FINITE-SIZE CORRECTIONS AND SCALING DIMENSIONS OF SOLVABLE LATTICE MODELS: AN ANALYTIC METHOD

Paul A. Pearce and Andreas Klümper

Mathematics Department, University of Melbourne,
Parkville Vic. 3052, Australia

A general analytic method for calculating finite-size corrections, central charges and scaling dimensions of solvable lattice models is presented. The approach is to solve the special functional equations or inversion identities satisfied by the commuting row transfer matrices of these lattice models at criticality. For purposes of illustration, the method is applied to calculate the central charge $c = 4/5$ and leading magnetic scaling dimension $x = 2/15$ of hard hexagons. These numbers are rational due to special values of Rogers dilogarithms.

PACS numbers: 05.50.+q, 75.10.Hk, 75.10.Jm

An outstanding success of conformal field theory is the precise quantitative prediction of critical exponents and scaling dimensions for many two-dimensional statistical systems. Critical behaviors are classified into universality classes according to the central charge c of the Virasoro algebra corresponding to the conformal group of symmetries at criticality. For models with $c < 1$, a complete classification of critical exponents can be given in terms of the unitary series with central charge [1,2]

$$c = 1 - \frac{6}{h(h-1)} \tag{1}$$

where $h = 4, 5, 6, \ldots$ In particular, the first few members of this series have been identified with the universality classes of the critical Ising model ($h = 4$), the tricritical Ising model ($h = 5$) and either the tetracritical Ising or three-state Potts models ($h = 6$). The scaling dimensions x give the large distance algebraic decay of correlations of the various scaling fields at criticality. Their values are determined by the conformal weights in the Kac table [3]. In two dimensions, the thermal and magnetic scaling dimensions, denoted by x_ϵ and x_σ respectively, are related to the usual critical exponents by the scaling relations

$$2 - \alpha = 2\nu = \frac{2}{2 - x_\epsilon}, \qquad \frac{2\beta}{2-\alpha} = x_\sigma. \tag{2}$$

The central charge and scaling dimensions thus completely encapsulate the critical behavior of these models.

The central charge and scaling dimensions of critical lattice systems are accessible [4,5] from finite-size corrections to row transfer matrix eigenvalues. Unfortunately, the widely adopted analytic methods for calculating the central charge [6,7,8,9,10] have proved too cumbersome to extend to calculations of scaling dimensions. Therefore much effort has been directed to numerical calculation of finite-size corrections (eg. [11,12,13,14]). In this letter, we present new methods to calculate *analytically* the finite-size corrections and scaling dimensions of critical lattice models. The approach is to solve special functional equations or inversion identities [15,16,17] and is quite general. For definiteness, however, we consider just the hard hexagon model [18,19,20,21] in this letter. A summary of further results is given in the concluding discussion and details will be published elsewhere [22,23].

The critical hard hexagon model is a special case of the generalized hard hexagon model. This is a lattice gas on the square lattice with nearest-neighbor exclusion. The spins or occupation numbers, denoted a, b etc., take the values 0 or 1 according to whether the site is empty or occupied. The spins on edges are restricted to the values $(a, b) = (0, 0), (1, 0)$ or $(0, 1)$. The weights of allowed faces are

$$W\begin{pmatrix} d & c \\ a & b \end{pmatrix} = \left[\frac{\sin(\lambda-u)}{\sin\lambda}\delta(a,c) + \frac{\sin u}{\sin\lambda}\sqrt{\frac{S_a S_c}{S_b S_d}}\;\delta(b,d)\right]$$

where $\lambda = \pi/5$ is the crossing parameter, $S_1 = \sin\lambda, S_0 = \sin 2\lambda$ and δ is the Kronecker delta. The spectral parameter u is related to spatial anisotropy and lies in the interval $-\pi/5 \le u \le 0$ in the physical regime with $u = -\pi/5$ for hard hexagons.

The face weights satisfy the Yang-Baxter equations [19], so the generalized hard hexagon model is exactly solvable and possesses a family of commuting row transfer matrices $\boldsymbol{V}(u)$ with elements

$$\langle\sigma|\boldsymbol{V}(u)|\sigma'\rangle = \prod_{j=1}^{N} W\begin{pmatrix} \sigma'_j & \sigma'_{j+1} \\ \sigma_j & \sigma_{j+1} \end{pmatrix}$$

where σ and σ' are two consecutive row configurations of an N column lattice with $\sigma_{N+1} = \sigma_1, \sigma'_{N+1} = \sigma'_1$. The eigenvalues of $\boldsymbol{V}(u)$ are entire functions of u and are determined by their zeros in the complex plane. The eigenvalues $T(u)$ of the normalized transfer matrices

$$\boldsymbol{T}(u) = \left[-\frac{\sin(2\lambda+u)\sin\lambda}{\sin(2\lambda-u)\sin(\lambda+u)}\right]^N \boldsymbol{V}(u)$$

satisfy [20] the inversion identity

$$T(u)T(u+\lambda) = 1 + T(u-2\lambda) \tag{3}$$

subject to the periodicity $T(u+\pi) = T(u)$ and crossing symmetry $\overline{T(u)} = T(\lambda - \overline{u})$. The inversion identity (3) is an exact equation for finite N and completely determines the eigenvalues $T(u)$.

Let $T(u) = \exp(-E_n)$ be an eigenvalue of $\boldsymbol{T}(u)$ and E_n the corresponding energy level where $n = 0, 1, 2, \ldots$ labels the levels. Then conformal invariance predicts that the leading finite-size corrections to the energy levels take the form [4,5]

$$\begin{aligned} E_0 &= Nf - \frac{\pi c}{6N}\sin\theta \\ E_n - E_0 &= \frac{2\pi}{N}\left(x_n \sin\theta + i s_n \cos\theta\right) \end{aligned} \tag{4}$$

where E_0 is the groundstate energy and f is the bulk free energy. The scaling dimensions and spins of the scaling fields are given by x_n and s_n. The angle θ is determined by the spatial anisotropy [24] and, for hard hexagons, is given by $\theta = -10u/3$.

To accommodate the groundstate of hard hexagons we restrict N to the values $N = 0 \pmod 3$. The largest eigenvalue $T(u)$ then has zeros which become dense on the lines $\mathrm{Re}(u) = -4\pi/10, \pi/10$ in the thermodynamic limit. To solve the inversion identity (3), we therefore distinguish two strips in the complex u plane where $T(u)$ is analytic and non-zero

$$\begin{array}{lrcl} \text{Strip 1}: & -\dfrac{4\pi}{10} < & \mathrm{Re}(u) & < \dfrac{\pi}{10} \\ \text{Strip 2}: & \dfrac{\pi}{10} < & \mathrm{Re}(u) & < \dfrac{6\pi}{10}. \end{array}$$

We represent $T(u)$ in these strips by the known bulk behavior [19] multiplied by some correction functions $l(u)$

$$\begin{aligned} T_1(u) &= z_1(u)^N l_1(u), \quad z_1(u) = \left[\frac{\sin(\frac{5u}{3} - \frac{\pi}{3})}{\sin(\frac{5u}{3} + \frac{\pi}{3})}\right] \\ T_2(u) &= z_2(u)^N l_2(u), \quad z_2(u) = \left[\frac{\sin\frac{5u}{3}}{\sin(\frac{5u}{3} - \frac{2\pi}{3})}\right] \end{aligned} \tag{5}$$

Inserting (5) into the inversion identity (3) we obtain

$$\begin{aligned} \frac{l_1(u)l_1(u+\lambda)}{l_2(u+3\lambda)} &= p_2(u) := 1 + \frac{1}{T_2(u+3\lambda)} \\ \frac{l_2(u)l_2(u+\lambda)}{l_1(u-2\lambda)} &= p_1(u) := 1 + \frac{1}{T_1(u-2\lambda)} \end{aligned} \tag{6}$$

where the subscripts refer to the relevant strip. The functions $l_1(u)$, $l_2(u)$, $p_1(u)$,and $p_2(u)$ are analytic, non-zero in their strips of analyticity and their logarithms tend to constants as $\mathrm{Im}(u) \to \pm\infty$. Therefore the derivatives of $\ln l_1(u)$ etc. possess Fourier transforms

$$\begin{aligned} L_1(k) &= \frac{1}{2\pi i}\int_{\mathrm{Re}(v)=x} dv [\ln l_1(v)]' e^{-kv} \\ [\ln l_1(v)]' &= \int_{-\infty}^{\infty} dk\, L_1(k)\, e^{ku} \end{aligned} \tag{7}$$

and so on, where the integration paths lie in the appropriate analyticity strips. Taking Fourier transforms of (6) we can solve the linear equations for $L_1(k)$ and $L_2(k)$ in terms of $P_1(k)$ and $P_2(k)$. Taking the inverse transform and evaluating the k integral then yields

$$\begin{aligned} [\ln l_1(u)]' &= \frac{1}{2\pi i}\frac{10}{\sqrt{3}}\int_{\mathrm{Re}(v)=\pi/4} dv\, [\ln p_1(v)]' \frac{\sin(\frac{5}{3}(u-v) + \frac{2\pi}{3})}{\sin 5(u-v)} \\ &+ \frac{1}{2\pi i}\frac{10}{\sqrt{3}}\int_{\mathrm{Re}(v)=-\pi/4} dv\, [\ln p_2(v)]' \frac{\sin(\frac{5}{3}(u-v) + \frac{\pi}{3})}{\sin 5(u-v)} \end{aligned}$$

$$\begin{aligned}[\ln l_2(u)]' &= \frac{1}{2\pi i}\frac{10}{\sqrt{3}}\int_{\mathrm{Re}(v)=\pi/4} dv\,[\ln p_1(v)]'\frac{\sin(\frac{5}{3}(u-v)+\frac{\pi}{3})}{\sin 5(u-v)} \\ &+ \frac{1}{2\pi i}\frac{10}{\sqrt{3}}\int_{\mathrm{Re}(v)=-\pi/4} dv\,[\ln p_2(v)]'\frac{\sin(\frac{5}{3}(u-v))}{\sin 5(u-v)} \qquad (8)\end{aligned}$$

Restricting the T and p functions to appropriate lines we define the following functions of a real variable x

$$\begin{aligned}\mathfrak{a}(x) &:= 1/T_1\Big(\frac{3i}{10}x-\frac{3\pi}{20}\Big), \qquad \mathfrak{A}(x) := p_1\Big(\frac{3i}{10}x+\frac{\pi}{4}\Big) = 1+\mathfrak{a}(x) \\ \mathfrak{b}(x) &:= 1/T_2\Big(\frac{3i}{10}x+\frac{7\pi}{20}\Big), \qquad \mathfrak{B}(x) := p_2\Big(\frac{3i}{10}x-\frac{\pi}{4}\Big) = 1+\mathfrak{b}(x). \qquad (9)\end{aligned}$$

After integrating (8) with respect to u and introducing these functions we obtain the coupled nonlinear integral equations

$$\begin{aligned}\ln\mathfrak{a}(x) &= -\ln z_1(3ix/10-3\pi/20)^N - s*\ln\mathfrak{A} - c*\ln\mathfrak{B} + D_1, \\ \ln\mathfrak{b}(x) &= -\ln z_2(3ix/10+7\pi/20)^N - c*\ln\mathfrak{A} - s*\ln\mathfrak{B} + D_2, \qquad (10)\end{aligned}$$

where $*$ denotes convolution and the kernels s and c are given by

$$s(y) = \frac{\sqrt{3}}{2\pi}\frac{\sinh\frac{1}{2}y}{\sinh\frac{3}{2}y}, \qquad c(y) = \frac{\sqrt{3}}{2\pi}\frac{\cosh\frac{1}{2}y}{\cosh\frac{3}{2}y}.$$

We next evaluate the constants D_1 and D_2 from the asymptotic behavior of $T(u)$

$$\lim_{\mathrm{Im}(u)\to\pm\infty} T(u) = (1+\sqrt{5})/2. \qquad (11)$$

Hence taking the limit $x\to\infty$ in (10) we deduce $D_1 = D_2 = 0$.

The characteristic length scale of the spread of zeros on the lines $\mathrm{Re}(u) = -4\pi/10, \pi/10$ is $\ln N$. If we replace x with $x+\ln N$ in (10), both of the first terms on the right side approach $-\sqrt{3}e^{-x}$ as $N\to\infty$. Assuming that a, b, A, and B scale accordingly, we define

$$\begin{aligned}a(x) &:= \lim_{N\to\infty}\mathfrak{a}(x+\ln N) \\ A(x) &:= \lim_{N\to\infty}\mathfrak{A}(x+\ln N) = 1+a(x) \qquad (12)\end{aligned}$$

with $la(x) := \ln a(x), lA(x) := \ln A(x)$ and analogous definitions for $b(x), lb(x), B(x)$ and $lB(x)$. From (10) we then obtain

$$\begin{aligned}la(x) &= -\sqrt{3}e^{-x} - s*lA - c*lB \\ lb(x) &= -\sqrt{3}e^{-x} - c*lA - s*lB. \qquad (13)\end{aligned}$$

We next turn to the eigenvalue in the physical strip $T_1(u)$. Taking the variable $u = 3ix/10 - 3\pi/20$, we see from (9) and (10) that the finite-size correction is

$$\begin{aligned}&s*\ln\mathfrak{A} + c*\ln\mathfrak{B} \\ &\quad= \int_0^\infty dy\,[s(x-y)+c(x+y)]\,\ln\mathfrak{A}(y) \\ &\quad+ \int_0^\infty dy\,[c(x-y)+s(x+y)]\,\ln\mathfrak{B}(y) \\ &\quad\simeq \frac{\sqrt{3}}{\pi N}\cosh x\int_{-\infty}^\infty dy\,e^{-y}\,[lA(y)+lB(y)]. \qquad (14)\end{aligned}$$

where we have used $\mathfrak{A}(-x) = \mathfrak{B}(x)$ and in the last step we have scaled the integration variable. The contribution to the integrals outside the scaling regime is exponentially small. Remarkably, the final integral in (14) can be calculated from (13) without explicitly solving this set of integral equations. For this purpose we first take the derivative of (13). Multiplying (13) with $lA'(x)$, $lB'(x)$ and its derivative with $lA(x)$, $lB(x)$, subtracting and integrating we obtain

$$\begin{aligned}
\sqrt{3} \quad & \int_{-\infty}^{\infty} dx\, e^{-x}\, [lA(x) + lA'(x) + lB(x) + lB'(x)] \\
= \quad & \int_{-\infty}^{\infty} dx [la'(x) lA(x) - la(x) lA'(x) + lb'(x) lB(x) - lb(x) lB'(x)]
\end{aligned} \tag{15}$$

where the contributions of the kernels cancel due to symmetry. We then integrate the left side by parts and change the variable of integration x to a and b on the right side to obtain

$$\begin{aligned}
& 2\sqrt{3} \int_{-\infty}^{\infty} dx\, e^{-x}\, [lA(x) + lB(x)] = \\
& \int_{a(-\infty)}^{a(\infty)} da \Big[\frac{\ln(1+a)}{a} - \frac{\ln a}{1+a}\Big] + \int_{b(-\infty)}^{b(\infty)} db \Big[\frac{\ln(1+b)}{b} - \frac{\ln b}{1+b}\Big] = \frac{4}{5}\frac{\pi^2}{3}
\end{aligned} \tag{16}$$

The last integrals are related to Rogers dilogarithms [25] and are simply evaluated for these special values of the terminals. The asymptotic values

$$a(\infty) = b(\infty) = (\sqrt{5} - 1)/2, \quad a(-\infty) = b(-\infty) = 0$$

follow from (5), (11) after recalling the definitions (9), (12). Combining (14) and (16) and using $\cosh x = -\sin 10u/3$ gives the desired result for the largest eigenvalue in the physical strip

$$E_0 = -\ln T(u) = -N \ln z_1(u) + \frac{4}{5}\frac{\pi}{6N} \sin\frac{10u}{3}$$

from which we obtain the central charge $c = 4/5$. Since the hard hexagon model is in the same universality class as the three-state Potts model, this agrees with the accepted value given by (1) with $h = 6$.

The two next-largest eigenvalues of $\boldsymbol{T}(u)$ are a complex conjugate pair with zeros distributed asymmetrically on the lines $\mathrm{Re}(u) = -4\pi/10, \pi/10$. These eigenvalues have the same analytic properties as the largest eigenvalue but different asymptotics given by

$$\lim_{\mathrm{Im}(u)\to\pm\infty} T(u) = (1 - \sqrt{5})/2$$

$$a(\infty) = b(\infty) = -(1 + \sqrt{5})/2, \quad a(-\infty) = b(-\infty) = 0.$$

Some care needs to be taken in treating the branches of logarithms. Picking one of the two complex conjugate eigenvalues, yields

$$\begin{aligned}
la(\infty) = \overline{lb(\infty)} &= \ln[(1 + \sqrt{5})/2] - \pi i \\
lA(\infty) = \overline{lB(\infty)} &= \ln[(\sqrt{5} - 1)/2] - \pi i
\end{aligned}$$

Evaluation of the constants in (10) then gives $D_1 = -D_2 = 2\pi i/3$.

Proceeding as before, we obtain (15) with the additional term

$$-\frac{2\pi i}{3}\int_{-\infty}^{\infty} dx\ [lA'(x) - lB'(x)] = -\frac{4\pi^2}{3}$$

on the right side. The dilogarithm integrals in (16) can still be evaluated but now they are along appropriate edges of the branch cuts. The contribution from these integrals is

$$4\int_0^1 dy\ \frac{\ln y}{1-y} + 2\int_1^{(1+\sqrt{5})/2} dy\Big[\frac{\ln(y-1)}{y} - \frac{\ln y}{y-1}\Big] + 2\pi^2 = \frac{16}{5}\frac{\pi^2}{3}.$$

Putting everything together finally leads to the result

$$\begin{aligned} E_1 &= -N\ln z_1(u) + \frac{2\pi i}{3} + (\frac{16}{5} - 4)\frac{\pi}{6N}\sin\frac{10u}{3} \\ &= E_0 + \frac{2\pi i}{3} - \frac{2\pi}{N}\frac{2}{15}\sin\frac{10u}{3} \end{aligned}$$

Hence the associated magnetic scaling dimension is $x = 2/15$. From (2) this agrees with the known critical exponents $\alpha = 1/3$ and $\beta = 1/9$.

The scaling dimensions of further excitations and their towers can be calculated by modifying the above arguments to allow for a finite number of zeros on the lines $\mathrm{Re}(u) = -\pi/5, -\pi/10, 3\pi/10$ and $2\pi/5$. In particular, the thermal scaling dimension $x = 4/5$ corresponding to $\alpha = 1/3$ is obtained by exciting a pair of zeros onto the line $\mathrm{Re}(u) = -\pi/5$ and another pair of zeros onto the line $\mathrm{Re}(u) = 2\pi/5$. Each isolated zero must satisfy an additional equation obtained by requiring that the right side of (3) vanishes. In the scaling limit, the logarithm of this equation admits many solutions differing in the choice of integer multiples of $2\pi i$. This corresponds to the many locations at which a particular zero can be excited and is the origin of the towers of integer spaced levels above the primary states.

The central charge and scaling dimensions of the tricritical hard squares model [21] can also be obtained by the above methods. The Boltzmann weights are identical to those of the generalized hard hexagon model only the physical regime is now $0 \le u \le \pi/5$. Since the eigenvalues cross at $u = 0$ different eigenvalues dominate. The largest eigenvalue in this case has zeros which accumulate on the lines $\mathrm{Re}(u) = -\pi/10, 3\pi/10$ and the excitations admit a finite number of zeros shifted onto the lines $\mathrm{Re}(u) = -\pi/5, \pi/10, 2\pi/5, 3\pi/5$ in the complex u plane. The central charge is found to be $c = 7/10$, as given by (1) with $h = 5$, so tricritical hard squares is in the same universality class as the tricritical Ising model as expected. In summary, our complete results [22] for these models are

Critical Hard Hexagons:

$$c = 4/5, \quad x = 2/15, 4/5, 17/15, 4/3, 9/5$$

Tricritical Hard Squares:

$$c = 7/10, \quad x = 3/40, 1/5, 7/8, 6/5.$$

Finally, the methods presented here can also been applied to the 6 and 19-vertex models and their related quantum spin chains. In particular, the central charges of the spin-$\frac{1}{2}$ and spin-1 XXZ chains considered previously [26] can now be obtained analytically [23] with the result

$$c = \frac{3s}{s+1}\left[1 - \frac{4(s+1)\phi^2}{\pi(\pi - 2s\gamma)}\right], \qquad s = \tfrac{1}{2}, 1$$

where γ is the anisotropy and ϕ is the boundary twist. In particular, for cyclic boundary conditions with no twist, this gives the familiar values $c = 1$ and $c = 3/2$ for spin-$\frac{1}{2}$ and spin-1 XXZ chains. On the other hand, if $\phi = \gamma = \pi/(m + 2s)$ this yields the minimal and superconformal series.

Acknowledgements

This work is supported by a grant from the Australian Research Council. A.K. acknowledges a Fellowship from Deutsche Forschungsgemeinschaft (DFG).

References

1. A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, *Nucl. Phys. B* **241** (1984), 333
2. D. Friedan, Z. Qiu and S. Shenker, *Phys. Rev. Lett.* **52** (1984), 1575 ; in "Vertex Operators in Mathematics and Physics", eds. J. Lepowsky, S. Mandelstam and I.M. Singer, Springer, 1984
3. V. G. Kac, *Lecture Notes in Physics* **94** (1979), 441–445
4. H. W. J. Blöte, J. L. Cardy and M. P. Nightingale, *Phys. Rev. Lett.* **56** (1986), 742
5. I. Affleck, *Phys. Rev. Lett.* **56** (1986), 746
6. C. J. Hamer, *J. Phys. A* **18** (1985), L1133
7. C. J. Hamer, *J. Phys. A* **19** (1986), 3335
8. H. J. de Vega and F. Woynarovich, *Nucl. Phys. B* **251** (1985), 439
9. A. N. Kirillov and N. Yu. Reshetikhin, *J. Phys. A* **19** (1986), 565
10. V.V. Bazhanov and N. Yu. Reshetikhin , *Int. J. Mod. Phys. A* **4** (1989), 115–42
11. G. von Gehlen and V. Rittenberg, *J. Phys. A* **20** (1987), 227
12. P. A. Pearce and D. Kim , *J. Phys. A* (1987), 20, 6471-85
13. F. C. Alcaraz, M. N. Barber and M. T. Batchelor , *Ann. Phys. (N.Y.)* **182** (1988), 280
14. F. C. Alcaraz and M. J. Martins, *J. Phys. A* **23** (1990), 1439-51
15. N. Yu. Reshetikhin , *Lett. Math. Phys.* **7** (1983), 205–13

16. P. A. Pearce , *Phys. Rev. Lett.* **58** (1987), 1502–4

17. P. A. Pearce , *J. Phys. A* **20** (1987), 6463–9

18. R. J. Baxter , *J. Phys. A* **13** (1980), L61–70

19. R. J. Baxter , "Exactly Solved Models in Statistical Mechanics", Academic Press, London, 1982.

20. R. J. Baxter and P. A. Pearce , *J. Phys. A* **15** (1982), 897

21. R. J. Baxter and P. A. Pearce , *J. Phys. A* **16** (1983), 2239

22. KlumP90 to be supplied.

23. A. Klümper, M. T. Batchelor and P. A. Pearce , to be published

24. D. Kim and P. A. Pearce , *J. Phys. A* **20** (1987), L451–6

25. L. Lewin, Dilogarithms and Associated Functions, MacDonald, London, 1958

26. A. Klümper and M. T. Batchelor , *J. Phys. A* **23** (1990), L189