CONFORMAL WEIGHTS OF RSOS LATTICE MODELS AND THEIR FUSION HIERARCHIES

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The finite-size corrections, central charges c and conformal weights Δ of L-state restricted solid-on-solid lattice models and their fusion hierarchies are calculated analytically. This is achieved by solving special functional equations, in the form of inversion identity hierarchies, satisfied by the commuting row transfer matrices at criticality. The results are all obtained in terms of Rogers dilogarithms. The RSOS models exhibit two distinct critical regimes. For the regime III/IV critical line, we find $c = \frac{3p}{p+2}\left[1 - \frac{2(p+2)}{r(r-p)}\right]$ where $L = r-1$ is the number of heights and $p = 1, 2, \ldots$ is the fusion level. The conformal weights are given by the generalized Kac formula $\Delta = \frac{[rt-(r-p)s]^2-p^2}{4pr(r-p)} + \frac{(s_0-1)(p-s_0+1)}{2p(p+2)}$ where $s = 1, 2, \ldots, r-1$; $t = 1, 2, \ldots, r-p-1$; $1 \le s_0 \le p+1$ and $s_0 - 1 = \pm(t-s) \bmod 2p$. For $p = 1, 2$ these models are described by the unitary minimal conformal series and the discrete superconformal series respectively. For the regime I/II critical line, we obtain $c = \frac{2(N-1)}{N+2}$ and $\Delta = \frac{l(l+2)}{4(N+2)} - \frac{m^2}{4N}$ for the conformal weights, independent of the fusion level p, where $N = L-1$, $l = 0, 1, \ldots, N-1$ and $m = -l, -l+2, \ldots, l-2, l$. In this critical regime the models are described by Z_N parafermion theories.

1 Introduction

In 1984 there appeared a remarkable series of papers [1,2,3,4]. In these papers, the unitary minimal conformal field theories with central charge $c < 1$ were completely classified into a discrete series with central charge

$$c = 1 - \frac{6}{L(L+1)}, \qquad L = 3, 4, 5, \ldots \tag{1.1}$$

and conformal weights given by the Kac formula [5]

$$\Delta = \frac{[(L+1)t - Ls]^2 - 1}{4L(L+1)}, \qquad 1 \le t \le L-1,\ 1 \le s \le L,\ s \le t. \tag{1.2}$$

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Moreover, it was shown that the members of this series are realized as the critical continuum limits of an exactly solvable family of L-state restricted solid-on-solid (RSOS) lattice models introduced by Andrews, Baxter and Forrester (ABF). More precisely, this observation applies to the critical line separating regimes III and IV of the RSOS models. These models lie in the universality classes of generic multicritical Ising models. The critical line separating regimes I and II, on the other hand, appears to be described by Z_N parafermion theories [6] with central charge

$$c = \frac{2(N-1)}{N+2}, \qquad N = L-1 = 2,3,4,\ldots \tag{1.3}$$

and conformal weights given by

$$\Delta = \frac{l(l+2)}{4(N+2)} - \frac{m^2}{4N} \tag{1.4}$$

where

$$l = 0,1,\ldots,N, \qquad m = -l,-l+2,\ldots,l-2,l. \tag{1.5}$$

It is important to extend these observations to the fusion RSOS(p,q) models [7,8,9]. These lattice models, obtained by fusing $p\times q$ blocks of face weights together, are related to coset conformal field theories obtained by the Goddard-Kent-Olive construction [10,11,12]. For the case $p = q$, Bazhanov and Reshetikhin [9] have obtained the central charge (1.3) independent of p on the regime I/II critical line and

$$c = \frac{3p}{p+2}\left[1 - \frac{2(p+2)}{r(r-p)}\right] \tag{1.6}$$

for the central charge on the III/IV critical line. Based on corner transfer matrix calculations and a correspondence principle, the Kyoto school [7] also conjecture the conformal weights are given by

$$\Delta = \frac{[rt-(r-p)s]^2-p^2}{4pr(r-p)} + \frac{(s_0-1)(p-s_0+1)}{2p(p+2)} \tag{1.7}$$

on the regime III/IV critical line where $r = L+1$, $p = 1,2,\ldots$ is the fusion level,

$$s = 1,2,\ldots,r-1, \;\; t = 1,2,\ldots r-p-1 \tag{1.8}$$

and s_0 is the unique integer determined by

$$1 \le s_0 \le p+1, \quad s_0 - 1 = \pm(t-s) \bmod 2p. \tag{1.9}$$

For the regime I/II critical line, their results suggest that (1.4) holds independent of the fusion level $p = q$.

In this paper we confirm all of the above results by direct analytic calculation of the finite size corrections [13,14] to the eigenvalue spectra of the row transfer matrices for these lattice models. This is achieved by generalizing the analytic methods recently introduced [15,16] to obtain the central charges and scaling dimensions of critical hard hexagons and tricritical hard squares. These particular models are a special case of the general RSOS models with $p = 1$ and $L = 4$. The key step in the calculation is to establish special functional equations for the eigenvalues in the form of inversion identity hierarchies. These equations, which are exact for finite size, extend the usual inversion identities [17,18,19] to the fusion RSOS(p, q) models.

The layout of the paper is as follows. In Section 1.1 we discuss conformal invariance and summarize the expected modular invariant partition functions for the RSOS(p, q) models. Explicit expressions are also given for the branching coefficients and string functions that enter these formulas. The significance of the modular invariant partition functions derive from the fact they are the generating functions of the finite-size scaling spectra of the row transfer matrices at criticality. They encapsulate complete information on all the eigenvalues and their degeneracies. Some preliminaries concerning the row transfer matrices of the RSOS(p, q) models are presented in Section 2. In particular, the critical RSOS(p, q) lattice models are defined in Section 2.1 and Section 2.2 introduces the inversion identity hierarchies satisfied by the row transfer matrices of these models. The zeros and poles of the row transfer matrices are discussed in Section 2.3 and their asymptotics in Section 2.4. The details of the calculations of the central charge and conformal weights in the critical regimes III/IV and I/II are presented respectively in Sections 3 and 4. The paper concludes, in Section 5, with a brief discussion.

1.1 Modular invariant partition functions

The partition function of a lattice model on a finite $M \times N$ periodic lattice or torus can be written as

$$Z_{M,N} = \exp(-MNf)Z(q) \tag{1.10}$$

where f is the bulk free energy and $Z(q)$ is the universal finite-size partition function with modular parameter q. In terms of the eigenvalues

$$\Lambda_n = \exp(-E_n), \quad n = 0, 1, 2, \ldots \tag{1.11}$$

of the transfer matrix $\boldsymbol{T}$, the partition function is

$$Z_{M,N} = \mathrm{Tr}\, \boldsymbol{T}^M = \sum_n \Lambda_n^M = \sum_n \exp(-ME_n). \tag{1.12}$$

From conformal invariance, the leading finite-size corrections to the energy levels E_n take the form [13,14]

$$\begin{aligned} E_0 &= Nf - \frac{\pi c}{6N}\sin\vartheta \\ E_n - E_0 &= \frac{2\pi}{N}\left(x_n \sin\vartheta + i s_n \cos\vartheta\right) \end{aligned} \tag{1.13}$$

where E_0 is the groundstate energy. The angle ϑ is determined by the spatial anisotropy [20]. For the III/IV critical line it is given by $\vartheta = ru$. The scaling dimensions x_n and spins s_n of the various levels, corresponding to different scaling fields, are given by

$$x_n = \Delta + \overline{\Delta} + k + \overline{k}, \quad s_n = \Delta - \overline{\Delta} + k - \overline{k} \tag{1.14}$$

where $k, \overline{k}$ are integers and $(\Delta, \overline{\Delta})$ are the conformal weights of the primary operators. In this paper, it turns out that the primary operators all have zero spin $s = 0$ so the scaling dimensions in this case are just given by $x = 2\Delta$.

The precise form of the finite-size partition function $Z(q)$ is constrained by modular invariance. For the ABF models on the regime III/IV critical line, this modular invariant is given [21] as a diagonal sesquilinear form in Virasoro characters

$$Z(q) = \sum_{\Delta} |\chi_\Delta(q)|^2 \tag{1.15}$$

where the sum is over the conformal weights in the Kac table (1.2) and the modular parameter is given by

$$q = \exp(2\pi i \tau), \qquad \tau = \frac{M}{N} \exp[i(\pi - \vartheta)], \qquad \vartheta = ru. \tag{1.16}$$

The Virasoro characters are defined by

$$\begin{aligned} \chi_\Delta(q) &= \chi_{\Delta_{t,s}}(q) \\ &= q^{-c/24} Q(q)^{-1} \sum_{n=-\infty}^{\infty} \left\{ q^{\frac{[2r(r-1)n+rt-(r-1)s]^2-1}{4r(r-1)}} - q^{\frac{[2r(r-1)n+rt+(r-1)s]^2-1}{4r(r-1)}} \right\} \end{aligned} \tag{1.17}$$

where

$$Q(q) = \prod_{n=1}^{\infty} (1 - q^n) \tag{1.18}$$

and the central charge is

$$c = 1 - \frac{6}{r(r-1)}. \tag{1.19}$$

The modular invariant partition function for the fusion RSOS(p,p) models on the regime III/IV critical line is a diagonal sesquilinear form in branching coefficients

$$Z(q) = \sum_{ts_0s} |b_{ts_0s}(q)|^2 \tag{1.20}$$

where the sum is over the generalized Kac table

$$s = 1, 2, \ldots, r-1, \quad t = 1, 2, \ldots r-p-1 \tag{1.21}$$

with the integer s_0 determined uniquely by the conditions

$$1 \le s_0 \le p+1, \quad s_0 - 1 = \pm(t-s) \bmod 2p. \tag{1.22}$$

The modular parameter q is independent of p and is given by (1.16) with $\vartheta = ru$. The branching coefficients are defined [7,8] by

$$b_{j_1j_2j_3}(q) = q^{\frac{j_1^2}{4m_1}+\frac{j_2^2}{4m_2}-\frac{j_1^2}{4m_3}-\frac{1}{8}} Q(q)^{-3} \sum_{\epsilon_1,\epsilon_2=\pm1} \left(\sum_{k,n_1,n_2}^{(1)} - \sum_{k,n_1,n_2}^{(2)} \right)$$

$$\times (-1)^{k+\frac{\epsilon_1+\epsilon_2}{2}} q^{k(k-1)/2+k(j_3+1)/2+\sum_{i=1}^{2}[k\epsilon_i(m_in_i+j_i/2)+m_in_i^2+j_in_i]} \tag{1.23}$$

where

$$m_1 = r-p, \quad m_2 = p+2, \quad m_3 = r \tag{1.24}$$

and the two sums are restricted to values of k, n_1, n_2 satisfying

$$\begin{aligned} \sum^{(1)}: &\quad k \ge \xi+1, \quad \eta \le \frac{j_3+1}{2} + \sum_{i=1}^{2} \epsilon_i(m_in_i + \frac{j_i}{2}) \in \mathsf{Z} \\ \sum^{(2)}: &\quad k \le \xi, \quad \eta-1 \ge \frac{j_3+1}{2} + \sum_{i=1}^{2} \epsilon_i(m_in_i + \frac{j_i}{2}) \in \mathsf{Z}\ . \end{aligned} \tag{1.25}$$

The integers $\xi = \xi(\epsilon_1, n_1)$ and $\eta = \eta(\epsilon_1, n_1)$ can be chosen arbitrarily for fixed ϵ_1 and n_1.

The modular invariant partition functions of the fusion RSOS(p,p) models on the regime I/II critical line are independent of p and given by the A type modular invariant of Gepner and Qiu [22]

$$Z(q) = \tfrac{1}{2} \sum_{l,m} |c_m^l(q)|^2 \tag{1.26}$$

where $c_m^l(q)$ are su(2) string functions of the Z_N parafermion algebra [6] and the sum is over l, m satisfying

$$0 \le l \le N, \quad 0 \le m < 2N, \quad l,m \in \mathsf{Z}\ , \quad l-m = 0 \bmod 2. \tag{1.27}$$

The modular parameter is again given by (1.16) only now the effective angle ϑ is given by

$$\vartheta = -\frac{2(N+2)u}{N} \tag{1.28}$$

where $-\frac{\pi}{2} + \frac{\pi}{N+2} \le u \le 0$. For $-l \le m \le l$, the su(2) string functions are given explicitly [23] by

$$\begin{aligned} c_m^l(q) &= Q(q)^{-3} q^{\frac{l(l+2)}{4(N+2)}-\frac{m^2}{4N}-\frac{c}{24}} \\ &\times \Bigg[\left(\sum_{s\ge0}\sum_{n\ge0} - \sum_{s<0}\sum_{n<0} \right) (-1)^s q^{s(s+1)/2+(l+1)n+(l+m)s/2+(N+2)(n+s)n} \\ &+ \left(\sum_{s>0}\sum_{n\ge0} - \sum_{s\le0}\sum_{n<0} \right) (-1)^s q^{s(s+1)/2+(l+1)n+(l-m)s/2+(N+2)(n+s)n} \Bigg] \end{aligned} \tag{1.29}$$

where the central charge is

$$c=\frac{2(N-1)}{N+2},\qquad N=L-1=2,3,4,\ldots \tag{1.30}$$

These definitions are extended to other values of m by the symmetries

$$c^l_m=c^l_{-m}=c^l_{m+2N}=c^{N-l}_{N-m}. \tag{1.31}$$

The conformal weights are immediately seen to be

$$\Delta=\frac{l(l+2)}{4(N+2)}-\frac{m^2}{4N}. \tag{1.32}$$

For $N=2$ and 3 these forms reproduce the known modular invariant partition functions and conformal weights of the Ising model and the hard hexagon model. In the latter case, the modular invariant partition function is precisely the same as for the 3-state Potts models as is expected on universality grounds. In particular, for the Z_3 models, the string functions coincide with the characters of the W_3 algebra and are simply related to the Virasoro characters

$$c^0_0=\chi_0+\chi_3,\quad c^2_0=\chi_{2/5}+\chi_{7/5},\quad c^0_2=\chi_{2/3},\quad c^1_1=\chi_{1/15} \tag{1.33}$$

so that

$$\begin{aligned}Z(q)&=|c^0_0|^2+|c^2_0|^2+2|c^0_2|^2+2|c^1_1|^2\\&=|\chi_0+\chi_3|^2+|\chi_{2/5}+\chi_{7/5}|^2+2|\chi_{2/3}|^2+2|\chi_{1/15}|^2.\end{aligned} \tag{1.34}$$

The modular invariant partition functions are the generating functions for the finite size scaling spectra of the row transfer matrices at criticality. When expanded in a small q expansion, these functions tell us not only the various energy levels that occur but also their degeneracies. The functional equations that we solve in this paper, on the other hand, give us the various energy levels but tell us nothing about the degeneracies. Of course, it is relatively easy to identify the degeneracies of the first few relevant levels by comparison with numerical results as was done, for example, in the study of hard squares and hexagons [15,16]. In general, however, it remains an open problem to calculate the degeneracy of arbitrary levels directly for solvable lattice models.

2 Transfer Matrices and Inversion Identity Hierarchies

2.1 Critical RSOS(p,q) lattice models

The restricted solid-on-solid models of Andrews, Baxter and Forrester [3] are IRF or interaction-round-a-face lattice models [24]. The heights or spins a,b,c,d *etc.* at each site of the square lattice take the values $1,2,\ldots,L$. These spins are subject to the nearest neighbour constraint $a-b=\pm1$ for each pair of adjacent spins a,b. The spins therefore take values on the Dynkin diagram of the classical Lie algebra A_L as shown in Figure 1. The statistical weight assigned to an elementary face of the lattice is zero unless all four pairs of adjacent spins on the edges are allowed. The weights of allowed faces are given by

$$W\begin{pmatrix} d & c \\ a & b \end{pmatrix}\!\!\left|\, u\right) = \frac{\sin(\lambda-u)}{\sin\lambda}\delta(a,c) + \frac{\sin u}{\sin\lambda}\sqrt{\frac{S_a S_c}{S_b S_d}}\,\frac{g_a}{g_c}\;\delta(b,d) \tag{2.1}$$

where

$$\lambda = \pi/(L+1), \qquad S_a = \sin a\lambda. \tag{2.2}$$

The spectral parameter u is related to spatial anisotropy, λ is the crossing parameter and δ is the Kronecker delta. The factors g_a are arbitrary gauge factors that cancel out on a periodic lattice. On the regime III/IV critical line the spectral parameter lies in the interval $0 \le u \le \lambda$ while on the regime I/II critical line the spectral parameter lies in the interval $-\frac{\pi}{2}+\lambda \le u \le 0$.

The face weights of the fused RSOS(p,q) models, with $\max(p,q) \le L-1$, are obtained by the fusion process illustrated graphically in Figure 1. This procedure only works for the special choice of gauge factors $g_a = (-1)^{a/2}$. Explicitly, the RSOS(p,q) face weights in this gauge are given by

$$W^{p,q}\left(\begin{matrix} a_{q+1} & b_{q+1} \\ a_1 & b_1 \end{matrix}\,\middle|\, u\right) = \prod_{k=0}^{q-2} s_k^p(u)^{-1} \sum_{a_1,\ldots,a_q}\prod_{k=1}^{q} W^{p,1}\left(\begin{matrix} a_{k+1} & b_{k+1} \\ a_k & b_k \end{matrix}\,\middle|\, u+(k-1)\lambda\right) \tag{2.3}$$

independent of the values of the edge spins $b_2,\ldots,b_q$ where

$$s_k^q(u) = \prod_{j=0}^{q-1}\frac{\sin[u+(k-j)\lambda]}{\sin\lambda} \tag{2.4}$$

and the $p\times 1$ face weights are given in turn by

$$W^{p,1}\left(\begin{matrix} b_1 & b_{p+1} \\ a_1 & a_{p+1} \end{matrix}\,\middle|\, u\right) = \sum_{a_2,\ldots,a_p}\prod_{k=1}^{p} W\left(\begin{matrix} b_k & b_{k+1} \\ a_k & a_{k+1} \end{matrix}\,\middle|\, u+(k-p)\lambda\right). \tag{2.5}$$

independent of the values of the edge spins $b_2,\ldots,b_p$. For the RSOS(p,q) models, adjacent spins or heights are subject to the constraints

$$0 \le (a_i - a_j + m)/2 \le m, \qquad m < a_i + a_j < 2L - m + 2 \tag{2.6}$$

where m is equal to p for a horizontal pair and q for a vertical pair.

2.2 Inversion identity hierarchies

The RSOS(p,q) lattice models are exactly solvable. In particular, the fused face weights satisfy the generalized Yang-Baxter equation

$$\begin{aligned}\sum_g W^{p,q}\left(\begin{matrix} f & g \\ a & b \end{matrix}\,\middle|\, u\right) W^{p,s}\left(\begin{matrix} e & d \\ f & g \end{matrix}\,\middle|\, u+v\right) W^{q,s}\left(\begin{matrix} d & c \\ g & b \end{matrix}\,\middle|\, v\right)\\ = \sum_g W^{q,s}\left(\begin{matrix} e & g \\ f & a \end{matrix}\,\middle|\, v\right) W^{p,s}\left(\begin{matrix} g & c \\ a & b \end{matrix}\,\middle|\, u+v\right) W^{p,q}\left(\begin{matrix} e & d \\ g & c \end{matrix}\,\middle|\, u\right).\end{aligned} \tag{2.7}$$

This is an immediate consequence of the elementary Yang-Baxter equation satisfied by the 1×1 face weights and leads to commuting transfer matrices. Suppose that $\boldsymbol{a}$ and $\boldsymbol{b}$ are allowed spin configurations of two consecutive rows of an N column lattice with periodic boundary conditions. Then the elements of the RSOS(p,q) row transfer matrix are given by

$$\langle \boldsymbol{a}|\boldsymbol{T}^{p,q}(u)|\boldsymbol{b}\rangle = \prod_{j=1}^{N} W^{p,q}\left(\begin{matrix} b_j & b_{j+1} \\ a_j & a_{j+1} \end{matrix}\,\middle|\, u\right) \tag{2.8}$$

where $a_{N+1} = a_1$ and $b_{N+1} = b_1$. Specifically, the Yang-Baxter equations imply the commutation relations

$$\boldsymbol{T}^{p,q}(u)\boldsymbol{T}^{p,q'}(v) = \boldsymbol{T}^{p,q'}(v)\boldsymbol{T}^{p,q}(u). \tag{2.9}$$

Thus if p is held fixed we obtain a hierarchy of commuting families of transfer matrices.

Bazhanov and Reshetikhin [9] have shown that the fusion hierarchy satisfies the functional equations

$$\boldsymbol{T}_0^{p,1}\boldsymbol{T}_1^{p,1} = f_{-1}^p f_1^p \boldsymbol{I} + f_0^p \boldsymbol{T}_0^{p,2} \tag{2.10}$$
$$\boldsymbol{T}_0^{p,q}\boldsymbol{T}_q^{p,1} = f_q^p \boldsymbol{T}_0^{p,q-1} + f_{q-1}^p \boldsymbol{T}_0^{p,q+1} \tag{2.11}$$

where

$$\boldsymbol{T}_k^{p,q} = \boldsymbol{T}^{p,q}(u+k\lambda), \qquad f_q^p = [s_q^p]^N. \tag{2.12}$$

The equation (2.10) is just the usual inversion identity [17,18,19]. It is just a special case of the other equations (2.11) if we define

$$\boldsymbol{T}_0^{p,-1} = 0, \quad \boldsymbol{T}_0^{p,0} = f_{-1}^p \boldsymbol{I}. \tag{2.13}$$

The RSOS(p,q) transfer matrices satisfy the symmetry

$$\boldsymbol{T}_0^{p,q} = \boldsymbol{Y}\boldsymbol{T}_{q+1}^{p,r-2-q}, \quad q = 1,2,\ldots,r-3 \tag{2.14}$$

where $r = L+1$ and $\boldsymbol{Y}$ is the height reflection operator

$$\langle \boldsymbol{a}|\boldsymbol{Y}|\boldsymbol{b}\rangle = \prod_{j=1}^{N} \delta(a_j, r-b_j), \qquad [\boldsymbol{T}^{p,q}(u), \boldsymbol{Y}] = 0. \tag{2.15}$$

It follows that the fusion hierarchy closes with $\boldsymbol{T}_0^{p,r-1} = 0$ and $\boldsymbol{T}_0^{p,r-2} = f_{r-2}^p \boldsymbol{Y}$.

Starting with the fusion hierarchy we derive new functional equations

$$\boldsymbol{T}_0^{p,q}\boldsymbol{T}_1^{p,q} = f_{-1}^p f_q^p \boldsymbol{I} + \boldsymbol{T}_0^{p,q+1}\boldsymbol{T}_1^{p,q-1}. \tag{2.16}$$

These functional equations are easily established by induction from the fusion hierarchy (2.11). For $q = 1$, (2.16) is just the usual inversion identity which is a special case of (2.11). We will assume that (2.16) holds with q replaced with $q-1$ and deduce that it also holds for q. Let us consider the equality

$$\boldsymbol{T}_0^{p,q}(\boldsymbol{T}_1^{p,q-1}\boldsymbol{T}_q^{p,1}) = (\boldsymbol{T}_0^{p,q}\boldsymbol{T}_q^{p,1})\boldsymbol{T}_1^{p,q-1} \tag{2.17}$$

which follows from commutativity. Applying the fusion hierarchy to the terms in brackets we obtain

$$\boldsymbol{T}_0^{p,q}(f_q^p\boldsymbol{T}_1^{p,q-2}+f_{q-1}^p\boldsymbol{T}_1^{p,q})=(f_q^p\boldsymbol{T}_0^{p,q-1}+f_{q-1}^p\boldsymbol{T}_0^{p,q+1})\boldsymbol{T}_1^{p,q-1}. \tag{2.18}$$

Expanding, rearranging and using the induction hypothesis now gives

$$\begin{aligned} f_{q-1}^p\boldsymbol{T}_0^{p,q}\boldsymbol{T}_1^{p,q} &= f_{q-1}^p\boldsymbol{T}_0^{p,q+1}\boldsymbol{T}_1^{p,q-1}+f_q^p\boldsymbol{T}_0^{p,q-1}\boldsymbol{T}_1^{p,q-1}-f_q^p\boldsymbol{T}_0^{p,q}\boldsymbol{T}_1^{p,q-2} \\ &= f_{q-1}^p f_{-1}^p f_q^p\boldsymbol{I}+f_{q-1}^p\boldsymbol{T}_0^{p,q+1}\boldsymbol{T}_1^{p,q-1} \end{aligned} \tag{2.19}$$

which yields the desired result after cancelling out f_{q-1}^p.

If we further define

$$\boldsymbol{t}_0^{p,q}=\frac{\boldsymbol{T}_1^{p,q-1}\boldsymbol{T}_0^{p,q+1}}{f_{-1}^p f_q^p},\quad 1\le q\le r-3. \tag{2.20}$$

Then these equations can be recast in the form

$$\begin{aligned} \boldsymbol{t}_0^{p,q}\boldsymbol{t}_1^{p,q} &= \frac{(\boldsymbol{T}_1^{p,q-1}\boldsymbol{T}_2^{p,q-1})(\boldsymbol{T}_0^{p,q+1}\boldsymbol{T}_1^{p,q+1})}{f_0^p f_q^p f_{-1}^p f_{q+1}^p} \\ &= \left[\boldsymbol{I}+\frac{\boldsymbol{T}_1^{p,q}\boldsymbol{T}_2^{p,q-2}}{f_0^p f_q^p}\right]\left[\boldsymbol{I}+\frac{\boldsymbol{T}_0^{p,q+2}\boldsymbol{T}_1^{p,q}}{f_{-1}^p f_{q+1}^p}\right] \\ &= [\boldsymbol{I}+\boldsymbol{t}_1^{p,q-1}][\boldsymbol{I}+\boldsymbol{t}_0^{p,q+1}] \end{aligned} \tag{2.21}$$

where

$$\boldsymbol{t}_0^{p,0}=\boldsymbol{t}_0^{p,r-2}=0. \tag{2.22}$$

We call the set of equations (2.21) for the quantities $\boldsymbol{t}^{p,q}=\boldsymbol{t}^q$ with p fixed and $q=1,2,\ldots,r-3$ an inversion identity hierarchy. We wish to solve these equations subject to periodicity

$$\boldsymbol{T}^{p,q}(u)=\boldsymbol{T}^{p,q}(u+\pi),\quad \boldsymbol{t}^{p,q}(u)=\boldsymbol{t}^{p,q}(u+\pi) \tag{2.23}$$

and the crossing symmetry

$$\boldsymbol{T}^{p,q}(u)=\overline{\boldsymbol{T}^{p,q}(-\overline{u}+(p-q)\lambda)},\quad \boldsymbol{t}^{p,q}(u)=\overline{\boldsymbol{t}^{p,q}(-\overline{u}+(p-q+1)\lambda)} \tag{2.24}$$

where the bars denote complex conjugation.

It is interesting to observe that equations of the form (2.21) have arisen in the work of Zamolodchikov [25,26] and others [27,28,29] on the thermodynamic Bethe ansatz. We also point out that, although in this paper we are concerned with the inversion identity hierarchy (2.21) at criticality, these same equations hold off-criticality for the elliptic solution [3] of the Yang-Baxter equations.

2.3 Zeros of the transfer matrices

In the previous subsection we dealt with the algebraic properties of the hierarchy of transfer matrices $\boldsymbol{T}^q(u) = \boldsymbol{T}^{p,q}(u)$ and associated matrices $\boldsymbol{t}^q(u) = \boldsymbol{t}^{p,q}(u)$. These relations are quite general. They apply to both critical regimes and to all eigenvalues which we denote by $T^q(u)$ and $t^q(u)$. Here we regard p as fixed and suppress it in the notation. We will continue to suppress the fixed index p in much of what follows. In the sequel the largest eigenvalue of $\boldsymbol{T}^p(u)$ and the related eigenvalues of the fusion hierarchy are referred to as the ground state of the model. The next-largest eigenvalues are called excitations.

To exploit the information provided by the inversion identity hierarchy we first have to characterize the thermodynamically dominant eigenvalues. These differ in the two critical regimes. This characterization is analytic in nature and consists of identifying the analyticity domains of the eigenvalues $T^q(u)$ of the ground state in regimes I/II and III/IV, respectively. We find that $T^q(u)$ is analytic in the following strips

regime I/II

$$\frac{p-q}{2}\lambda - \frac{r-1}{2}\lambda < \mathrm{Re}\, u < \frac{p-q}{2}\lambda + \frac{\lambda}{2} \tag{2.25}$$

regime III/IV

$$\frac{p-q}{2}\lambda - \frac{\lambda}{2} < \mathrm{Re}\, u < \frac{p-q}{2}\lambda + 3\frac{\lambda}{2}. \tag{2.26}$$

For $q = p$ these contain the physical strips of the model in the respective regimes. Inside these analyticity strips the ground state eigenvalues $T^q(u)$ do not possess any zeros apart from those which are imposed by the parametrization of the Boltzmann weights. These zeros appear for all states. They are of order N and have fixed locations independent of the state under consideration

$$\begin{array}{ll} \{\lambda, ..., (p-q)\lambda\} & \text{for} \quad 0 \le q \le p-1 \\ \{-q\lambda, ..., -(p'+1)\lambda\} & \text{for} \quad p'+1 \le q \le r-2 \end{array} \tag{2.27}$$

where $p' = r-2-p$. This is derived recursively from $T^0(u) = f^p_{-1} = f_{-1} = f(u-\lambda)$ where

$$f(u) = \left[\prod_{j=0}^{p-1} \frac{\sin(u-j\lambda)}{\sin\lambda}\right]^N \tag{2.28}$$

by employing the relation

$$T_0^{q+1} = \frac{T_0^q T_1^q - f_{-1} f_q}{T_1^{q-1}} \tag{2.29}$$

The second line in (2.27) is obtained from the first one by recalling that $T_0^q = \pm T_{q+1}^{q'}$ where $q' = r-2-q$ which follows from the symmetry (2.14).

From (2.25),(2.26) and the definition (2.20) we find the following analyticity strips for $t^q(u)$:

regime I/II

$$\frac{p-q}{2}\lambda - \frac{r}{2}\lambda < \operatorname{Re} u < \frac{p-q}{2}\lambda \tag{2.30}$$

regime III/IV

$$\frac{p-q}{2}\lambda - \lambda < \operatorname{Re} u < \frac{p-q}{2}\lambda + \lambda. \tag{2.31}$$

For the zeros and poles of $t^q(u)$ we have to distinguish several cases. If $p < \frac{r}{2} - 1$ we have five subcases

(I) $1 \le q \le p-1$

$$\begin{array}{ll} \text{zeros:} & \emptyset \\ \text{poles:} & \{-q\lambda, ..., -\lambda\} \cup \{(p-q+1)\lambda, ..., p\lambda\} \end{array} \tag{2.32}$$

(II) $q = p$

$$\begin{array}{ll} \text{zeros:} & \{0\} \\ \text{poles:} & \{-p\lambda, ..., -\lambda\} \cup \{\lambda, ..., p\lambda\} \end{array} \tag{2.33}$$

(III) $p+1 \le q \le p'-1$

$$\begin{array}{ll} \text{zeros:} & \emptyset \\ \text{poles:} & \{-q\lambda, ..., (p-q-1)\lambda\} \cup \{\lambda, ..., p\lambda\} \end{array} \tag{2.34}$$

(IV) $q = p'$

$$\begin{array}{ll} \text{zeros:} & \{-(p'+1)\lambda\} \\ \text{poles:} & \{-p'\lambda, ..., (p-p'-1)\lambda\} \cup \{\lambda, ..., p\lambda\} \end{array} \tag{2.35}$$

(V) $p'+1 \le q \le r-3$

$$\begin{array}{ll} \text{zeros:} & \emptyset \\ \text{poles:} & \{-p'\lambda, ..., (p-q-1)\lambda\} \cup \{\lambda, ..., (r-q-2)\lambda\} \end{array} \tag{2.36}$$

This is compatible with the symmetry of $t^{p,q}$

$$t_0^{p,q} = (-1)^{pN} t_{q+1}^{p,q'}, \qquad q' = r-2-q. \tag{2.37}$$

For the case $p > \frac{r}{2} - 1$ the zeros and poles of order N can be obtained from the previous case and the relation

$$t_0^{p,q} = t_{p'+1}^{p',q}, \qquad p' = r-2-p \tag{2.38}$$

The zeros and poles in the marginal case $p = \frac{r}{2} - 1$ are also given by the above list. In this case we have to observe that range (III) does not exist. Also ranges (II) and (IV) coincide. The set of poles in this case is as given above but the set of zeros is $\{0, -(p+1)\lambda\}$.

2.4 Asymptotics of t^q

The first information we will gain from the inversion identity hierarchy are explicit formulas for the asymptotics $t^q(\pm i\infty)$ which we will denote by t^q_∞ as a shorthand. In the limit under consideration, (2.21) reduces to

$$(t^q_\infty)^2 = \left[1 + t^{q-1}_\infty\right]\left[1 + t^{q+1}_\infty\right] \tag{2.39}$$

with the closure condition

$$t^0_\infty = t^{r-2}_\infty = 0 \tag{2.40}$$

Let us write t^1_∞ as

$$t^1_\infty = \frac{\sin 3\theta}{\sin\theta} \tag{2.41}$$

with θ to be determined. Using (2.39) as a recursion relation for t^{q+1}_∞ we then derive

$$t^q_\infty = \frac{\sin q\theta \sin(q+2)\theta}{\sin^2\theta}, \quad 1 + t^q_\infty = \frac{\sin^2(q+1)\theta}{\sin^2\theta} \tag{2.42}$$

for all $q \geq 0$. The closure condition (2.40) imposes the quantization

$$\theta = \frac{m_j\pi}{r}, \quad m_j = 1, 2, \ldots, r-1. \tag{2.43}$$

This is consistent with the braid limit [30]

$$\lim_{\operatorname{Im} u\to\pm\infty} \left(\frac{\sin\lambda}{\sin(u-\lambda/2)}\right)^N T^{p,1}(u) = 2\cos(\frac{m_j\pi}{r}) \tag{2.44}$$

where m_j are the Coxeter exponents. Indeed this gives

$$\begin{aligned} t^1_\infty &= \lim_{\operatorname{Im} u\to\pm\infty} \frac{T^0_1 T^2_0}{f^p_{-1} f^p_1} \\ &= \lim_{\operatorname{Im} u\to\pm\infty} \frac{T^1_0 T^1_1}{f^p_{-1} f^p_1} - 1 = 4\cos^2\theta - 1 = \frac{\sin 3\theta}{\sin\theta} \end{aligned} \tag{2.45}$$

with θ given by (2.43).

3 Calculation of finite size corrections for regime III/IV

In this section we treat the regime III/IV critical RSOS models. These series contain the minimal unitary series and the superconformal series and are therefore of considerable interest. Perhaps more importantly, regime III/IV is much simpler to handle, mathematically speaking, than regime I/II which we treat in the following section. First off in this section we determine the bulk behaviour of the functions $t^q(u)$ in the analyticity strips (2.31) for the ground state. We also derive integral equations for the correction functions from a closer inspection of the inversion identity hierarchy. Next we extend this analysis to all relevant excited states. We identify the amplitude of the $1/N$ finite-size corrections and derive an expression for it in terms of Rogers dilogarithms. The asymptotic values of the eigenvalue functions derived in Section 2.4 play a special role. An explicit formula for the central charge is obtained in Section 3.4 and the analogous calculation for the primary excited states is given in Section 3.5.

3.1 Bulk behaviour and integral equations for the largest eigenvalue

Reference to Section 2.3 shows that the analyticity strip for $t^p(u)$ contains a zero of order N at $u=0$ and poles of order N at $u=\pm\lambda$. All other functions $t^q(u)$ are *a*nalytic and *n*on-*z*ero (ANZ) in their analyticity strips. We find that the leading bulk contributions to the eigenvalues for large N are given by

$$t^q_{\rm bulk}(u)=\begin{cases}\text{constant}, & q\neq p,\\ \text{constant}\cdot\left[\tan\frac{r}{2}u\right]^N, & q=p.\end{cases}\tag{3.1}$$

Clearly this is consistent with the observed zeros and poles. The constants are to be adjusted so that (2.21) is satisfied. The calculations are similar to those performed for the asymptotics of the eigenvalues. For $1\le q\le p-1$ we find

$$t^q_{\rm bulk}=\frac{\sin q\sigma\sin(q+2)\sigma}{\sin^2\sigma},\quad 1+t^q_{\rm bulk}=\frac{\sin^2(q+1)\sigma}{\sin^2\sigma}\tag{3.2}$$

with the quantization

$$\sigma=\frac{m'_j\pi}{p+2},\quad m'_j=1,2,\ldots,p+1.\tag{3.3}$$

Analogously, for $p+1\le q\le r-3$ we obtain

$$t^q_{\rm bulk}=\frac{\sin(q-p)\tau\sin(q-p+2)\tau}{\sin^2\tau},\quad 1+t^q_{\rm bulk}=\frac{\sin^2(q-p+1)\tau}{\sin^2\tau}\tag{3.4}$$

where

$$\tau=\frac{m''_j\pi}{r-p},\quad m''_j=1,2,\ldots,r-p-1.\tag{3.5}$$

Lastly, we find that

$$t^p_{\rm bulk}(u)=\pm4\cos\sigma\cos\tau\left[\tan\frac{r}{2}u\right]^N\tag{3.6}$$

For the largest eigenvalue, the appropriate choices are $\theta=\frac{\pi}{r}$, $\sigma=\frac{\pi}{p+2}$ and $\tau=\frac{\pi}{r-p}$ for (2.42), (3.2) and (3.4). This is consistent with the application of the Perron-Frobenius theorem and positive asymptotics. The other choices will correspond to excited states.

It is useful to introduce functions of a real variable by restricting the eigenvalue functions to certain lines in the complex plane

$$\mathfrak{a}^q(x)\;:=\;t^q\left(\frac{i}{r}x+\frac{p-q}{2}\lambda\right)$$

$$\mathfrak{A}^q(x) \quad := \quad 1 + \mathfrak{a}^q(x). \tag{3.7}$$

The inversion identity hierarchy (2.21) can then be rewritten in terms of the new functions as

$$\mathfrak{a}^q\left(x - i\frac{\pi}{2}\right)\mathfrak{a}^q\left(x + i\frac{\pi}{2}\right) = \mathfrak{A}^{q-1}(x)\mathfrak{A}^{q+1}(x) \tag{3.8}$$

We introduce finite-size correction terms $l^q(x)$ by writing $\mathfrak{a}^q(x)$ as

$$\mathfrak{a}^q(x) = \begin{cases} l^q(x), & q \neq p, \\ \tanh^N\left(\frac{x}{2}\right) \cdot l^p(x), & q = p. \end{cases} \tag{3.9}$$

All the functions $l^q(x)$ are *a*nalytic, *n*on-*z*ero in $-\pi < \operatorname{Im} x < \pi$ and possess *c*onstant asymptotics for $\operatorname{Re} x \to \pm\infty$ (ANZC). They satisfy the functional equation

$$l^q\left(x - i\frac{\pi}{2}\right)l^q\left(x + i\frac{\pi}{2}\right) = \mathfrak{A}^{q-1}(x)\mathfrak{A}^{q+1}(x) \tag{3.10}$$

Due to the ANZC properties of l^q and $\mathfrak{A}^q$ we can introduce Fourier transforms of the logarithmic derivatives

$$\begin{aligned} L^q(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, [\ln l^q(x)]'\, e^{-ikx} \\ [\ln l^q(x)]' &= \int_{-\infty}^{\infty} dk\, L^q(k) e^{ikx} \end{aligned} \tag{3.11}$$

with analogous equations for $\mathfrak{A}^q$.

Applying Fourier transforms to the logarithmic derivative of (3.10) we obtain

$$L^q(k) = \frac{1}{e^{\frac{\pi}{2}k} + e^{-\frac{\pi}{2}k}} A^{q-1}(k) + \frac{1}{e^{\frac{\pi}{2}k} + e^{-\frac{\pi}{2}k}} A^{q+1}(k). \tag{3.12}$$

Transforming back we find a double integral whose order can be interchanged. The k-integral can then be evaluated using

$$\int_{-\infty}^{\infty} dk \frac{e^{ik(x-y)}}{e^{\frac{\pi}{2}k} + e^{-\frac{\pi}{2}k}} = \frac{1}{\cosh(x-y)}. \tag{3.13}$$

Thus we obtain

$$[\ln l^q(x)]' = \int_{-\infty}^{\infty} dy \frac{\left[\ln \mathfrak{A}^{q-1}\right]'(y)}{2\pi\cosh(x-y)} + \int_{-\infty}^{\infty} dy \frac{\left[\ln \mathfrak{A}^{q+1}\right]'(y)}{2\pi\cosh(x-y)}. \tag{3.14}$$

Integrating this equation and recalling (3.9) we obtain the nonlinear integral equation

$$\ln \mathfrak{a}^q = \ln \mathfrak{e}^q + k * \ln \mathfrak{A}^{q-1} + k * \ln \mathfrak{A}^{q+1} + D^q \tag{3.15}$$

where

$$\mathfrak{e}^q(x) := \begin{cases} 1, & q \neq p, \\ \tanh^N \frac{x}{2}, & q = p \end{cases} \tag{3.16}$$

and the kernel $k(x)$ is defined by

$$k(x) := \frac{1}{2\pi \cosh x}. \tag{3.17}$$

The convolution $f * g$ of two functions f and g is defined by

$$(f * g)(x) := \int_{-\infty}^{\infty} f(x-y)g(y)dy = \int_{-\infty}^{\infty} f(y)g(x-y)dy \tag{3.18}$$

If we are only interested in the bulk behaviour we can substitute the functions $\mathfrak{A}^q$ by constants. In this limit, the integral equation (3.15) reduces to (3.1) as is expected.

The set of integral equations (3.15) with $q = 1, 2, \ldots, r-3$ can be cast into a more compact matrix form

$$l\vec{\mathfrak{a}} = l\vec{\mathfrak{e}} + K * l\vec{\mathfrak{A}} + \vec{D} \tag{3.19}$$

where the $(r-3) \times (r-3)$ matrix kernel

$$K = k(x) \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \tag{3.20}$$

has the symmetry

$$K^T(y-x) = K(x-y). \tag{3.21}$$

This symmetry is a keystone of our analytic treatment of finite-size corrections.

3.2 Excitations

Low-lying excitations have the same bulk behaviour as the ground state. The difference lies in the fact that the eigenvalue functions now possess a finite number of zeros in the analyticity strips which were free of zeros in the ground state. Let us consider one such zero of one particular function $T^q(u)$. Empirically, this zero is located on the line $\operatorname{Re} u = \frac{p-q+1}{2}\lambda$ which is a symmetry axis. We write this zero as $u_0 + \frac{1}{2}\lambda$. According to (2.20) the functions $t^{q-1}(u)$ and $t^{q+1}(u)$ have zeros at $u_0 + \frac{1}{2}\lambda$ and $u_0 - \frac{1}{2}\lambda$, respectively. From (3.7) we see that $\mathfrak{a}^{q-1}(x)$ and $\mathfrak{a}^{q+1}(x)$ each have a zero at the same real number x_0. Similarly, (2.16) which takes the form

$$T_0^q T_1^q = f_{-1} f_q \left[1 + t_0^q\right] \tag{3.22}$$

implies that $1 + t^q(u)$ has a pair of zeros $u_0 \pm \frac{1}{2}\lambda$. Using (3.7) again, we see that $\mathfrak{A}^q(x)$ has a pair of zeros $x_0 \pm \frac{\pi}{2}i$.

Next we describe the derivation of integral equations for excited states. Let us consider (3.8) and assume that $\mathfrak{a}^q(x)$ has a zero x_0 and $\mathfrak{A}^{q-1}(x)$ or $\mathfrak{A}^{q+1}(x)$ has a pair of zeros $x_0 \pm \frac{\pi}{2}i$. We follow the reasoning of the last subsection, but we have to take care since the simple ANZC properties do not hold anymore. It is still possible to define the Fourier transforms $L^q(k)$ and $A^q(k)$ by the first formula of (3.11). However, we have to use a modified integration path $\mathcal{L}$ which follows the axis $\operatorname{Im} x = \pi/2$ closely from below and avoids $x_0 + \frac{\pi}{2}i$ from above as shown in Figure 2. Applying this to the logarithmic derivative of (3.10) we obtain on the left hand side

$$\begin{aligned} & e^{\frac{\pi}{2}k} \int_{\mathcal{L}_-} dx \, [\ln l^q(x)]' \, e^{-ikx} + e^{-\frac{\pi}{2}k} \int_{\mathcal{L}_+} dx \, [\ln l^q(x)]' \, e^{-ikx} \\ & = \left(e^{\frac{\pi}{2}k} + e^{\frac{\pi}{2}k} \right) \int_{\mathcal{L}} dx \, [\ln l^q(x)]' \, e^{-ikx} \end{aligned} \tag{3.23}$$

where Cauchy's theorem can be applied since $l^q(x)$ is ANZC between $\mathcal{L}_+$ and $\mathcal{L}_-$. As before we derive (3.14) where x is a point on $\mathcal{L}$ and the y-integration is along $\mathcal{L}$. The order of the intermediate double integral can be interchanged because $|\operatorname{Im}(x-y)| < \pi/2$ for any x and y on $\mathcal{L}$ which renders the k-integral convergent. Finally, we obtain (3.15) and (3.16) but the convolution is modified to

$$(f * g)(x) := \int_{\mathcal{L}} f(x-y) g(y) dy = \int_{x-\mathcal{L}} f(y) g(x-y) dy \tag{3.24}$$

Here we have treated the case where one function $\mathfrak{a}^q$ has a zero x_0. The generalization to the case where several functions possess zeros is straightforward.

3.3 Finite-size corrections

So far we have transformed functional equations for $t^q(u)$ into integral equations for the finite-size corrections. Before continuing the investigation of these equations let us pause to express the finite-size corrections for T^q in terms of the functions $\mathfrak{A}^q$. To achieve this we treat the relation

$$T^q(u) T^q(u+\lambda) = f(u-\lambda) f(u+q\lambda) \, [1 + t^q(u)] \tag{3.25}$$

in the manner developed in previous subsections. Let $T^q_{\text{bulk}}(u)$ describe the bulk behaviour of $T^q(u)$ in its analyticity strip (2.26). It has to satisfy the relation

$$T^q_{\text{bulk}}(u) T^q_{\text{bulk}}(u+\lambda) = f(u-\lambda) f(u+q\lambda) \tag{3.26}$$

We define the finite-size correction $T^q_{\text{finite}}(u)$ by

$$T^q(u) = T^q_{\text{bulk}}(u) \cdot T^q_{\text{finite}}(u). \tag{3.27}$$

Inserting (3.27) into (3.25) and respecting (3.26) we find

$$T^q_{\text{finite}}(u) T^q_{\text{finite}}(u+\lambda) = 1 + t^q(u). \tag{3.28}$$

Again it is useful to introduce functions of a real variable by restricting the eigenvalue functions to certain lines

$$\mathfrak{b}^q(x) := T^q_{\text{finite}} \left(\frac{i}{r} x + \frac{p-q+1}{2} \lambda \right). \tag{3.29}$$

Applying (3.28) we obtain

$$\mathfrak{b}^q\left(x - i\frac{\pi}{2}\right)\mathfrak{b}^q\left(x + i\frac{\pi}{2}\right) = \mathfrak{A}^q(x). \tag{3.30}$$

In the case of the ground state the functions $\mathfrak{A}^q$ and $\mathfrak{b}^q$ are ANZC. Taking the logarithmic derivative of (3.30), and introducing Fourier transforms, we are able to express $\mathfrak{b}^q$ in terms of $\mathfrak{A}^q$

$$\ln \mathfrak{b}^q = k * \ln \mathfrak{A}^q + C^q \tag{3.31}$$

where C^q are integration constants. In the case of excited states we have to take care of zeros in the analyticity strips. Let x_0 be a zero of $\mathfrak{b}^q(x)$ and $x_0 \pm i\frac{\pi}{2}$ be the zeros of $\mathfrak{A}^q(x)$. We can still derive (3.31) but with (3.24) as the definition of the convolution. The integration constants in (3.31) can be calculated from the asymptotics of $\mathfrak{A}^q$ and $\mathfrak{b}^q$. In this asymptotic limit (3.31) becomes

$$\ln \mathfrak{b}^q_\infty = \frac{1}{2}\ln \mathfrak{A}^q_\infty + C^q \tag{3.32}$$

where we have used $\int_{-\infty}^{\infty} k(x)dx = 1/2$. For each q, C^q is a multiple of πi so it does not contribute to the $1/N$ corrections.

We return to the discussion of the nonlinear integral equations (3.15). The only place where the system size N enters is through the function $\mathfrak{e}^p(x) = \tanh^N \frac{x}{2}$. This function has three asymptotic regimes with transitions in scaling regimes when x is of the order of $-\ln N$ or $\ln N$. In these scaling regimes the function takes the form of a double exponential

$$e^p_\pm(x) := \lim_{N\to\infty} \mathfrak{e}^p\left(\pm(x + \ln N)\right) = \exp(-2e^{-x}). \tag{3.33}$$

It is natural to assume that the functions $\mathfrak{a}^q$ and $\mathfrak{A}^q$ scale similarly. We therefore expect that the following scaling limits exist

$$\begin{aligned} a^q_\pm(x) &:= \lim_{N\to\infty} \mathfrak{a}^q\left(\pm(x + \ln N)\right) \\ A^q_\pm(x) &:= \lim_{N\to\infty} \mathfrak{A}^q\left(\pm(x + \ln N)\right) = 1 + a^q_\pm(x). \end{aligned} \tag{3.34}$$

In this scaling limit, (3.15) takes the form

$$la^q = le^q + k * lA^{q-1} + k * lA^{q+1} + D^q \tag{3.35}$$

where we suppress the subscripts $\pm$ and use the abbreviations

$$\begin{aligned} la^q(x) &:= \ln a^q(x) \\ lA^q(x) &:= \ln A^q(x) \\ le^q(x) &:= \begin{cases} 0, & q \neq p, \\ -2e^{-x}, & q = p. \end{cases} \end{aligned} \tag{3.36}$$

For the moment we disregard the integration constants in (3.31) and focus on the finite-size corrections to $\mathfrak{b}^p(x)$ which, for fixed x, are of order $1/N$

$$\begin{aligned}\ln \mathfrak{b}^p(x) &= (k * \ln \mathfrak{A}^p)(x) \\ &= \frac{1}{2\pi}\int_{-\ln N}^{\infty}\left[\frac{\ln \mathfrak{A}^p(y+\ln N)}{\cosh(x-y-\ln N)} + \frac{\ln \mathfrak{A}^p(-y-\ln N)}{\cosh(x+y+\ln N)}\right] dy + o\left(\frac{1}{N}\right) \\ &= \frac{e^x}{\pi N}\int_{-\infty}^{\infty} e^{-y} lA_+^p(y)dy + \frac{e^{-x}}{\pi N}\int_{-\infty}^{\infty} e^{-y} lA_-^p(y)dy + o\left(\frac{1}{N}\right) \end{aligned} \tag{3.37}$$

Two remarks concerning (3.37) are in order. First the last integrals exist since, for $y \to -\infty$ the function $lA^p(y) = \ln(1 + a^p(y))$ tends to zero faster than any exponential, cf. (3.35) for $q = p$. Second, if we replace p by some $q \neq p$ then the last integrals will not converge because $lA^q(-\infty) \neq 0$. Even a subtraction $lA^q(y) \to lA^q(y) - lA^q(-\infty)$ in the second line does not help since the rate of decay is unlikely to be sufficient. So the finite-size corrections of $\mathfrak{b}^q(x)$ with $q \neq p$ are almost surely of lower order than $1/N$.

In the case $q = p$ the integrals not only exist but it is actually possible to evaluate them explicitly without solving the integral equations! This remarkable fact is essentially a consequence of the special symmetry (3.21) of the kernel in the integral equations. Multiplying the derivative of (3.35) with lA^q, and (3.35) itself with $(lA^q)'$, taking the difference, summing over q, and finally integrating we find

$$\begin{aligned} &\sum_{q=1}^{r-3}\int_{-\infty}^{\infty}\left[(la^q)' \, lA^q - la^q \, (lA^q)'\right] dx \\ &= \sum_{q=1}^{r-3}\int_{-\infty}^{\infty}\left[(le^q)' \, lA^q - (le^q + D^q)\,(lA^q)'\right] dx \end{aligned} \tag{3.38}$$

where the contributions of the kernel cancel due to the symmetry. The right hand side simplifies to

$$\begin{aligned} &\int_{-\infty}^{\infty}\left[(le^p)' \, lA^p - le^p \, (lA^p)'\right] dx - \sum_{q=1}^{r-3}\int_{-\infty}^{\infty} D^q \, (lA^q)' \, dx \\ &= 4\int_{-\infty}^{\infty} e^{-x} lA^p(x)dx - \sum_{q=1}^{r-3} D^q lA^q \Big|_{-\infty}^{\infty} \end{aligned} \tag{3.39}$$

The left hand side of (3.38) can be evaluated after changing the variable of integration from x to a, hence

$$\begin{aligned} 2\int_{-\infty}^{\infty} e^{-x} lA^p(x)dx &= \frac{1}{2}\sum_{q=1}^{r-3}\int\left[\frac{\ln(1+a^q)}{a^q} - \frac{\ln a^q}{1+a^q}\right] da^q + \frac{1}{2}\sum_{q=1}^{r-3} D^q lA^q \Big|_{-\infty}^{\infty} \\ &= \sum_{q=1}^{r-3} L_+(a^q)\Big|_{-\infty}^{\infty} + \frac{1}{2}\sum_{q=1}^{r-3} D^q lA^q \Big|_{-\infty}^{\infty} \end{aligned} \tag{3.40}$$

We have introduced the dilogarithmic function $L_+(a)$ related to the standard Rogers dilogarithm $L(a)$ by

$$L_+(a) = L\left(\frac{a}{1+a}\right) = L\left(1 - \frac{1}{1+a}\right) = L(1) - L\left(\frac{1}{1+a}\right) \tag{3.41}$$

Lastly, we obtain

$$2\int_{-\infty}^{\infty} e^{-x} lA^p(x)dx = -\sum_{q=1}^{r-3} L\left(\frac{1}{A^q}\right)\Big|_{-\infty}^{\infty} + \frac{1}{2}\sum_{q=1}^{r-3} D^q lA^q\Big|_{-\infty}^{\infty} \tag{3.42}$$

where the constants D^q are given in terms of the asymptotics by

$$D^q = la_\infty^q - \frac{1}{2}lA_\infty^{q-1} - \frac{1}{2}lA_\infty^{q+1} \tag{3.43}$$

3.4 The central charge

Before evaluating the right hand side of (3.42) for all primary excitations it is a worthwhile exercise to first calculate the central charge. The asymptotics of a^q and A^q are given by those of t^q and its bulk behaviour, i.e. (2.42), (3.2) and (3.4) with $\theta = \frac{\pi}{r}$, $\sigma = \frac{\pi}{p+2}$, and $\tau = \frac{\pi}{r-p}$. From (3.43) the integration constants vanish, $D^q = 0$ for all q. Hence

$$\begin{aligned}
2\int_{-\infty}^{\infty} e^{-x} lA^p(x)dx &= \sum_{q=1}^{p-1} L\left(\frac{\sin^2\sigma}{\sin^2(q+1)\sigma}\right) + L(1) + \sum_{q=p+1}^{r-3} L\left(\frac{\sin^2\tau}{\sin^2(q+1-p)\tau}\right) \\
&\quad - \sum_{q=1}^{r-3} L\left(\frac{\sin^2\theta}{\sin^2(q+1)\theta}\right) \\
&= \left(\frac{3p}{p+2} - \frac{6p}{r(r-p)}\right)\frac{\pi^2}{6}
\end{aligned} \tag{3.44}$$

where we have used the identity [31]

$$\sum_{k=2}^{n-2} L\left(\frac{\sin^2\frac{\pi}{n}}{\sin^2 k\frac{\pi}{n}}\right) = \left(2 - \frac{6}{n}\right)L(1), \quad n \geq 4 \tag{3.45}$$

which is a special case of a more general identity established in Appendix A. Inserting this result into (3.37) we find

$$\ln \mathfrak{b}^p(x) = \left(\frac{3p}{p+2} - \frac{6p}{r(r-p)}\right)\frac{\pi}{6N}\cosh x \tag{3.46}$$

Taking into account the geometrical factor $\cosh x = \sin(ru)$ which is identified in Appendix B we obtain the result (1.6) for the central charge c.

3.5 The primary conformal weights

In (3.42) and (3.43) the finite-size scaling amplitudes are determined by the terminals of the functions a^q and A^q. The precise quantitative dependence of $a^q(x)$ on x is unimportant. However, qualitative features of the paths in the complex plane such as the way singularities at 0 and -1 are encircled do play a role as these single out different branches of the dilogarithms. Our empirical observations for tricritical hard squares [15,16] reveals quite a simple pattern for the primary excitations. The points 0 and -1 are surrounded alternately where the winding is in a clockwise sense. Moreover, the asymptotics (2.42), (3.2), and (3.4) are given by angles

$$\theta = s\frac{\pi}{r}, \quad \sigma = \overline{s}\frac{\pi}{p+2}, \quad \tau = t\frac{\pi}{r-p} \tag{3.47}$$

where s, $\overline{s}$ and t are integers. These integers are not in fact all independent.

To disentangle this problem let us focus on the functions A^q which appear explicitly in (3.42). These functions interpolate between the asymptotics

$$\begin{array}{lll} \dfrac{\sin^2[(q+1)\sigma+\omega]}{\sin^2\sigma} \longrightarrow \dfrac{\sin^2[(q+1)\theta+\omega]}{\sin^2\theta}, & & q=1,...,p \\ \dfrac{\sin^2[(q+1-p)\tau]}{\sin^2\tau} \longrightarrow \dfrac{\sin^2[(q+1)\theta+\omega]}{\sin^2\theta}, & & q=p,...,r-3 \end{array} \tag{3.48}$$

where we have introduced an angle ω which is a multiple of π. This was done for technical reasons and ω will be determined by certain matching conditions. The homotopy class of A^q is given by the concatenation of two paths $d_2 \cdot d_1^{-1}$ (first d_1 in reversed sense, then d_2). The path d_1 is defined for $1 \le q \le p$ by the formula

$$\frac{\sin^2[(q+1)\sigma+\omega]}{\sin^2\sigma}. \tag{3.49}$$

The initial point is given by $\sigma = \pi/r$ and $\omega = -\pi/r$. We first increase ω along a path in the upper half plane to its final value. Then σ is increased likewise to the final value (3.47). The path d_2 is defined similarly through

$$\frac{\sin^2[(q+1)\theta+\omega]}{\sin^2\theta}. \tag{3.50}$$

In the case $p \le q \le r-3$ we use the second line of (3.48) with the initial point given by $\tau = \theta = \pi/r$ and $\omega = -p\pi/r$

This construction may seem strange, but it is a natural way to encode our findings for tricritical hard squares in compact formulas. The application will be straightforward as the above prescription will uniquely define the analytic continuations of dilogarithmic functions of several arguments where the initial points always lie in the interval $(0,1)$.

We have introduced four quantities θ, σ, τ, and ω describing the homotopy classes, although just two of them are independent parameters. The constraints are given by matching conditions. For $q = p$ we want identical results for the homotopy class of A^q by the two prescriptions. So we have

$$\text{phase of } \frac{\sin^2[(q+1)\sigma+\omega]}{\sin^2\sigma} = \text{phase of } \frac{\sin^2\tau}{\sin^2\tau} = 0 \tag{3.51}$$

If we set

$$n := \left\lfloor \frac{\sigma}{\pi} \right\rfloor = \left\lfloor \frac{\overline{s}}{p+2} \right\rfloor \tag{3.52}$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x, then this relation implies

$$\omega = (2n + 1 - \overline{s})\pi. \tag{3.53}$$

Lastly we want the above prescription to hold also in the case $q = r - 2$. The closure condition $A^{r-2} \equiv 1$ imposes the constraint

$$\text{phase of } \frac{\sin^2[(r-p-1)\tau]}{\sin^2\tau} = \text{phase of } \frac{\sin^2[(r-1)\theta+\omega]}{\sin^2\theta}. \tag{3.54}$$

From this we get an additional equation for ω and a relation between $\overline{s}$, t and s

$$\begin{aligned} \omega &= (t-s)\pi \\ \overline{s} &= s - t + 2n + 1 \\ n &= \left\lfloor \frac{s-t}{p} \right\rfloor. \end{aligned} \tag{3.55}$$

We are now able to calculate the constants D^q from (3.43)

$$D^q = \begin{cases} \ln\left\{\dfrac{\sin(\theta+\omega)}{\sin\theta}\right\}, & q = 1 \\ 0, & 1 < q < r-3 \\ \ln\left\{\dfrac{\sin[(r-1)\theta+\omega]}{\sin\theta}\right\}, & q = r-3 \end{cases} \tag{3.56}$$

Inserting this and the terminals into (3.42) we obtain

$$\begin{aligned} &2\int_{-\infty}^{\infty} e^{-x} l A^p(x) dx \\ = \; &\sum_{k=1}^{p+1} L\left(\frac{\sin^2\sigma}{\sin^2(k\sigma+\omega)}\right) + \sum_{k=1}^{r-p-1} L\left(\frac{\sin^2\tau}{\sin^2 k\tau}\right) - \sum_{k=1}^{r-1} L\left(\frac{\sin^2\theta}{\sin^2(k\theta+\omega)}\right) \\ &+L\left(\frac{\sin^2\theta}{\sin^2(\theta+\omega)}\right) + L\left(\frac{\sin^2\theta}{\sin^2[(r-1)\theta+\omega]}\right) \\ &-L\left(\frac{\sin^2\sigma}{\sin^2(\sigma+\omega)}\right) - L\left(\frac{\sin^2\tau}{\sin^2(r-p-1)\tau}\right) - L(1) \\ &+\ln\frac{\sin(\theta+\omega)}{\sin\theta}\left[\ln\frac{\sin(2\theta+\omega)}{\sin\theta} - \ln\frac{\sin(2\sigma+\omega)}{\sin\sigma}\right] \\ &+\ln\frac{\sin((r-1)\theta+\omega)}{\sin\theta}\left[\ln\frac{\sin((r-2)\theta+\omega)}{\sin\theta} - \ln\frac{\sin(r-p-2)\tau}{\sin\tau}\right] \end{aligned} \tag{3.57}$$

We now apply the identity (A.1) to the sums in this equation. In the resulting expression we find several simplifications due to the identities (A.6) – (A.13) and are left with

$$\begin{aligned} &2\int_{-\infty}^{\infty} e^{-x} l A^p(x) dx = 2(p+2)\sigma\omega + (p+1)(p+2)\sigma^2 \\ &+ (r-p-1)(r-p)\tau^2 - 2r\theta\omega - (r-1)r\theta^2 - 2\pi^2 n(n+1) - 3L(1). \end{aligned} \tag{3.58}$$

Inserting (3.47) into (3.58) we obtain the final expression

$$2\int_{-\infty}^{\infty} e^{-x} lA^p(x)dx = \left[\frac{s^2}{r} - \frac{\overline{s}^2}{p+2} - \frac{t^2}{r-p} + \frac{1}{2} + 2n(n+1)\right]\pi^2 \tag{3.59}$$

Therefore, from (3.37) and refeqham9, we see that the conformal weights are given by

$$\begin{aligned}\Delta &= \frac{1}{2\pi^2}\left[\int_{-\infty}^{\infty} e^{-x} lA^p(x)dx\right]\Bigg|_{s,\overline{s},t}^{s=\overline{s}=t=1} \\ &= \frac{\overline{s}^2-1}{4(p+2)} + \frac{t^2-1}{4(r-p)} - \frac{s^2-1}{4r} - \frac{1}{2}n(n+1).\end{aligned} \tag{3.60}$$

This can be cast into the more standard form

$$\Delta = \frac{\left[rt-(r-p)s\right]^2 - p^2}{4pr(r-p)} + \frac{\nu(p-\nu)}{2p(p+2)} \tag{3.61}$$

where s and t are integers satisfying

$$1 \le s \le r-1, \qquad 1 \le t \le r-p-1 \tag{3.62}$$

and

$$\nu := s - t - \left\lfloor \frac{s-t}{p} \right\rfloor p. \tag{3.63}$$

This coincides with (1.7) since

$$(s_0-1)(p-s_0+1) = \nu(p-\nu). \tag{3.64}$$

We mention that the constant C^q in (3.31) is equal to $i\omega$. So the lattice momentum of the state is

$$P_0 = \omega = (t-s)\pi. \tag{3.65}$$

4 Treatment of finite-size corrections for regime I/II

This regime corresponds to the Z_N invariant Fateev-Zamolodchikov models. The organization of this section is similar to that of Section 3. The derivation of the integral equations, however, requires more effort and the structure of the primary excitations is more involved. Nevertheless, the finite-size corrections to all eigenvalues of the hierarchy can be calculated in terms of Rogers dilogarithms.

4.1 Bulk behaviour and integral equations for the largest eigenvalue

The main difference between regime I/II and regime III/IV is the occurence of poles of order N for $t^q(u)$ in the strips (2.30). This entails the dominance of t^q over 1 on the right hand side of the inversion identity hierarchy (2.21). It is therefore necessary to rewrite the inversion identity hierarchy in the form

$$\frac{t_0^q t_1^q}{t_1^{q-1} t_0^{q+1}} = \left[1+\frac{1}{t_1^{q-1}}\right]\left[1+\frac{1}{t_0^{q+1}}\right], \qquad 2 \le q \le r-4. \tag{4.1}$$

For $q=1$ and $q=r-3$ we have

$$\begin{aligned} \frac{t_0^1 t_1^1}{t_0^2} &= 1+\frac{1}{t_0^2} \\ \frac{t_0^{r-3} t_1^{r-3}}{t_1^{r-4}} &= 1+\frac{1}{t_1^{r-4}}. \end{aligned} \tag{4.2}$$

Setting $t^q = (z^q)^N$ for the bulk behaviour we obtain the functional relations

$$\frac{z_0^q z_1^q}{z_1^{q-1} z_0^{q+1}} = 1, \qquad 1 \le q \le r-3 \tag{4.3}$$

where we have defined $z^0 = z^{r-2} \equiv 1$. Obviously the last equation is satisfied by

$$z^q(u) = \prod_{j=0}^{p-1} \frac{\sin\left[\frac{r}{r-2}\left(u - j\lambda\right)\right]}{\sin\left[\frac{r}{r-2}\left(u-(q-j)\lambda\right)\right]}. \tag{4.4}$$

It also respects the closure condition in the cases $q=0$ and $r-2$, and shows the right pole structure as given in Section 2.3 for the strips (2.30). We conclude that the solution (4.4) gives the correct bulk terms. Indeed these same expressions are obtained as the bulk solution of the integral equations we are about to derive. Before proceeding we note the complex conjugation symmetry

$$z^q(u) = (-1)^p \overline{z^{q'}\left(-\overline{u}+(p'+1)\lambda\right)} \tag{4.5}$$

which is shared by the functions $t^q(u)$.

As in the previous section we introduce functions of a real variable

$$\begin{aligned} \mathfrak{a}^q(x) &:= \left[t^q\left(\frac{r-2}{2r}ix - \left(q-p+\frac{r}{2}\right)\frac{\lambda}{2}\right)\right]^{-1} \\ \mathfrak{A}^q(x) &:= 1+\mathfrak{a}^q(x) \end{aligned} \tag{4.6}$$

where $1 \le q \le r-3$. Inserting this into the inversion identity hierarchy (2.21) we find

$$\frac{\mathfrak{a}^{q-1}(x)\mathfrak{a}^{q+1}(x)}{\mathfrak{a}^q\left(x-i\frac{\pi}{r-2}\right)\mathfrak{a}^q\left(x+i\frac{\pi}{r-2}\right)} = \mathfrak{A}^{q-1}(x)\mathfrak{A}^{q+1}(x) \tag{4.7}$$

with the closure conditions $\mathfrak{a}^0 = \mathfrak{a}^{r-2} = \mathfrak{A}^0 = \mathfrak{A}^{r-2} \equiv 1$. We introduce the set of functions $l^q(x)$ which give the finite-size corrections to $\mathfrak{a}^q(x)$

$$\mathfrak{a}^q(x) = l^q(x)/\left[z^q(x)\right]^N. \tag{4.8}$$

In terms of these functions the functional equations (4.7) read

$$\frac{l^{q-1}(x)l^{q+1}(x)}{l^q\left(x-i\frac{\pi}{r-2}\right)l^q\left(x+i\frac{\pi}{r-2}\right)}=\mathfrak{A}^{q-1}(x)\mathfrak{A}^{q+1}(x) \tag{4.9}$$

where $l^0=l^{r-2}\equiv 1$. The functions $l^q(x)$ are ANZC in the strip

$$-\left(\frac{\pi}{2}+\frac{\pi}{r-2}\right)<\operatorname{Im}x<\left(\frac{\pi}{2}+\frac{\pi}{r-2}\right). \tag{4.10}$$

Hence, the logarithmic derivatives admit Fourier transforms as in (3.11).

Proceeding as in the previous section we obtain

$$L^{q-1}(k)-2\cosh\frac{\pi k}{r-2}L^q(k)+L^{q+1}(k)=A^{q-1}(k)+A^{q+1}(k) \tag{4.11}$$

where we now have the closure conditions $L^0=L^{r-2}=A^0=A^{r-2}\equiv 0$. This set of $r-3$ linear equations can be recast in matrix form as

$$\begin{pmatrix} K_0 & 1 & 0 & \dots & 0 & 0\\ 1 & K_0 & 1 & \dots & 0 & 0\\ 0 & 1 & K_0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & K_0 & 1\\ 0 & 0 & 0 & \dots & 1 & K_0 \end{pmatrix}\begin{pmatrix} L^1\\ L^2\\ L^3\\ \vdots\\ L^{r-4}\\ L^{r-3}\end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0\\ 1 & 0 & 1 & \dots & 0 & 0\\ 0 & 1 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 0 & 1\\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}\begin{pmatrix} A^1\\ A^2\\ A^3\\ \vdots\\ A^{r-4}\\ A^{r-3}\end{pmatrix} \tag{4.12}$$

where $K_0=-2\cosh[\pi k/(r-2)]$. Solving for $\vec{L}$ in terms of $\vec{A}$ gives

$$\vec{L}=K\cdot\vec{A} \tag{4.13}$$

If we write

$$K=I+\widetilde{K} \tag{4.14}$$

the matrix $\widetilde{K}$ is essentially the inverse of the matrix on the left hand side of (4.12). The solution is a symmetric matrix whose entries in the upper right triangle are given by

$$\widetilde{K}_l^j=-2\coth\left(\frac{\pi k}{r-2}\right)\frac{\sinh\left[(r-2-l)\frac{\pi k}{r-2}\right]\sinh\left[j\frac{\pi k}{r-2}\right]}{\sinh\pi k},\qquad j\le l. \tag{4.15}$$

Next we define the matrix $K(x)$ as the transform

$$K_l^j(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dkK_l^j(k)e^{ikx}. \tag{4.16}$$

Transforming back in (4.13) we derive the nonlinear integral equation

$$l\vec{\mathfrak{a}}=l\vec{\mathfrak{e}}+K*l\vec{\mathfrak{A}}+\vec{D} \tag{4.17}$$

where

$$\mathfrak{e}^q(u)=\left[\prod_{j=0}^{p-1}\frac{\sinh\left[\frac{x}{2}-\left(\frac{p+q}{2}-\frac{r}{4}-j\right)\frac{\pi i}{r-2}\right]}{\sinh\left[\frac{x}{2}-\left(\frac{p-q}{2}-\frac{r}{4}-j\right)\frac{\pi i}{r-2}\right]}\right]^N. \tag{4.18}$$

The integral kernel K is symmetric in the sense (3.21). Note that K is now a short-hand for $K(x)$, it does not denote the matrix in (4.13).

4.2 Excitations

In this subsection we extend (4.17) to low-lying excitations which have the same bulk behaviour as the ground state. The difference lies in the occurence of zeros in the analyticity strips. Let us consider one particular function $T^q(u)$ with a zero which we write as $u_0 + \frac{\lambda}{2}$. According to (2.20) the functions $t^{q-1}(u)$ and $t^{q+1}(u)$ have zeros at $u_0 + \frac{1}{2}\lambda$ and $u_0 - \frac{1}{2}\lambda$, respectively. From (4.6) we see that $\mathfrak{a}^{q-1}(x)$ and $\mathfrak{a}^{q+1}(x)$ each have a pole at the same number z_0. Similarly, (3.22) implies that $1 + t^q(u)$ has a pair of zeros $u_0 \pm \frac{1}{2}\lambda$. From (4.6) again we see that $\mathfrak{A}^q(x)$ has a pair of zeros $x_0 \pm \frac{\pi}{r-2}i$. If these zeros happen to lie in the strip $-\pi/2 < \operatorname{Im} x < \pi/2$ we can derive some qualitative information about the location of the poles of $\mathfrak{A}^q(x)$. The bulk behaviour of $\mathfrak{A}^q(x)$ is given by a constant 1 which implies that $\mathfrak{A}^q(x)$ has a pair of poles close to $x_0 \pm \frac{\pi}{r-2}i$. Then of course $\mathfrak{a}^q(x)$ also has a pair of poles in the neighbourhood of $x_0 \pm \frac{\pi}{r-2}i$.

From these conditions we derive two patterns for the poles x^q of $\mathfrak{a}^q(x)$. The first pattern is

$$x^q_\pm = x_0 + \frac{\pi}{2}i - (q \pm 1)\frac{\pi}{r-2}i \tag{4.19}$$

where x_0 is a real parameter. These zeros mainly occur in pairs except for $q = 1$ and $r-3$ where the highest and the lowest pole, respectively, are not allowed. Zeros of the function $\mathfrak{A}^q(x)$ are located at all $x^q_\pm$, even at x^1_- and x^{r-3}_+. An additional termination condition reads $\operatorname{Im} x^1_- = \pi/2$ and $\operatorname{Im} x^{r-3}_+ = -\pi/2$ which, in fact, is satisfied by (4.19). The second allowed pattern is derived from (4.19) by taking the complex conjugate on the right hand side.

Next we describe the derivation of integral equations for excited states. This follows closely the procedure of the previous section. We define Fourier transforms $L^q(k)$ and $A^q(k)$ by the first formula of (3.11). However, we use a modified integration path $\mathcal{L}$ which follows the axis $\operatorname{Im} x = \pi/2$ closely from below and avoids $x_0 + \frac{\pi}{2}i$ from above as in Figure 3. Applying this to the logarithmic derivative of (4.9) we obtain on the left hand side

$$\begin{aligned}
&\int_{\mathcal{L}} dx \left[\ln l^{q-1}(x)\right]' e^{-ikx} + \int_{\mathcal{L}} dx \left[\ln l^{q+1}(x)\right]' e^{-ikx} \\
&-e^{\frac{\pi}{r-2}k} \int_{\mathcal{L}_-} dx \left[\ln l^q(x)\right]' e^{-ikx} - e^{-\frac{\pi}{r-2}k} \int_{\mathcal{L}_+} dx \left[\ln l^q(x)\right]' e^{-ikx} \\
&= \int_{\mathcal{L}} dx \left[\ln l^{q-1}(x)\right]' e^{-ikx} + \int_{\mathcal{L}} dx \left[\ln l^{q+1}(x)\right]' e^{-ikx} \\
&-2\cosh \frac{\pi}{r-2}k \int_{\mathcal{L}} dx \left[\ln l^q(x)\right]' e^{-ikx}
\end{aligned} \tag{4.20}$$

where Cauchy's theorem can be applied since $l^q(x)$ is ANZC between $\mathcal{L}_+$ and $\mathcal{L}_-$. As before we derive (4.17) where the convolution is modified as in (3.24). The generalization to the case of multiple zeros is again straightforward.

4.3 Finite-size corrections

Before continuing the investigation of the nonlinear integral equations we express the finite-size corrections of T^q through the functions $\mathfrak{A}^q$. Instead of (3.25) the functions

T^q have to satisfy

$$\frac{T^q(u)T^q(u+\lambda)}{T^{q-1}(u+\lambda)T^{q+1}(u)} = f(u-\lambda)f(u+q\lambda)\left[1+\frac{1}{t^q(u)}\right]. \tag{4.21}$$

Let $T^q_{\rm bulk}(u)$ describe the bulk behaviour of $T^q(u)$ in the analyticity strip (2.25). It has to satisfy

$$\frac{T^q_{\rm bulk}(u)T^q_{\rm bulk}(u+\lambda)}{T^{q-1}_{\rm bulk}(u+\lambda)T^{q+1}_{\rm bulk}(u)} = f(u-\lambda)f(u+q\lambda). \tag{4.22}$$

More interesting than the bulk terms are the finite-size corrections which are defined by

$$T^q(u) = T^q_{\rm bulk}(u)T^q_{\rm finite}(u). \tag{4.23}$$

Inserting (4.23) into (4.21) and respecting (4.22) we are left with

$$\frac{T^q_{\rm finite}(u)T^q_{\rm finite}(u+\lambda)}{T^{q-1}_{\rm finite}(u+\lambda)T^{q+1}_{\rm finite}(u)} = \left[1+\frac{1}{t^q(u)}\right]. \tag{4.24}$$

Again we introduce functions of a real variable x

$$\mathfrak{b}^q(x) := T^q_{\rm finite}\left(\frac{r-2}{2r}ix - \left(q-p+\frac{r-2}{2}\right)\frac{\lambda}{2}\right). \tag{4.25}$$

Applying this to (4.24) we derive

$$\frac{\mathfrak{b}^q\left(x-i\frac{\pi}{r-2}\right)\mathfrak{b}^q\left(x+i\frac{\pi}{r-2}\right)}{\mathfrak{b}^{q-1}(x)\mathfrak{b}^{q+1}(x)} = \mathfrak{A}^q(x). \tag{4.26}$$

Taking the logarithmic derivative of (4.26) and introducing Fourier transforms yields a set of linear equations

$$\begin{pmatrix} K_0 & 1 & 0 & \dots & 0 & 0 \\ 1 & K_0 & 1 & \dots & 0 & 0 \\ 0 & 1 & K_0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & K_0 & 1 \\ 0 & 0 & 0 & \dots & 1 & K_0 \end{pmatrix}\begin{pmatrix} B^1 \\ B^2 \\ B^3 \\ \vdots \\ B^{r-4} \\ B^{r-3}\end{pmatrix} = -\begin{pmatrix} A^1 \\ A^2 \\ A^3 \\ \vdots \\ A^{r-4} \\ A^{r-3}\end{pmatrix} \tag{4.27}$$

where again $K_0 = -2\cosh[\pi k/(r-2)]$. This can be solved for $\vec{B}$ in terms of $\vec{A}$

$$\vec{B} = \widehat{K}\cdot\vec{A} \tag{4.28}$$

where the matrix $\widehat{K}$ is symmetric and the upper right elements are given by

$$\widehat{K}^j_l = \frac{\sinh\left[(r-2-l)\frac{\pi k}{r-2}\right]\sinh\left[j\frac{\pi k}{r-2}\right]}{\sinh\frac{\pi k}{r-2}\sinh\pi k}, \qquad j\le l. \tag{4.29}$$

Transforming back (4.28) we express $\mathfrak{b}^q$ in terms of $\mathfrak{A}^q$

$$l\vec{\mathfrak{b}} = \widehat{K} * l\vec{\mathfrak{A}} + \vec{C} \tag{4.30}$$

where $\widehat{K}$ is the Fourier transform of (4.29) and $\vec{C}$ is a vector of integration constants which can be evaluated from the asymptotics of $\mathfrak{A}^q$ and $\mathfrak{b}^q$. The components of $\vec{C}$ are multiples of $2\pi i/(r-2)$ and therefore $\vec{C}$ does not contain $1/N$ corrections. The kernel $\widehat{K}(x)$ is symmetric with the large x asymptotic behaviour

$$\widehat{K}_l^j(x) \simeq \frac{\sin j\frac{\pi}{r-2}\sin l\frac{\pi}{r-2}}{\pi\sin\frac{\pi}{r-2}} e^{-|x|} \tag{4.31}$$

Let us now return to the nonlinear integral equations (4.17). The only place where the system size N enters is through $\mathfrak{e}^q(x)$ which scales like

$$\ln \mathfrak{e}^q(\pm(x+\ln N)) = \mp\frac{Npq}{r-2}\pi i - 2\frac{\sin q\frac{\pi}{r-2}\sin p\frac{\pi}{r-2}}{\sin\frac{\pi}{r-2}} e^{-x} + o\left(\frac{1}{N}\right) \tag{4.32}$$

In the following we only consider the case when Npq is an even multiple of $r-2$. In fact N must be a multiple of $r-2$ to accommodate the ground state. In the scaling limit the integral equations (4.17) take the form

$$l\vec{a} = l\vec{e} + K * l\vec{A} + \vec{D} \tag{4.33}$$

where we drop the subscripts $\pm$ and use the definitions (3.34) and (3.36) with the obvious modification

$$le^q(x) = -2\frac{\sin q\frac{\pi}{r-2}\sin p\frac{\pi}{r-2}}{\sin\frac{\pi}{r-2}} e^{-x}. \tag{4.34}$$

For fixed x the finite-size corrections of $\mathfrak{b}^q(x)$ are of order $1/N$

$$\begin{aligned}
\ln \mathfrak{b}^q(x) &= \sum_{j=1}^{r-3} (\widehat{K}_j^q * l\mathfrak{A}^j)(x) + C^q \\
&= \sum_{j=1}^{r-3} \int_{-\ln N}^{\infty} [\widehat{K}_j^q(x-y-\ln N) l\mathfrak{A}^j(y+\ln N) \\
&\qquad + \widehat{K}_j^q(x+y+\ln N) l\mathfrak{A}^j(-y-\ln N)]dy + C^q \\
&= \frac{\sin q\frac{\pi}{r-2}}{N\pi\sin\frac{\pi}{r-2}} \Bigg[e^x \sum_{j=1}^{r-3} \sin j\frac{\pi}{r-2} \int_{-\infty}^{\infty} e^{-y} lA_+^j(y)dy + \\
&\qquad + e^{-x}\sum_{j=1}^{r-3} \sin j\frac{\pi}{r-2} \int_{-\infty}^{\infty} e^{-y} lA_-^j(y)dy \Bigg] + C^q + o\left(\frac{1}{N}\right)
\end{aligned} \tag{4.35}$$

In contrast to regime III/IV the integrals exist for all q. In fact they do not depend on q at all. The amplitude of the $1/N$ corrections can be evaluated as in the previous section. Manipulating (4.33) as before we find

$$\sum_{q=1}^{r-3} \int_{-\infty}^{\infty} \left[(la^q)'\, lA^q - la^q\,(lA^q)' \right] dx$$

$$= \sum_{q=1}^{r-3} \int_{-\infty}^{\infty} \left[(le^q)' \, lA^q - (le^q + D^q) \, (lA^q)' \right] dx. \tag{4.36}$$

Changing the variable of integration x to a on the left hand side, integrating by parts and inserting (4.34) we obtain

$$\begin{aligned} &2\frac{\sin p\frac{\pi}{r-2}}{\sin\frac{\pi}{r-2}} \sum_{q=1}^{r-3} \sin q\frac{\pi}{r-2} \int_{-\infty}^{\infty} e^{-x} lA^q(x) dx \\ &= \frac{1}{2}\sum_{q=1}^{r-3} \int \left[\frac{\ln(1+a^q)}{a^q} - \frac{\ln a^q}{1+a^q} \right] da^q + \frac{1}{2}\sum_{q=1}^{r-3} D^q lA^q|_{-\infty}^{\infty} \\ &= \sum_{q=1}^{r-3} L\left(\frac{a_\infty^q}{A_\infty^q}\right) + \frac{1}{2}\sum_{q=1}^{r-3} D^q lA_\infty^q \end{aligned} \tag{4.37}$$

where we have used (3.41) as well as $a_{-\infty}^q = 0$ and $A_{-\infty}^q = 1$. The constants are determined by the asymptotics

$$\vec{D} = l\vec{a_\infty} - K(k=0) \cdot l\vec{A_\infty} \tag{4.38}$$

where K is the matrix in (4.13).

Next we calculate the central charge from the finite-size corrections to the ground state. As argued in Appendix C we have

$$c = \frac{6}{\pi^2} \left[\sum_{q=1}^{r-3} L\left(\frac{a_\infty^q}{A_\infty^q}\right) + \frac{1}{2}\sum_{q=1}^{r-3} D^q lA_\infty^q \right]. \tag{4.39}$$

The asymptotics a_∞^q and A_∞^q for this eigenvalue are identical for both scaling regimes and are given by (2.42) with $\theta = \pi/r$. From (4.38) we see that $D^q = 0$ so that finally using (3.45) we obtain the central charge

$$c = \frac{6}{\pi^2} \sum_{q=1}^{r-3} L\left(\frac{\sin^2\frac{\pi}{r}}{\sin^2(q+1)\frac{\pi}{r}} \right) = 2 - \frac{6}{r} \tag{4.40}$$

4.4 The primary conformal weights

The amplitudes of the $1/N$ finite-size corrections of all excitations are given by the right hand side of (4.37) and by (4.38) in terms of the asymptotics of $a^q(x)$ and $A^q(x)$. The precise dependence of these functions on x is again unimportant. However, the way singularities at 0 and -1 are encircled in the complex plane by the values of $a^q(x)$ as x varies along the integration path does matter. Here the situation is not as clear cut as in regime III/IV where $a^q(x)$ is real on the real axis and, starting from this, the behaviour of $a^q(x)$ on the path $\mathcal{L}$ could be deduced. In regime I/II, $a^q(x)$ is not real for real x. Such a property is only observed if $\mathrm{Im}\, x = \pm\left(\frac{\pi}{2} + \frac{\pi}{r-2}\right)$. Unfortunately, the analyticity strip of $a^q(x)$ ends on these lines where poles are densely distributed.

Nevertheless, we follow as closely as possible the treatment of regime III/IV. The asymptotics of $a^q(x)$ and $A^q(x)$ are given by $a^q_{-\infty}=0$, $A^q_{-\infty}=1$, and

$$a^q_\infty = \frac{\sin^2\theta}{\sin q\theta \sin(q+2)\theta}, \qquad A^q_\infty = \frac{\sin^2(q+1)\theta}{\sin q\theta\sin(q+2)\theta} \tag{4.41}$$

where θ is a multiple of $\frac{\pi}{r}$, but is considered as a complex parameter for the time being. The homotopy classes of $a^q(x)$ and $A^q(x)$ are constructed as outlined in Section 3.5, namely, from (4.41) by increasing θ along a path in the complex plane avoiding all singularities and ending at $s\frac{\pi}{r}$ with integer $s\in\{1,2,...,r-1\}$. Bearing our treatment of regime III/IV in mind, the most obvious path of θ would run in the upper half plane or the lower one. However, we observe in contrast to the previous case that for this regime new primary excitations are obtained by allowing for a change from the upper half plane to the lower one as shown in Figure 4.

We next derive the constants D^q from (4.38)

$$\vec{D} = \begin{pmatrix} -2 & 1 & \dots & 0 \\ 1 & -2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -2 \end{pmatrix}^{-1} \left[\begin{pmatrix} -2 & 1 & \dots & 0 \\ 1 & -2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -2 \end{pmatrix} l\vec{a}_\infty - \begin{pmatrix} 0 & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} l\vec{A}_\infty \right] \tag{4.42}$$

where all matrices are tridiagonal and the matrix K was substituted by the matrices appearing in (4.12). Inserting (4.41) into (4.42) we find for the term in square brackets

$$[...] = \begin{cases} 0, & \text{for } 1\le q < r-3 \\ 2\ln\dfrac{\sin(r-1)\theta}{\sin\theta}, & \text{for } q = r-3. \end{cases} \tag{4.43}$$

Multiplying this with the inverse matrix in (4.42) which is equal to $\frac{1}{2}\widetilde{K}(k=0)$, cf. (4.15), we obtain

$$D^q = -\frac{2q}{r-2}\ln\frac{\sin(r-1)\theta}{\sin\theta} \tag{4.44}$$

On the right hand side of (4.37) we have two contributions. The sum of dilogarithmic functions can be simplified by applying (A.1)

$$\begin{aligned}\sum_{q=1}^{r-3} L\left(\frac{\sin^2\theta}{\sin^2(q+1)\theta}\right) &= (r-1)r\theta^2 - L(1) - \ln\frac{\sin(r-1)\theta}{\sin\theta}\ln\frac{\sin r\theta}{\sin\theta} \\ -L_2\left(-\frac{\sin(r-1)\theta}{\sin\theta}e^{ir\theta}\right) &- L_2\left(-\frac{\sin(r-1)\theta}{\sin\theta}e^{-ir\theta}\right) - L\left(\frac{\sin^2\theta}{\sin^2(r-1)\theta}\right).\end{aligned} \tag{4.45}$$

The second contribution yields

$$\frac{1}{2}\sum_{q=1}^{r-3} D^q lA_\infty = -\frac{1}{r-2}\ln\frac{\sin(r-1)\theta}{\sin\theta}\ln\frac{\sin^{r-2}(r-2)\theta}{\sin\theta\sin^{r-3}(r-1)\theta}. \tag{4.46}$$

If we take (4.39) as a definition of a quantity c for excited states we obtain

$$\frac{\pi^2}{6}c = (r-1)r\theta^2 - L(1) - L_2\left(-\frac{\sin(r-1)\theta}{\sin\theta}e^{ir\theta}\right) - L_2\left(-\frac{\sin(r-1)\theta}{\sin\theta}e^{-ir\theta}\right)$$

$$-L_2\left(\frac{\sin^2\theta}{\sin^2(r-1)\theta}\right)-\frac{r-1}{r-2}\left[\ln\frac{\sin(r-1)\theta}{\sin\theta}\right]^2 \tag{4.47}$$

This can be evaluated explicitly where we have to respect the concrete way θ is increased from small positive numbers along a path in the complex plane to $s\frac{\pi}{r}$. By m_1 and m_2 we denote the number of multiples of $\frac{\pi}{r-1}$ which are encircled in a clockwise and anticlockwise sense, respectively. Note that $m_1+m_2=s-1$. We have the identities

$$\begin{aligned} L_2\left(-\frac{\sin(r-1)\theta}{\sin\theta}e^{ir\theta}\right) &= L(1)+2\pi^2 m_2(m_2+1) \\ L_2\left(-\frac{\sin(r-1)\theta}{\sin\theta}e^{-ir\theta}\right) &= L(1)+2\pi^2 m_1(m_1+1) \end{aligned} \tag{4.48}$$

and

$$L_2\left(\frac{\sin^2\theta}{\sin^2(r-1)\theta}\right)=L(1) \tag{4.49}$$

where $L(1)$ is the value of the standard branch. We find

$$\frac{\pi^2}{6}c=2L(1)-\frac{\pi^2}{r}s^2+\frac{\pi^2}{r-2}(m_1-m_2)^2. \tag{4.50}$$

From this and (B.9) we finally conclude, after introducing $l:=s-1$, $m:=m_1-m_2$, that

$$\Delta=\frac{l(l+2)}{4r}-\frac{m^2}{4(r-2)}$$

where

$$l=0,1,...,r-2, \qquad m=-l,-l+2,...,l-2,l \tag{4.51}$$

Note that most of the conformal weights are at least twofold degenerate since $\Delta(l,m)=\Delta(l,-m)$.

After this derivation two questions remain open. First one may wonder if the prescription of the analytic continuation in θ as given above is allowed. This is actually possible since (4.17) is equivalent to the functional equations (2.21) with appropriate phase factors depending on θ. If θ specializes to a multiple of π/r these phase factors are reduced to 1 and the original equations (2.21) are reproduced. The second question concerns paths different from the particular one given in Figure 4. We have done calculations for such cases and have found conformal weights belonging to towers above the conformal weights in (4.51). It therefore appears that all primary conformal weights are given by (4.51). We also mention that the constant $\vec{C}$ in (4.30) is equal to $-\frac{1}{2}\vec{D}$ for the primary weights

$$C^q=\frac{q}{r-2}\ln\frac{\sin(r-1)\theta}{\sin\theta}=m\frac{\pi i}{r-2}q. \tag{4.52}$$

From this we derive the lattice momentum of the state

$$P_0=\frac{p\pi}{r-2}\cdot m \tag{4.53}$$

where only the quantum number m enters.

5 Discussion

In this paper we have calculated the central charges and primary conformal weights for RSOS models and their fusion hierarchies via finite-size corrections to the eigenvalues of the row transfer matrices. This was achieved by solving special functional equations in the form of inversion identity hierarchies. Using Fourier transforms, these functional equations were transformed into nonlinear integral equations for the finite-size corrections. The nonlinear integral equations cannot be solved in general. Nevertheless, it is possible to extract analytically the values of certain integrals. This is possible due to the special symmetry of the kernel which results by setting up the functional equations in the form of an inversion identity hierarchy. Remarkably, the only integrals that can be calculated are precisely the integrals that give the physically interesting quantities, namely, the central charges and conformal weights! This unexpected benevolence of nature is quite striking.

Our calculations have been carried through for the ground state and all low-lying excitations in both regimes I/II and III/IV. We have shown how to deform the integration paths for the integral equations to include the excitations. The final results for the central charge c and the conformal weights Δ are given in terms of asymptotics involving the Coxeter exponents of A_L and Rogers dilogarithms. These integrals have been evaluated explicitly for the primary conformal weights. The branches of the dilogarithms were fixed by determining the homotopy classes of the relevant contours in the complex plane. Essentially, the finite-size corrections are given by the analytic continuation of $c = c(\theta)$ to values of θ which are multiples of $\frac{\pi}{r}$.

The arguments presented here need to be supplemented to apply to those cases where the asymptotics of the eigenvalue functions vanish. The key to handling these cases is the regularization scheme using analytic continuation. After performing the intermediate calculations the troublesome set of parameters may be approached continuously to obtain the final result. This was tacitly done in Sections 3 and 4.

Finally, it is of interest to extend the work of this paper in several directions. First, it appears that inversion identity hierarchies are satisfied [30] by all the A–D–E lattice models. In general for these models, the asymptotics in the upper and lower half planes can be different with the result that $\Delta \neq \overline{\Delta}$ so scaling operators with spin appear in the theory. It would be interesting to look more closely at the solution of the inversion identity hierarchy in such cases and for the 3-state Potts model (D_4) in particular. Second, it should also be possible to generalize [32,33] the inversion identity hierarchies to the higher rank face models of Jimbo, Miwa and Okado [34].

Appendix A

In the main text of this paper we have employed several identities for sums of dilogarithmic functions. All of them can be regarded as special cases of the general identity

$$\sum_{k=1}^{r-1} L\left(\frac{\sin^2\theta}{\sin^2(\phi+k\theta)}\right) = 2(r-1)\theta\phi + (r-1)r\theta^2$$

$$+ \left[L_2\left(-\frac{\sin\phi}{\sin\theta}e^{i(\phi+\theta)}\right) + L_2\left(-\frac{\sin\phi}{\sin\theta}e^{-i(\phi+\theta)}\right) + \ln\frac{\sin\phi}{\sin\theta}\ln\frac{\sin(\phi+\theta)}{\sin\theta}\right]\Bigg|^{\phi}_{\phi+(r-1)\theta} \qquad \text{(A.1)}$$

where

$$\begin{aligned} L(x) &= L_2(x) + \frac{1}{2}\ln x \ln(1-x) \\ L_2(x) &= -\int_0^x dy \frac{\ln(1-y)}{y} \end{aligned} \qquad \text{(A.2)}$$

The two parameters θ and ϕ are generically small positive numbers, but may be continued to any complex values as long as all singularities of the logarithms and dilogarithms are avoided. If we set $\theta = \frac{\pi}{r}$ and $\phi = 0$ in (A.1) we find the identity (3.45) with $n = r$. Other identities related to (A.1) have been obtained by Kirillov [35].

To prove (A.1) for all r it is sufficient to show it for $r = 2$

$$L\left(\frac{\sin^2\theta}{\sin^2(\phi+\theta)}\right) = 2\theta\phi + 2\theta^2 + \left[L_2\left(-\frac{\sin\phi}{\sin\theta}e^{i(\phi+\theta)}\right) + L_2\left(-\frac{\sin\phi}{\sin\theta}e^{-i(\phi+\theta)}\right) + \ln\frac{\sin\phi}{\sin\theta}\ln\frac{\sin(\phi+\theta)}{\sin\theta}\right]\Bigg|^{\phi}_{\phi+\theta} \qquad \text{(A.3)}$$

This is done by establishing the equality of the derivatives with respect to ϕ and finally proving the equation for a particular value of ϕ, say 0. The derivative of the left hand side of (A.3) with respect to ϕ is

$$\begin{aligned} \text{lhs}' &= \frac{\cos(\phi+\theta)}{\sin(\phi+\theta)}\ln\frac{\sin\phi\sin(\phi+2\theta)}{\sin^2(\phi+\theta)} \\ &\quad + \frac{\sin^2\theta\cos(\phi+\theta)}{\sin(\phi+\theta)\sin\phi\sin(\phi+2\theta)}\ln\frac{\sin^2\theta}{\sin^2(\phi+\theta)} \end{aligned} \qquad \text{(A.4)}$$

In fact, after some tedious calculations the derivative of the right hand side of (A.3) yields the same term. Lastly we set $\phi = 0$ and show

$$L(1) = 2\theta^2 - L_2\left(-e^{2i\theta}\right) - L_2\left(-e^{-2i\theta}\right) \qquad \text{(A.5)}$$

by taking derivatives and specializing to $\theta = \pi/2$.

For the parameters θ, σ, τ, and ω as given by (3.47) and (3.55) we have a list of identities which hold if first ω is increased in the upper half plane to its final value and then θ, σ, and τ:

$$L\left(\frac{\sin^2\theta}{\sin^2(\theta+\omega)}\right) - L\left(\frac{\sin^2\sigma}{\sin^2(\sigma+\omega)}\right) = \omega i \ln\left[\frac{\sin 2\theta}{\sin^2\theta}\frac{\sin^2\sigma}{\sin 2\sigma}\right] \qquad \text{(A.6)}$$

$$L_2\left(-\frac{\sin\omega}{\sin\sigma}e^{i(\omega+\sigma)}\right) - L_2\left(-\frac{\sin\omega}{\sin\theta}e^{i(\omega+\theta)}\right) = 0 \tag{A.7}$$

$$L_2\left(-\frac{\sin\omega}{\sin\sigma}e^{-i(\omega+\sigma)}\right) - L_2\left(-\frac{\sin\omega}{\sin\theta}e^{-i(\omega+\theta)}\right) = -2\omega i\ln\left[\frac{\sin\sigma}{\sin\theta}e^{i(\sigma-\theta)}\right] \tag{A.8}$$

$$\ln\frac{\sin\omega}{\sin\sigma}\ln\frac{\sin(\omega+\sigma)}{\sin\sigma} - \ln\frac{\sin\omega}{\sin\theta}\ln\frac{\sin(\omega+\theta)}{\sin\theta} = -\omega i\ln\frac{\sin\theta}{\sin\sigma} \tag{A.9}$$

$$\begin{aligned} L\left(\frac{\sin^2\theta}{\sin^2((r-1)\theta+\omega)}\right) \quad &- \quad L\left(\frac{\sin^2\tau}{\sin^2(r-p-1)\tau}\right) \\ &= \quad \pi i(t-1)\ln\left[\frac{\sin 2\theta}{\sin^2\theta}\frac{\sin^2\tau}{\sin 2\tau}\frac{\sin(r\theta+\omega)}{\sin(r-p)\tau}\right] \end{aligned} \tag{A.10}$$

$$\begin{aligned} &L\left(\frac{\sin^2\theta}{\sin^2((r-1)\theta+\omega)}\right) - L\left(\frac{\sin^2\tau}{\sin^2(r-p-1)\tau}\right) \\ &+\ln\frac{\sin(\omega+(r-1)\theta)}{\sin\theta}\ln\frac{\sin(\omega+r\theta)}{\sin\theta} - \ln\frac{\sin(r-p-1)\tau}{\sin\tau}\ln\frac{\sin(r-p)\tau}{\sin\tau} \\ &= \pi i(t-1)\ln\left[\frac{\sin 2\theta}{\sin\theta}\frac{\sin\tau}{\sin 2\tau}\right] \end{aligned} \tag{A.11}$$

$$L_2\left(-\frac{\sin(\omega+(p+1)\sigma)}{\sin\sigma}e^{i(\omega+(p+2)\sigma)}\right) = L(1) \tag{A.12}$$

$$L_2\left(-\frac{\sin(\omega+(p+1)\sigma)}{\sin\sigma}e^{-i(\omega+(p+2)\sigma)}\right) = 4\pi^2(1+2+...+n)+L(1) = 2\pi^2 n(n+1)+L(1) \tag{A.13}$$

Appendix B

In this appendix we consider the Hamiltonian limit and dispersion relation in regime III/IV. The Hamiltonian limit of $\boldsymbol{T}^p(u)$ is taken at $u=0$ or λ. At these points the transfer matrix degenerates to a shift operator and the logarithmic derivative is a sum of local hermitian operators

$$\begin{aligned} P &\propto \ln T^p(u)\Big|_{u=0 \text{ or } \lambda} \\ H &\propto [\ln T^p(u)]'\Big|_{u=0 \text{ or } \lambda} \end{aligned} \tag{B.1}$$

Energies and momenta can be obtained from the eigenvalues of $\boldsymbol{T}^p(u)$. Neglecting bulk terms it is sufficient to use $T^p_{\text{finite}}(u)$ or $\mathfrak{b}^p(x)$, cf. (3.29)

$$\begin{aligned} P &= \mp i\ln\mathfrak{b}^p(x)\Big|_{x=\pm i\frac{\pi}{2}} \\ E &= \pm i\,[\ln\mathfrak{b}^p(x)]'\Big|_{x=\pm i\frac{\pi}{2}} \end{aligned} \tag{B.2}$$

Here we have chosen the normalization to have real P and E, and to render the sound velocity equal to +1. To show this we note that (3.30) implies the following form for the excitations in the thermodynamic limit

$$\mathfrak{b}^p(x) = \prod_{\text{zeros } x_j} \tanh\left(\frac{x-x_j}{2}\right) \tag{B.3}$$

Focussing on one 'elementary excitation' x_0 we find

$$\begin{aligned} P_0(x_0) &= \mp i \ln\tanh\left(\frac{x-x_0}{2}\right)\Big|_{x=\pm i\frac{\pi}{2}} \\ E_0(x_0) &= \pm i \left[\ln\tanh\left(\frac{x-x_0}{2}\right)\right]'\Big|_{x=\pm i\frac{\pi}{2}} = \frac{1}{\cosh x_0}. \end{aligned} \tag{B.4}$$

After eliminating the rapidity x_0 we get

$$E_0(P_0) = \sin P_0, \qquad 0 \le P_0 \le \pi. \tag{B.5}$$

So the excitations are indeed positive and have sound velocity equal to +1. As we already know from (3.37), the finite-size corrections of $\ln \mathfrak{b}^p(x)$ are given by exponentials e^x and e^{-x}. For the energy-momentum spectrum of H, as given by (B.2), to be conformally invariant we must have

$$\ln \mathfrak{b}^p(x) = \frac{c\pi}{6N}\cosh x - \frac{2\pi}{N}\left[(\Delta+\bar{\Delta})\cosh x + (\Delta-\bar{\Delta})\sinh x\right] \tag{B.6}$$

or

$$\ln T^p_{\text{finite}}(u) = \frac{c\pi}{6N}\sin ru - \frac{2\pi}{N}\left[(\Delta+\bar{\Delta})\sin ru + i(\Delta-\bar{\Delta})\cos ru\right] \tag{B.7}$$

From (B.6) and (3.37) we find for the ground state

$$c = \frac{12}{\pi^2}\int_{-\infty}^{\infty} e^{-x} lA^p(x)dx. \tag{B.8}$$

If we take the last equation as the definition of c for any state we obtain the corresponding conformal weight by

$$\Delta = \frac{c(\text{ground state}) - c(\text{excitation})}{24} \tag{B.9}$$

Appendix C

In this appendix we consider the Hamiltonian limit and dispersion relation in regime I/II. The Hamiltonian limit of $\boldsymbol{T}^p(u)$ is taken at $u=0$ or λ. Neglecting bulk terms, the energies and momenta can be obtained from $\mathfrak{b}^p(x)$ at $x=-\frac{\pi}{2}i$ or $x=-\frac{\pi}{2}i-2\frac{\pi}{r-2}i$, cf. (4.25). The last point, however, does not lie in the analyticity strip, so we take

$$P = -i\ln \mathfrak{b}^q(x)\Big|_{x=-i\frac{\pi}{2}}$$

$$E = +i\left[\ln \mathfrak{b}^q(x)\right]'\Big|_{x=-i\frac{\pi}{2}} \tag{C.1}$$

where we are interested in the case $q = p$ eventually. This normalization renders the sound velocity $+1$. To show this we focus on one 'elementary excitation' which is given by the poles (4.19) of $\mathfrak{a}^q(x)$ and implies the zero

$$x^q = x_0 + \frac{\pi}{2}i - q\frac{\pi}{r-2}i \tag{C.2}$$

for $\mathfrak{b}^q(x)$. The corresponding solution of (4.26) in the thermodynamic limit is

$$\mathfrak{b}^q(x) = \frac{\sinh\left[\frac{1}{2}(x-x_0) + \frac{\pi}{4}i - \frac{\pi}{2(r-2)}qi\right]}{\sinh\left[\frac{1}{2}(x-x_0) + \frac{\pi}{4}i + \frac{\pi}{2(r-2)}qi\right]} \tag{C.3}$$

After eliminating x_0 we get

$$E_1(P_1) = \frac{\cos\frac{\pi}{r-2}q - \cos P_1}{\sin\frac{\pi}{r-2}q}, \qquad -\frac{\pi}{r-2}q \leq P_1 \leq \frac{\pi}{r-2}q \tag{C.4}$$

Here P_1 denotes the momentum of the elementary excitation. The second type of elementary excitations obtained from (4.19) and (C.2) by complex conjugation is

$$E_2(P_2) = \frac{\cos P_2 - \cos\frac{\pi}{r-2}q}{\sin\frac{\pi}{r-2}q}, \qquad \frac{\pi}{r-2}q \leq P_2 \leq 2\pi - \frac{\pi}{r-2}q \tag{C.5}$$

So the excitations are indeed positive and have sound velocity $+1$. As we already know from (4.35) the finite-size corrections of $\ln \mathfrak{b}^p(x)$ are given by exponentials e^x and e^{-x}. For the energy-momentum spectrum, as given by (C.1), to be conformally invariant we must have

$$\ln \mathfrak{b}^p(x) = -\frac{c\pi}{6N}\cosh x + \frac{2\pi}{N}\left[(\Delta+\bar{\Delta})\cosh x + (\Delta-\bar{\Delta})\sinh x\right] \tag{C.6}$$

or

$$\ln T^p_{\text{finite}}(u) = -\frac{c\pi}{6N}\sin\frac{2r}{r-2}u - \frac{2\pi}{N}\left[(\Delta+\bar{\Delta})\sin\frac{2r}{r-2}u + i(\Delta-\bar{\Delta})\cos\frac{2r}{r-2}u\right] \tag{C.7}$$

From (C.6) and (4.35) we find for the ground state

$$\begin{aligned} c &= \frac{12}{\pi^2}\frac{\sin p\frac{\pi}{r-2}}{\sin\frac{\pi}{r-2}}\sum_{l=1}^{r-3}\sin l\frac{\pi}{r-2}\int_{-\infty}^{\infty} e^{-y} l\mathfrak{A}^l(y)dy \\ &= \frac{6}{\pi^2}\left[\sum_{q=1}^{r-3} L\left(\frac{a^q_\infty}{A^q_\infty}\right) + \frac{1}{2}\sum_{q=1}^{r-3} d^q l A^q_\infty\right] \end{aligned} \tag{C.8}$$

If we take the last equation as the definition of c for any state we obtain the corresponding conformal weight by (B.9).

The $1/N$ corrections for general T^q are given by

$$\ln \mathfrak{b}^q(x) = \frac{\sin q\frac{\pi}{r-2}}{\sin p\frac{\pi}{r-2}}\ln \mathfrak{b}^p(x) \tag{C.9}$$

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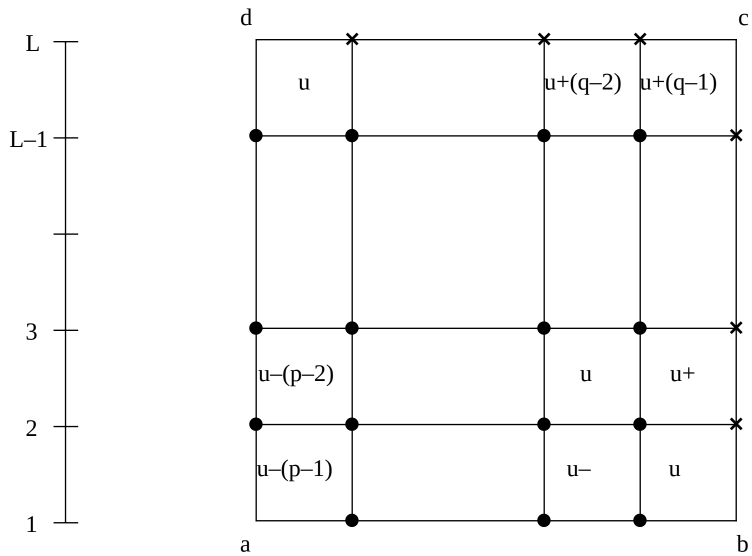

Figure 1. The Dynkin diagram for A_L giving the allowed states of adjacent sites on the square lattice for the ABF or RSOS$(1,1)$ models. Also shown is a face weight for the RSOS(p,q) models. The spin states at sites indicated with a solid circle are summed over and the result is independent of the state of the spins at sites marked with a cross.

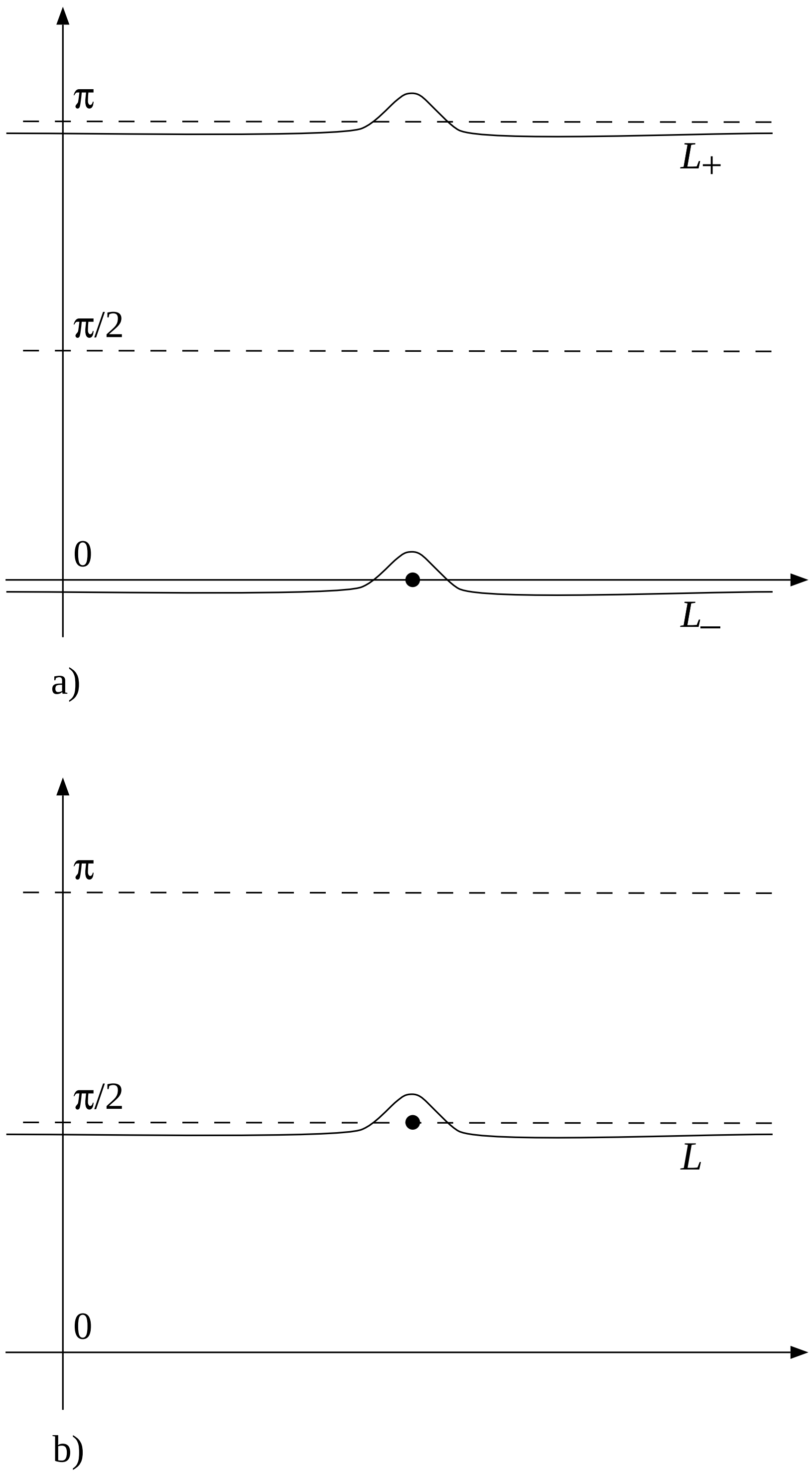

Figure 2. Representation of typical paths $\mathcal{L}$, $\mathcal{L}_+$, and $\mathcal{L}_-$ in the complex plane for regime III/IV. A typical zero of the function $\mathfrak{a}^q(x)$ is depicted as a full dot in part a). The corresponding zero of $\mathfrak{A}^{q-1}(x)$ or $\mathfrak{A}^{q+1}(x)$ is shown in part b). Note that $\mathfrak{a}^q(x)$ is ANZC between $\mathcal{L}_+$ and $\mathcal{L}_-$.

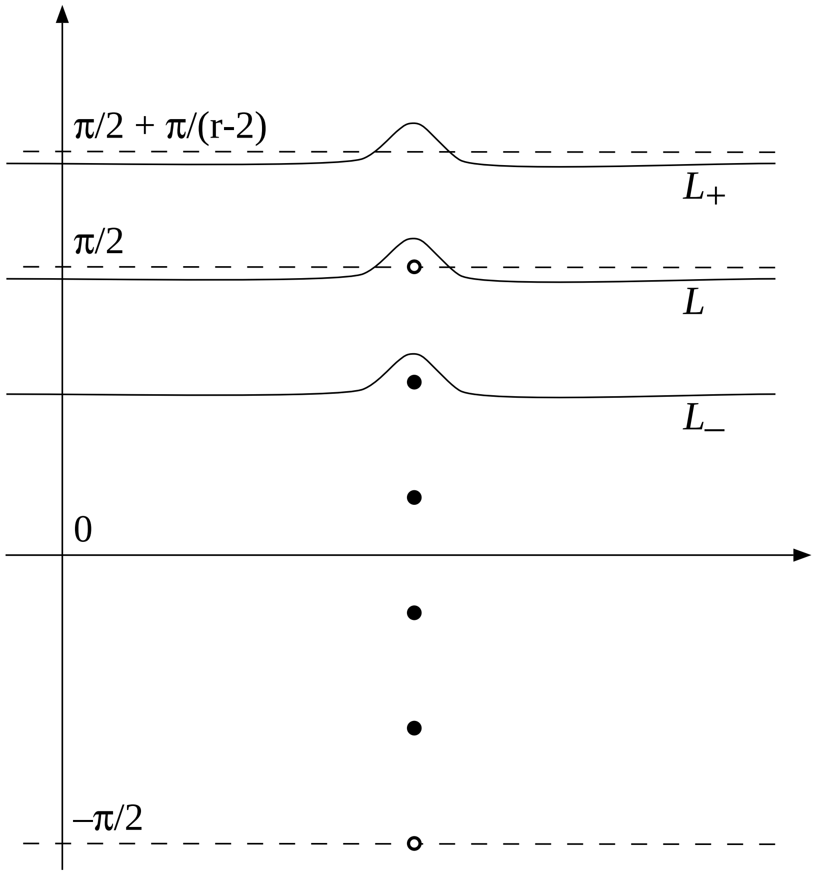

Figure 3. Representation of typical paths $\mathcal{L}$, $\mathcal{L}_+$, and $\mathcal{L}_-$ in the complex plane for regime I/II. Full dots represent poles of the functions $\mathfrak{a}^q(x)$, and poles as well as zeros of the functions $\mathfrak{A}^q(x)$. The open dots represent zeros of the functions $\mathfrak{A}^1(x)$ and $\mathfrak{A}^{r-3}(x)$ only. Note that $\mathfrak{a}^q(x)$ is ANZC between $\mathcal{L}_+$ and $\mathcal{L}_-$.

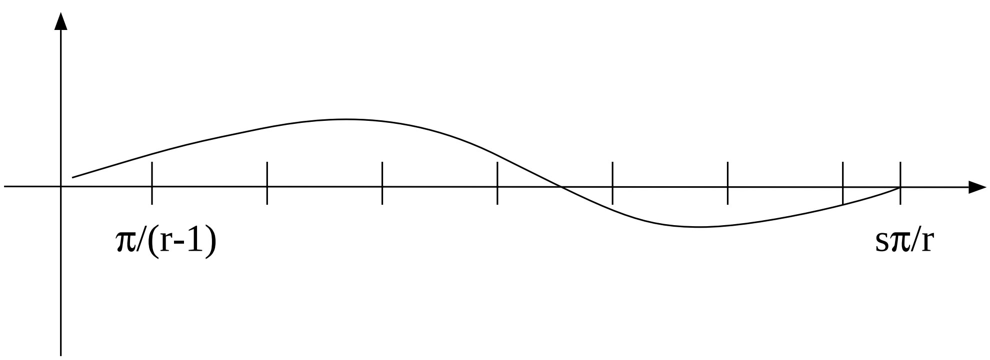

Figure 4. Depiction of the path in the complex plane along which the analytic continuation of $c = c(\theta)$ in (4.47) is performed. The endpoint is a multiple s of $\frac{\pi}{r}$. The path surrounds certain multiples of $\frac{\pi}{r-1}$ clockwise and counterclockwise. These numbers m_1 and m_2, respectively, are related to s by $s = m_1 + m_2 + 1$.

Captions

Figure 1: The Dynkin diagram for A_L giving the allowed states of adjacent sites on the square lattice for the ABF or RSOS$(1,1)$ models. Also shown is a face weight for the RSOS(p,q) models. The spin states at sites indicated with a solid circle are summed over and the result is independent of the state of the spins at sites marked with a cross.

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