in the complex plane $\mathbf{C}$ with

$$q = e^{2\pi i\tau}, \qquad \operatorname{im}\tau > 0. \qquad (3.25)$$

The vertices of the torus are $0, 1, \tau$ and $1 + \tau$. The generators T and S map this torus into itself. Under the generators of the modular group the Virasoro characters transform as

$$T \quad : \qquad \chi_\Delta(q) \mapsto e^{2\pi i(\frac{c}{24}-\Delta)} \tag{3.26}$$

$$S \quad : \qquad \chi_\Delta(q) \mapsto \sum_{\Delta'} \mathcal{A}(\Delta, \Delta')\chi_{\Delta'}(q) \tag{3.27}$$

where the elements of the $\frac{1}{2}(h-1)(h-2) \times \frac{1}{2}(h-1)(h-2)$ matrix $\mathcal{A}$ are given by

$$\mathcal{A}(\Delta_{r,s}, \Delta'_{r',s'}) = \sqrt{\frac{8}{h(h-1)}}(-1)^{(r+s)(r'+s')} \sin \frac{\pi r r'}{h-1} \sin \frac{\pi s s'}{h}. \tag{3.28}$$

The matrix $\mathcal{A}$ is real symmetric, orthogonal and an involution

$$\mathcal{A}^{\mathrm{T}} = \mathcal{A} = \mathcal{A}^{-1}. \tag{3.29}$$

Hence the requirement that the sesquilinear form in Virasoro characters giving the partition function is modular invariant becomes

$$\mathcal{A}\mathcal{N} = \mathcal{N}\mathcal{A}, \qquad \Delta - \bar{\Delta} \in \mathbf{N}. \tag{3.30}$$

Since $\mathcal{N}$ is a nonnegative integer matrix this is a system of diophantine equations. One family of solutions to these equations for $h = 4, 5, \ldots$ is given by

$$\mathcal{A}(\Delta, \bar{\Delta}) = \delta(\Delta, \bar{\Delta}). \tag{3.31}$$

Clearly, this gives the operator content of the ABF [14] or classical A_L models with $h = L+1$. For $h = 4$ and 5 these are the only solutions corresponding to the Ising and tricritical Ising models. For $h = 6$, however, there are two solutions corresponding to the tetracritical Ising model and the three-state Potts models. The modular invariant partition function of the three-state Potts or D_4 model is

$$Z(q) = |\chi_0(q) + \chi_3(q)|^2 + |\chi_{2/5}(q) + \chi_{7/5}(q)|^2 + 2|\chi_{1/15}(q)|^2 + 2|\chi_{2/3}(q)|^2. \tag{3.32}$$

This solution, which involves only a subset of the allowed conformal weights, exhibits both degenerate primary operators and operators with spin.

The diophantine equations have in fact been solved in complete generality by Capelli, Itzykson and Zuber [31,32,33]. Remarkably, they found that the modular invariant sesquilinear forms in Virasoro characters, giving the universal partition functions of unitary conformal theories with $c < 1$, are in a one-to-one correspondence with the classical A–D–E Lie algebras. More specifically they found two infinite hierarchies corresponding to the A and D series and three exceptional solutions. Their results are tabulated in Table 4. The occurrence of the Coxeter numbers and Coxeter exponents in these expressions is striking. This classification has been derived from purely theoretical coniderations based on conformal and modular invariance. However, Pasquier [11] obtained precisely the same operator content and modular invariant partition functions by direct computation on the critical A–D–E lattice models.

Algebra	Modular Invariant Partition Function
A_L	$Z = \frac{1}{2}\sum_{r=1}^{h-2}\sum_{s=1}^{h-1}\lvert\chi_{r,s}\rvert^2$
D_L (L even)	$Z = \frac{1}{2}\sum_{r=1}^{h-2}\left\{\sum_{\substack{s=1\\ s\ \text{odd}}}^{\frac{h}{2}-2}\lvert\chi_{r,s}+\chi_{r,h-s}\rvert^2 + 2\lvert\chi_{r,h/2}\rvert^2\right\}$
D_L (L odd)	$Z = \frac{1}{2}\sum_{r=1}^{h-2}\left\{\sum_{\substack{s=1\\ s\ \text{odd}}}^{h-1}\lvert\chi_{r,s}\rvert^2 + \lvert\chi_{r,h/2}\rvert^2 + \sum_{\substack{s=2\\ s\ \text{even}}}^{\frac{h}{2}-2}(\chi_{r,s}\bar\chi_{r,h-s}+\bar\chi_{r,s}\chi_{r,h-s})\right\}$
E_6	$Z = \frac{1}{2}\sum_{r=1}^{h-2}\left\{\lvert\chi_{r,1}+\chi_{r,7}\rvert^2 + \lvert\chi_{r,4}+\chi_{r,8}\rvert^2 + \lvert\chi_{r,5}+\chi_{r,11}\rvert^2\right\}$
E_7	$Z = \frac{1}{2}\sum_{r=1}^{h-2}\Big\{\lvert\chi_{r,1}+\chi_{r,17}\rvert^2 + \lvert\chi_{r,5}+\chi_{r,13}\rvert^2 + \lvert\chi_{r,7}+\chi_{r,11}\rvert^2 + \lvert\chi_{r,9}\rvert^2$ $+ [(\chi_{r,3}+\chi_{r,15})\bar\chi_{r,9} + (\bar\chi_{r,3}+\bar\chi_{r,15})\chi_{r,9}]\Big\}$
E_8	$Z = \frac{1}{2}\sum_{r=1}^{h-2}\left\{\lvert\chi_{r,1}+\chi_{r,11}+\chi_{r,19}+\chi_{r,29}\rvert^2 + \lvert\chi_{r,7}+\chi_{r,13}+\chi_{r,17}+\chi_{r,23}\rvert^2\right\}$

Table 4. Modular invariant partition functions of classical A–D–E lattice models expressed as sesquilinear forms in Virasoro characters. Here $c = 1 - \frac{6}{h(h-1)}$ where h is the Coxeter number. The index s is ranges over the Coxeter exponents.

4. $c = 1$ Theories

It is natural to ask whether it is possible to classify the $c = 1$ two-dimensional conformal field theories [34] and critical lattice models [11] along the same lines as for the $c < 1$ theories. In particular, we would like to know if the critical affine A–D–E models give an exhaustive classification of $c = 1$ theories. The $c < 1$ theories possess a finite number of scaling operators with fixed rational scaling dimensions. By contrast, the $c = 1$ theories admit continuously varying scaling dimensions and an infinite number of scaling operators. The $c = 1$ theories are therefore intrinsically more difficult. Some progress has be made, however, for the $c = 1$ rational conformal field theories which we consider in this section.

4.1 Gaussian partition functions

The canonical example of a $c = 1$ theory is the Gaussian model (Coulomb gas, six-vertex model, scalar free boson compactified on a circle) [35,36] with the modular invariant partition function on a torus given by

$$Z(q,r)=\frac{1}{\eta(q)\eta(\bar{q})}\sum_{m,n\in\mathbb{Z}} q^{\Delta_{m,n}}\bar{q}^{\bar{\Delta}_{m,n}} \tag{4.1}$$

where

$$\eta(q)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \tag{4.2}$$

is the Dedekind eta function and r is a free parameter called the radius of compactifiation. The conformal weights are

$$\Delta_{m,n}=\tfrac{1}{2}(\frac{m}{2r}+nr)^2,\qquad \bar{\Delta}_{m,n}=\tfrac{1}{2}(\frac{m}{2r}-nr)^2. \tag{4.3}$$

so the scaling dimensions and spins are

$$x_{m,n} = \Delta_{m,n}+\bar{\Delta}_{m,n}=\frac{m^2}{4r^2}+n^2r^2 \tag{4.4}$$

$$s_{m,n} = \Delta_{m,n}-\bar{\Delta}_{m,n}=mn. \tag{4.5}$$

The theory is called rational if r^2 is rational. In this case all the conformal weights and scaling dimensions are also rational. It is straightforward to verify that $Z(q,r)$ is indeed modular invariant and satisfies the duality relation

$$Z(q,r)=Z(q,\frac{1}{2r}) \tag{4.6}$$

so that the Gaussian partition function is self-dual at $r=1/\sqrt{2}$.

The $c=1$ Virasoro characters are given by

$$\chi_\Delta(q)=\begin{cases} q^\Delta/\eta(q), & \Delta\neq n^2/4,\quad \Delta\geq 0\\ \left[q^{n^2/4}-q^{(n+2)^2/4}\right]/\eta(q), & \Delta\neq n^2/4,\quad n=1,2,3,\ldots. \end{cases} \tag{4.7}$$

It is therefore clear that $Z(q,r)$ can only be decomposed as a sesquilinear form in Virasoro characters with an infinite number of terms. Fortunately, the $c=1$ theories have a chiral or Kac-Moody symmetry in addition to the Virasoro symmetry. For the rational theories, it is then possible [37] to decompose the modular invariant partition function as a *finite* sesquilinear form in characters of the larger chiral or Kac-Moody algebra. Suppose that

$$r^2=\frac{p}{2p'},\quad p,p' \text{ coprime}. \tag{4.8}$$

The chiral or Kac-Moody characters at level $N=pp'$ are then defined by

$$\chi^{(1)}_{N,k}(q)=\frac{1}{\eta(q)}\sum_{n=-\infty}^{\infty} q^{N(n+\frac{k}{2N})^2},\quad N,k\in\mathbb{Z}. \tag{4.9}$$

Algebra	Modular Invariant Partition Function
$A^{(1)}_{L-1}$	Z_L (L odd), $Z_{L/2}$ (L even)
$D^{(1)}_{L-1}$	$\frac{1}{2}(Z_{L-3}+2Z_2-Z_1)$
$E^{(1)}_6$	$\frac{1}{2}(2Z_3+Z_2-Z_1)$
$E^{(1)}_7$	$\frac{1}{2}(Z_4+Z_3+Z_2-Z_1)$
$E^{(1)}_8$	$\frac{1}{2}(Z_5+Z_3+Z_2-Z_1)$
A_L	$\frac{1}{2}(Z_{L+1}-Z_1)$
D_L	$\frac{1}{2}(Z_{2(L-1)}-Z_{L-1}+Z_2-Z_1)$
E_6	$\frac{1}{2}(Z_{12}-Z_6-Z_4+Z_3+Z_2-Z_1)$
E_7	$\frac{1}{2}(Z_{18}-Z_9-Z_6+Z_3+Z_2-Z_1)$
E_8	$\frac{1}{2}(Z_{30}-Z_{15}-Z_{10}-Z_6+Z_5+Z_3+Z_2-Z_1)$

Table 5. Modular invariant partition functions of critical classical and affine A–D–E lattice models expressed as linear combinations of Gaussian partition functions. The modular parameter q is given by $q=\exp(2\pi i\tau), \tau=(\ell'/\ell)\exp[i(\pi-\theta)]$ with $\theta=\pi u$ for the affine models and $\theta=\pi u/\lambda$ for the classical models.

and the result of Kiritsis [37] states that the modular invariant partition functions of $c=1$ rational conformal field theories can all be written as

$$Z(q)=\sum_{\substack{(N,k)\\(\bar N,\bar k)}}\mathcal{N}_{N,k}^{\bar N,\bar k}\chi^{(1)}_{N,k}(q)\chi^{(1)}_{\bar N,\bar k}(\bar q). \tag{4.10}$$

where $\mathcal{N}_{n,k}^{\bar N,\bar k}$ are nonegative integers giving the operator content of the theory. In addition, Kiritsis also showed that the only modular invariants at level N are the Gaussian partition functions of the form

$$Z(q,\sqrt{\frac{p}{2p'}})=\sum_{a=0}^{p-1}\sum_{b=0}^{p'-1}\left\{\chi^{(1)}_{N,ap'+bp}(q)\chi^{(1)}_{N,ap'-bp}(\bar q)+\chi^{(1)}_{N,N+ap'+bp}(q)\chi^{(1)}_{N,N+ap'-bp}(\bar q)\right\}. \tag{4.11}$$

It therefore follows that the modular invariant partition function of a $c=1$ rational conformal field theory can also be written as a finite linear combination of Gaussian partition functions. Indeed, if we set

$$Z_n=Z(q,n/\sqrt{2}) \tag{4.12}$$

then the modular invariant partition functions of the critical affine A–D–E models are given by direct calculation [36,11] as the linear combinations of Gaussian partition functions shown in Table 5 where the modular parameter q is given by (3.13) with

$$\theta=\pi u. \tag{4.13}$$

Dijkgraaf, Verlinde and Verlinde [38] have shown that the A–D–E models exhaust all the conformal field theories that can be built from Gaussian partition functions. Under the further assumptions, that the operator algebra is consistent and that the identity operator has multiplicty 1, Kiritsis [37] has shown the subset of $c=1$ rational conformal field theories given by the critical affine A–D–E models is complete. A detailed discussion of the operator algebra of the rational conformal field theories can be found in Dijkgraaf, Vafa, Verlinde and Verlinde [39].

The $c<1$ characters can be written simply as differences of chiral or Kac-Moody characters at level $N=h(h-1)$

$$\chi_{r,s}(q)=\chi^{(1)}_{h(h-1),hr-(h-1)s}(q)-\chi^{(1)}_{h(h-1),hr+(h-1)s}(q). \tag{4.14}$$

From this identity it follows that the modular invariant partition functions of the classical A–D–E models can also be written [36] as a linear combinations of Gaussian partition functions as shown in Table 5.

5. $c>1, SU(n)$ and Beyond

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