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ROW TRANSFER MATRIX FUNCTIONAL EQUATIONS FOR A–D–E LATTICE MODELS

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ABSTRACT

Determinantal functional equations satisfied by the row transfer matrix eigenvalues of critical A–D–E lattice spin models are presented. These are obtained for models associated with the Lie algebras $A^{(1)}_{L-1}$, $D^{(1)}_{L-1}$, A_L, D_L and $E_{6,7,8}$ by exploiting connections with functional equations satisfied by the row transfer matrix eigenvalues of the six-vertex model at rational values of the crossing parameter $\lambda{=}s\pi/h$ where h is the Coxeter number. In addition, fusion is used to derive special functional equations, called inversion identity hierarchies, which provide the key to the direct calculation of finite-size corrections, central charges and conformal weights for the critical A–D–E lattice models.

Keywords: A–D–E lattice models, row transfer matrices, functional equations.

1. Introduction

The A–D–E lattice models are interaction-round-a-face or IRF models [1] that generalize the restricted solid-on-solid (RSOS) models solved by Andrews, Baxter and Forrester [2] in 1984. In 1987, Pasquier [3] realized that the L heights of the RSOS model could be associated with the Dynkin diagram of the classical Lie algebra A_L and he introduced solvable critical lattice models associated with each of the remaining classical and affine A–D–E Dynkin diagrams. Off-critical extensions exist [2,4,5,6] for the classical and affine A and D models but not for the exceptional E models [7]. Reviews of the critical A–D–E models emphasizing connections with conformal field theory can be found, for example, in [8,9,10]. It is now known that the critical A–D–E lattice models are related to the six-vertex model. For instance, the free energy of a critical A–D–E model is obtained from the free energy of the six-vertex model by fixing the crossing parameter λ to the appropriate rational fraction of π. Typically, this value is given by $\lambda = \pi/h$ where h is the Coxeter

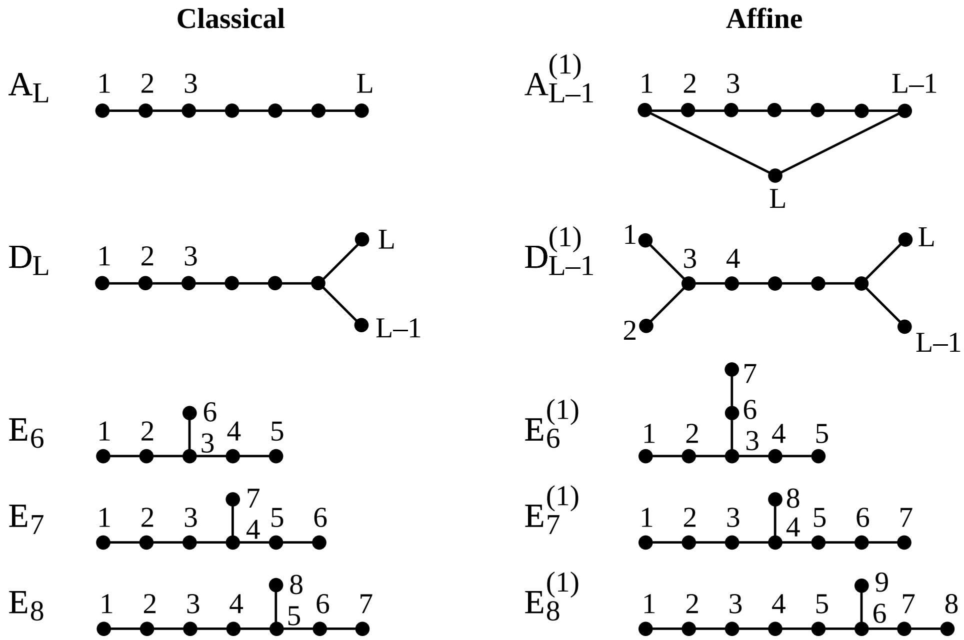

Fig. 1. Dynkin diagrams of the A–D–E classical and affine Lie algebras.

Table 1. The Coxeter number h and the Coxeter exponents m_j for A–D–E classical and affine Lie algebras. The eigenvalues of the adjacency matrices for the corresponding Dynkin diagrams are given by $T{=}2\cos(m_j\pi/h)$.

Lie Algebra	h	m_j
A_L	$L+1$	$1,2,3,\ldots,L$
D_L	$2L-2$	$L-1,1,3,5,\ldots,2L-3$
$E6$	12	$1,4,5,7,8,11$
$E7$	18	$1,5,7,9,11,13,17$
$E8$	30	$1,7,11,13,17,19,23,29$
$A^{(1)}_{L-1}$	L	$0,2,4,\ldots,2L-2$
$D^{(1)}_{L-1}$	$2(L-3)$	$0,2,4,\ldots,2(L-3),L-3,L-3$
$E^{(1)}_6$	6	$0,2,2,3,4,4,6$
$E^{(1)}_7$	12	$0,3,4,6,6,8,9,12$
$E^{(1)}_8$	30	$0,6,10,12,15,18,20,24,30$

number of the A–D–E model. However, the relation goes much deeper than this. In fact, many eigenvalues of the transfer matrices of the related six-vertex and A–D–E models are *exactly* in common for a *finite* system. In this paper, this relationship is exploited to derive determinantal functional equations satisfied by the row transfer matrix eigenvalues of the critical A–D–E lattice models. In turn, these functional equations are used to derive special functional equations, called inversion identity hierarchies, which can be solved [11] for the finite-size corrections, central charges and conformal weights of the critical A–D–E lattice models.

The allowed states of the A–D–E lattice models take L discrete values represented by the vertices of the classical and affine A–D–E Dynkin diagrams as shown in Figure 1. States at adjacent sites of the square lattice must be adjacent on the Dynkin diagram. The face weights of faces not satisfying this adjacency condition for each pair of adjacent sites are zero. The eigenvalues of the adjacency matrices of the Dynkin diagrams play a special role and their values are given by

$$T = 2\cos(m_j\pi/h) \tag{1.1}$$

where the number h and the exponents m_j are as shown in Table 1. Following Pasquier [12] we will refer to h as the Coxeter number and m_j as the Coxeter exponents.

The nonzero face weights of the classical A–D–E models are given by [3]

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = \frac{\sin(\lambda-u)}{\sin\lambda}\delta(a,c) + \frac{\sin u}{\sin\lambda}\sqrt{\frac{S_aS_c}{S_bS_d}}\,\delta(b,d) \tag{1.2}$$

where u is the spectral parameter, λ is the crossing parameter and S_a are the components of the Perron-Frobenius vector of the adjacency matrix. The nonzero face weights of the affine $A_{L-1}^{(1)}$ or critical cyclic solid-on-solid models [13,5,6] are

$$\begin{aligned} W\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) &= \frac{\sin(\lambda-u)}{\sin\lambda}\delta(a,c) + \frac{\sin u}{\sin\lambda}\delta(b,d) \\ &\quad + \left(1 - \frac{\sin(\lambda-u)}{\sin\lambda} - \frac{\sin u}{\sin\lambda}\right)\delta(a,c)\delta(b,d) \end{aligned} \tag{1.3}$$

where the values of the states are taken mod L and $\lambda = s\pi/L$. Similarly, the nonzero weights of the affine $D_{L-1}^{(1)}$ models are determined by [6]

$$\begin{aligned} W\left(\begin{matrix} a-1 & a \\ a & a+1 \end{matrix}\,\middle|\, u\right) &= \frac{\sin(\lambda-u)}{\sin\lambda}, \qquad a \neq 1,2,L-1,L \\ W\left(\begin{matrix} a & a-1 \\ a+1 & a \end{matrix}\,\middle|\, u\right) &= \epsilon_{a-1}^L\epsilon_{a+1}^L\,\frac{\sin u}{\sin\lambda} \qquad a \neq 1,2,L-1,L \\ W\left(\begin{matrix} a & a\pm1 \\ a\pm1 & a \end{matrix}\,\middle|\, u\right) &= 1, \qquad a, a\pm1 \neq 1,2,L-1,L \end{aligned} \tag{1.4b}$$

$$\begin{aligned}
W\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\!\!\left|\, u \right) &= \frac{1}{2}\left(1+\frac{\sin(\lambda-u)}{\sin\lambda}\right)\\
W\begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}\!\!\left|\, u \right) &= \frac{1}{2}\left(1-\frac{\sin(\lambda-u)}{\sin\lambda}\right)\\
W\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\!\!\left|\, u \right) &= 1-\frac{\sin u}{\sin\lambda}\\
W\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}\!\!\left|\, u \right) &= 1+\frac{\sin u}{\sin\lambda}
\end{aligned} \tag{1.4b}$$

and so on where $\lambda = \pi/(L-3)$ and

$$\epsilon_a^L = \begin{cases} 1/\sqrt{2}, & \text{if } a = 1, 2, L-1, L \\ 1, & \text{otherwise.} \end{cases} \tag{1.5}$$

The face weights of all the critical A–D–E models are invariant under symmetries of the Dynkin diagrams and satisfy the crossing symmetry

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = \sqrt{\frac{S_a S_c}{S_b S_d}}\; W\left(\begin{matrix} c & b \\ d & a \end{matrix}\,\middle|\, \lambda-u\right). \tag{1.6}$$

The elements of the A–D–E row transfer matrix $\boldsymbol{V}(u)$ are given by

$$\langle\sigma|\boldsymbol{V}(u)|\sigma'\rangle = \prod_{j=1}^{N} W\left(\begin{matrix} \sigma_j' & \sigma_{j+1}' \\ \sigma_j & \sigma_{j+1} \end{matrix}\,\middle|\, u\right) \tag{1.7}$$

where $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_N\}$ and $\sigma' = \{\sigma_1', \sigma_2', \ldots, \sigma_N'\}$ are the states of two consecutive periodic rows of N spins. For simplicity, we will always assume that N is even. Since the A–D–E face weights satisfy the Yang-Baxter equations, the row transfer matrices form a commuting family

$$\boldsymbol{V}(u)\boldsymbol{V}(v) = \boldsymbol{V}(v)\boldsymbol{V}(u) \tag{1.8}$$

which can be simultaneously diagonalized. We denote the eigenvalues of $\boldsymbol{V}(u)$ by $V(u)$.

2. Six-Vertex Model

Let $V(u)$ be an eigenvalue of the row transfer matrix $\boldsymbol{V}(u)$ of the six-vertex model with periodic boundary conditions and N faces in a row where N is even. Then it has been shown by Baxter [14,1] that the eigenvalue $V(u)$ satisfies the functional equation

$$V(u)Q(u) = \phi(u-\lambda)Q(u+\lambda) + \phi(u)Q(u-\lambda) \tag{2.1}$$

where $Q(u)$ is the corresponding eigenvalue of an auxiliary matrix family $\boldsymbol{Q}(u)$ which commutes with $\boldsymbol{V}(u)$, λ is the crossing parameter and

$$\phi(u) = \left(\frac{\sin u}{\sin\lambda}\right)^N. \tag{2.2}$$

Here $V(u)$, $Q(u)$ and $\phi(u)$ are entire and periodic functions

$$V(u+\pi)=\pm V(u),\qquad Q(u+\pi)=\pm Q(u),\qquad \phi(u+\pi)=\phi(u). \tag{2.3}$$

An immediate consequence of these functional equations is the inversion identity [15,16,17]

$$V(u)V(u+\lambda)=\phi(u-\lambda)\phi(u+\lambda)+\phi(u)V^2(u) \tag{2.4}$$

where

$$V^2(u)=Q(u-\lambda)Q(u+2\lambda)\sum_{n=-1}^{1}\frac{\phi(u+n\lambda)}{Q(u+n\lambda)Q(u+(n+1)\lambda)} \tag{2.5}$$

is itself entire. In the thermodynamic limit $N\to\infty$, the second term on the right side of (2.4) which is responsible for the finite-size corrections can be neglected and we obtain the inversion relation

$$V(u)V(u+\lambda)=\phi(u-\lambda)\phi(u+\lambda) \tag{2.6}$$

which is satisfied by all the eigenvalues. In particular, if $\kappa(u)$ is the Nth root of the largest eigenvalue of $\boldsymbol{V}(u)$ giving the partition function per site in the strip $0<\text{Re }u<\lambda$, then $\kappa(u)$ satisfies

$$\begin{aligned}\kappa(u)\kappa(u+\lambda)&=\frac{\sin(\lambda+u)\sin(\lambda-u)}{\sin^2\lambda}\\ \kappa(u)&=\kappa(\lambda-u)\end{aligned} \tag{2.7}$$

where the second relation is the crossing or rotation symmetry. These equations can be solved subject to appropriate analyticity assumptions [1] to yield the formula

$$\ln\kappa(u)=\int_{-\infty}^{\infty}\frac{\cosh(\pi-2\lambda)t\sinh ut\sinh(\lambda-u)t}{t\sinh\pi t\cosh\lambda t}\,dt. \tag{2.8}$$

This integral cannot be simplified in general but, if $\lambda=\pi/L$ where L is odd, the integral must simplify to give

$$\kappa(u)=\frac{\sin(u+\lambda)\sin(u+3\lambda)\ldots\sin(u+(L-2)\lambda)}{\sin\lambda\sin(u+2\lambda)\sin(u+4\lambda)\ldots\sin(u+(L-3)\lambda)} \tag{2.9}$$

since it is easily verified that this is the solution of the functional equations with the required analyticity.

The matrix $\boldsymbol{V}^2(u)$ appearing in the inversion identity can be identified as the row transfer matrix of the 1×2 fusion of the six-vertex model [19]. More generally, let $\boldsymbol{V}^q(u)$ be the row transfer matrix of the $1\times q$ fusion model obtained from the six-vertex model. Furthermore, let $V^q(u)$ be the eigenvalue of $\boldsymbol{V}^q(u)$ and define

$$V_n^q=V^q(u+n\lambda),\qquad f_n=\phi(u+n\lambda) \tag{2.10}$$

where $V_n^1 = V_n = V(u+n\lambda)$ and $V_n^0 = f_{n-1}$. Then it is possible to establish the fusion hierarchy [20]

$$V_0^q V_q^1 = f_q V_0^{q-1} + f_{q-1} V_0^{q+1} \quad q = 1, 2, \ldots \tag{2.11}$$

where the eigenvalues V_n^q are again entire and, from (2.1),

$$V_n^q = Q(u-\lambda)Q(u+q\lambda) \sum_{n=-1}^{q-1} \frac{\phi(u+n\lambda)}{Q(u+n\lambda)Q(u+(n+1)\lambda)} \tag{2.12}$$

For $q=1$, (2.11) reduces to the inversion identity with $V_0^0 = f_{-1}$.

Using the fusion hierarchy equations (2.11) and induction it can be shown [11] that

$$V_0^q V_1^q = f_{-1} f_q + V_0^{q+1} V_1^{q-1}, \quad q = 1, 2, \ldots \tag{2.13}$$

Lastly, if we define

$$t_n^q = \frac{V_{n+1}^{q-1} V_n^{q+1}}{f_{n-1} f_{n+q}} \tag{2.14}$$

then we obtain the simple functional equations

$$t_0^q t_1^q = (1 + t_1^{q-1})(1 + t_0^{q+1}), \quad q = 1, 2, \ldots \tag{2.15}$$

with $t_n^0 = 0$. Equations of this form are called [11] inversion identity hierarchies. Remarkably, these equations appear to be satisfied by all the A–D–E models. Such equations play a special role in the calculation of the finite-size corrections at criticality because t_0^1 gives the correction term in the inversion identity (2.4). For the classical A–D–E models this system is truncated to a finite system of equations with $t_0^{h-2} = 0$ where h is the Coxeter number. In the affine case, however, the fusion hierarchy (2.11) and the inversion identity hierarchy (2.15) yield an infinite hierarchy of equations for $q = 1, 2, \ldots$ It is interesting to observe that precisely such equations have also arisen in the work of Zamolodchikov [21,22] on the thermodynamic Bethe ansatz.

3. Critical Affine *A*–*D*–*E* Models

The six-vertex model is dual to an unrestricted solid-on-solid model with an infinite number of heights. If the crossing parameter $\lambda = s\pi/h$ is a rational fraction of π, however, the solid-on-solid model can be restricted to $h = L$ heights by identifying heights mod L. In this way the six-vertex model is simply related to the critical cyclic solid-on-solid (CSOS) models [13,5,6]. It follows from this relation that the row transfer matrices of the CSOS models satisfy the same functional equations [23] as the corresponding six-vertex model. So consider the rational six-vertex model with $\lambda = s\pi/h$ and suppose

$$Q(u + L\lambda) = rQ(u) \tag{3.1}$$

where $r=\pm 1$ is a quantum number of the six-vertex row transfer matrix. Then the functional equations (2.1) close and, by repeatedly incrementing u by λ, it is seen that the eigenvalues $V(u)$ satisfy

$$\begin{pmatrix} -V_0 & f_{h-1} & 0 & \dots & 0 & rf_0 \\ f_1 & -V_1 & f_0 & \dots & 0 & 0 \\ 0 & f_2 & -V_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -V_{h-2} & f_{h-3} \\ rf_{h-2} & 0 & 0 & \dots & f_{h-1} & -V_{h-1} \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ \vdots \\ Q_{h-2} \\ Q_{h-1} \end{pmatrix} = 0 \qquad (3.2)$$

where $Q_n=Q(u+n\lambda)$. Since the vector of Q functions is nonzero the determinant of the matrix must vanish, that is,

$$\begin{vmatrix} -V_0 & f_{h-1} & 0 & \dots & 0 & rf_0 \\ f_1 & -V_1 & f_0 & \dots & 0 & 0 \\ 0 & f_2 & -V_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -V_{h-2} & f_{h-3} \\ rf_{h-2} & 0 & 0 & \dots & f_{h-1} & -V_{h-1} \end{vmatrix} = 0. \qquad (3.3)$$

This determinantal equation has been obtained by Bazhanov and Reshetikhin [24] in studying the A_L RSOS models. Some simple functional equations of this form have also been discussed by Baxter [25].

If we take the braid limits Im $u\to\pm i\infty$ with

$$\lim_{\mathrm{Im} u\to\pm i\infty}\left(\frac{\sin\lambda}{\sin(u-\frac{\lambda}{2})}\right)^N V(u)=T \qquad (3.4)$$

then the determinantal equation (3.3) reduces to

$$\begin{vmatrix} -T & 1 & 0 & \dots & 0 & r \\ 1 & -T & 1 & \dots & 0 & 0 \\ 0 & 1 & -T & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -T & 1 \\ r & 0 & 0 & \dots & 1 & -T \end{vmatrix} = 0 \qquad (3.5)$$

so the asymptotic values T of the eigenvalues $V(u)$ are given by

$$T=2\cos(m_j/h) \qquad (3.6)$$

where

$$m_j=\begin{cases} 0,2,4,\dots,2h-2, & r=1 \\ 1,3,5,\dots,2h-1, & r=-1 \end{cases} \qquad (3.7)$$

are the even and odd Coxeter exponents.

3.1. CSOS models: $A_{L-1}^{(1)}$, $h = L$

It turns out that the determinantal functional equations (3.3), with $r = 1$, are indeed satisfied by the eigenvalues $V(u)$ of the CSOS row transfer matrices. The form of the determinant can be simplified in the case when $\lambda = \pi/h$ and $h = L$ is odd. In this case we define dimensionless transfer matrices by

$$T(u) = \frac{V(u)}{\kappa(u)^N}, \qquad T_n = T(u + n\lambda). \tag{3.8}$$

It then follows, after multiplying rows and columns by suitable factors, that the determinantal equation simplifies to

$$\begin{vmatrix} -T_0 & 1 & 0 & \dots & 0 & r \\ 1 & -T_1 & 1 & \dots & 0 & 0 \\ 0 & 1 & -T_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -T_{h-2} & 1 \\ r & 0 & 0 & \dots & 1 & -T_{h-1} \end{vmatrix} = 0. \tag{3.9}$$

For example, for $h = L = 3$, $\lambda = \pi/3$ and $r = 1$ this functional equation becomes

$$T_0 T_1 T_2 = 2 + T_0 + T_1 + T_2. \tag{3.10}$$

In this case the CSOS model corresponds to a three colouring problem. In general, if $\lambda = s\pi/h$, we obtain a polynomial functional equation of degree h in $T(u)$.

The determinantal equation (3.3) implies the fusion hierarchy (2.11) and hence the inversion identity hierarchy (2.15). To see how this works, consider the three colouring problem with $L = 3$ and $r = 1$. If we divide by V_2, the cubic determinantal equation can be written as

$$V_0 V_1 = f_1 f_2 + f_0 V_0^2 \tag{3.11}$$

where

$$V_0^2 = (2 f_1 f_2 + f_2 V_0 + f_1 V_1)/V_2 = (V_0 V_1 - f_1 f_2)/f_0. \tag{3.12}$$

Despite appearances, V_0^2 is entire. Using $V_0^0 = f_2$, we have obtained the first equation of the fusion hierarchy with V_0^2 identified as the transfer matrix of the 1×2 fusion. Continuing in this way we find that

$$\begin{aligned} V_0^2 V_2 &= f_2 V_0 + f_1 V_0^3 \\ V_0^3 V_3 &= f_0 V_0^2 + f_2 V_0^4 \\ V_0^4 V_4 &= f_1 V_0^3 + f_0 V_0^5 \end{aligned} \tag{3.13}$$

and so on where

$$\begin{aligned} V_0^3 &= 2 f_2 + V_1 \\ V_0^4 &= 2 V_0 + f_1 \\ V_0^5 &= 2(V_0 V_1 - f_1 f_2)/f_0. \end{aligned} \tag{3.14}$$

This process can be continued indefinitely and leads to the identifications

$$\begin{aligned} t_0^1 &= (V_0V_1 - f_1f_2)/f_1f_2 \\ t_0^2 &= (2f_2 + V_1)V_1/(f_2)^2 \\ t_0^3 &= (2V_0 + f_1)(V_1V_2 - f_0f_2)/f_0f_1f_2 \end{aligned} \tag{3.15}$$

etc. with no closure.

3.2. $D_{L-1}^{(1)}$ models: $h = 2(L-3)$

The crossing parameter of the $D_{L-1}^{(1)}$ models is $\lambda = 2\pi/h = \pi/(L-3)$ where $h = 2(L-3)$. The eigenvalues of the row transfer matrices of these models satisfy the same determinantal equation (3.3) as the CSOS models subject to the extra conditions

$$V_{L-3} = rV_0, \qquad f_{L-3} = f_0. \tag{3.16}$$

The size of this determinant can be reduced by identifying the states a and $L+1-a$. In this reduced representation the eigenvalues must satisfy one of the equations

$$\begin{vmatrix} \mp V_0 & f_{h'-1} & 0 & \dots & 0 & f_0 \\ f_1 & \mp V_1 & f_0 & \dots & 0 & 0 \\ 0 & f_2 & \mp V_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mp V_{h'-2} & f_{h-3} \\ rf_{h'-2} & 0 & 0 & \dots & f_{h'-1} & \mp V_{h'-1} \end{vmatrix} = 0. \tag{3.17}$$

where $h' = h/2$. For $h = L = 6$ and $\lambda = \pi/3$, corresponding to magnetic hard squares [26], this yields the previously derived equation

$$T_0T_1T_2 = \pm(1+r) + T_0 + rT_1 + T_2 \tag{3.18}$$

where $T_n = T(u + n\pi/3)$ is the dimensionless transfer matrix (3.8) with h replaced by h' in the crossing parameter. Similarly, for $L = 8$, $h = 10$ and $h' = 5$, we find

$$\begin{aligned} T_0T_1T_2T_3T_4 &= 1 + r - T_0 - rT_1 - T_2 - rT_3 - T_4 \\ &+ T_0T_1T_2 + T_0T_1T_4 + rT_1T_2T_3 + T_0T_3T_4 + T_2T_3T_4. \end{aligned} \tag{3.19}$$

For h' even, we cannot introduce a dimensionless transfer matrix. So for $L = 7$ and $h = 8$ the best we can do is

$$\begin{aligned} V_0V_1V_2V_3 &= (1+r)f_0f_1f_2f_3 - f_1^2f_3^2 - rf_0^2f_2^2 \\ &+ f_1f_3V_0V_1 + f_0f_2V_0V_3 + f_1f_3V_2V_3 + rf_0f_2V_1V_2 \end{aligned} \tag{3.20}$$

and so on.

There is as yet no known trigonometric solution to the Yang-Baxter equations for the IRF models whose adjacency conditions are given by the Dynkin diagrams of $E_6^{(1)}$,$E_7^{(1)}$ and $E_8^{(1)}$ so we do not consider these models here.

4. Critical Classical *A–D–E* Models

The row transfer matrix eigenvalues of the restricted solid-on-solid (RSOS) models or classical A–D–E models also satisfy the determinantal functional equation (3.3) at criticality with crossing parameter $\lambda = \pi/h$ where $h = L+1$ is the Coxeter number. More specifically, these models satisfy the much stronger condition that all $(h-1)\times(h-1)$ minors vanish

$$\underset{(h-1)\times(h-1)}{\text{Minors}}\begin{pmatrix} -V_0 & f_{h-1} & 0 & \dots & 0 & rf_0 \\ f_1 & -V_1 & f_0 & \dots & 0 & 0 \\ 0 & f_2 & -V_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -V_{h-2} & f_{h-3} \\ rf_{h-2} & 0 & 0 & \dots & f_{h-1} & -V_{h-1} \end{pmatrix} = 0. \tag{4.1}$$

As an immediate consequence, all of the eigenvalues of the classical A–D–E models satisfy

$$\begin{vmatrix} -V_1 & f_0 & 0 & \dots & 0 & 0 \\ f_2 & -V_2 & f_1 & \dots & 0 & 0 \\ 0 & f_3 & -V_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -V_{h-2} & f_{h-3} \\ 0 & 0 & 0 & \dots & f_{h-1} & -V_{h-1} \end{vmatrix} = 0. \tag{4.2}$$

Here the asymptotic values (3.6) in the braid limit are given by the Coxeter exponents

$$m_j = 1, 2, 3, \dots, h-1. \tag{4.3}$$

The determinantal equation (4.2) is satisfied by all of the classical A–D–E models. Again the fusion hierarchy and inversion identity hierarchy can be deduced from this determinantal equation only this time with the closure conditions $V_0^{h-1} = 0$ and $t_0^{h-2} = 0$ where h is the Coxeter number. Some details will be given below in the case of D_4 corresponding to the 3-state Potts model.

4.1. A_L: L even, $h = L+1$ odd

Let us consider the critical Andrews-Baxter-Forrester models [2] or A_L models, first for L even and then for L odd. The Coxeter number is $h = L+1$. In the case that L is even, the determinantal equation (4.2) for the A_L models takes the simple form

$$\begin{vmatrix} -T_1 & 1 & 0 & \dots & 0 & 0 \\ 1 & -T_2 & 1 & \dots & 0 & 0 \\ 0 & 1 & -T_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -T_{h-2} & 1 \\ 0 & 0 & 0 & \dots & 1 & -T_{h-1} \end{vmatrix} = 0 \tag{4.4}$$

where T_n is an eigenvalue of the dimensionless transfer matrix (3.8). Lower degree polynomial equations for these eigenvalues can then be obtained by setting other $(h-1)\times(h-1)$ minors to zero. In particular, removing the row $(h+1)/2$ and the column h in (4.1) or (3.9) yields the functional equations

$$T_0T_1 = 1 - rT_3 \tag{4.5}$$

$$T_0T_1T_2 = 1 + rT_0 + rT_2 - T_4T_5 \tag{4.6}$$

$$T_0T_1T_2T_3 = -1 + rT_5 + rT_7 + T_0T_1 + T_0T_3 + T_2T_3 - rT_5T_6T_7 \tag{4.7}$$

for $h = 5, 7, 9$ and so on where the eigenvalues, whose asymptotics are given by even (odd) Coxeter exponents, satisfy the equations with $r = 1$ $(r = -1)$ respectively. The equations with $r = -1$, corresponding to eigenvalues with odd Coxeter exponents, are in fact satisfied by the reduced lattice-gas representation of the RSOS models with adjacency condition given by tadpole diagrams. For L even, these are the orbifold duals of the A_L models in the sense of Fendley and Ginsparg [27]. For example (4.5) is the functional equation satisfied by the generalized hard hexagon models as obtained by Baxter and Pearce [28]. In this case, Andrews, Baxter and Forrester [2] observed that the 5×5 matrix in (3.2) is of rank 3 and so obtained the functional equation (4.5).

4.2. A_L: L odd, $h = L + 1$ even

In the case when L is odd the eigenvalues still fall into sectors with even $(r = 1)$ or odd $(r = -1)$ Coxeter exponents. However, this time equating the $(h-1)\times(h-1)$ minors (4.1) to zero gives two independent equations corresponding to removing either the $h/2 + 1$ row and h column or the $h/2$ row and h column. Hence, for $L = 5$ and $h = 6$, the eigenvalues must satisfy both of the equations

$$V_0V_1V_2 = f_0f_2V_0 + f_1f_5V_2 - rf_0f_1V_4 \tag{4.8}$$

$$f_3V_0V_1 = f_1f_3f_5 + rf_0f_2f_4 - rf_0V_3V_4. \tag{4.9}$$

Similarly, for $L = 7$ and $h = 8$, we find

$$\begin{aligned} f_5V_0V_1V_2V_3 &= rf_0f_1f_2f_4f_6 - f_1^2f_3f_5f_7 \\ &+ f_1f_3f_5V_0V_1 + f_0f_2f_5V_0V_3 + f_1f_5f_7V_2V_3 - rf_0f_1f_2V_5V_6 \end{aligned} \tag{4.10}$$

$$f_4f_5V_0V_1V_2 = f_0f_2f_4f_5V_0 + f_1f_4f_5f_7V_2 + rf_0f_1f_4f_6V_4 + rf_0f_1f_3f_5V_6 - rf_0f_1V_4V_5V_6 \tag{4.11}$$

and so on.

4.3. D_L and $E_{6,7,8}$

The D_L models with $h = 2L-2$ have eigenvalues whose asymptotics have odd Coxeter exponents. Accordingly, it is found that the row transfer matrix eigenvalues of these models satisfy the same equations as the A_{2L-3} models but with $r=-1$. This is not surprising since the A_{2L-3} and D_L models are orbifold duals of each other [27]. For illustration, let us consider the fusion equations for $L=4$ and $h=6$ in some detail. The D_4 model is the critical 3-state Potts model! In this case, the equations (4.8) and (4.9) with $r=-1$ are satisfied by the row transfer matrix eigenvalues of the critical 3-state Potts model. Precisely the same equations have been obtained recently by Albertini [29] and studied by Albertini, Dasmahapatra and McCoy in these proceedings [30]. From these equations, using $V_0^0 = f_5$, we obtain the fusion equations

$$\begin{aligned} V_0 V_1 &= f_1 V_0^0 + f_0 V_0^2, \quad V_0^2 V_2 = f_2 V_0 + f_1 V_0^3 \\ V_0^3 V_3 &= f_3 V_0^2 + f_2 V_0^4, \quad V_0^4 V_4 = f_4 V_0^3 + f_3 V_0^5 \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} V_0^2 &= (V_3V_4 - f_2 f_4)/f_3 = (f_2 V_0 + f_1 V_4)/V_2 \\ V_0^3 &= V_4, \quad V_0^4 = f_4, \quad V_0^5 = 0 \end{aligned} \tag{4.13}$$

and each V_0^q is entire. The inversion identity hierarchy follows from the fusion equations with the truncation $t_0^4 = 0$. The matrices corresponding to the eigenvalues V_0^q can in fact be taken as *defining* the $1\times q$ fusion for the D_L models. Explicit expressions can be written down for the face weights of these fusion models but they are complicated and will not be given here. The cases with larger values of L are similar. In particular, for larger values of L, the inversion identity hierarchy truncates with $t_0^{L-4} = 0$.

The row transfer matrix eigenvalues of Pasquier's E_6, E_7 and E_8 models also appear to satisfy the determinantal equation (3.3) with Coxeter numbers $h = 12$, 18 and 30 respectively. The fusion and inversion identity hierarchies are therefore also expected to hold in these cases.

5. Conclusion

It has been argued that the row transfer matrix eigenvalues of critical A–D–E lattice models satisfy certain determinantal functional equations and, consequently, also the fusion and inversion identity hierarchies (2.11) and (2.15). The inversion identity hierarchy is finite in the classical case and infinite in the affine case. Although only the critical case has been considered in this paper, these functional equations also extend to the off-critical elliptic A and D lattice models. At criticality, these equations provide the key to calculating the finite-size corrections, central charges and conformal weights. Indeed, for the RSOS or A_L models of Andrews, Baxter and Forrester, these equations have been completely solved for the finite-size corrections in recent work with Andreas Klümper [11,31]. In particular, we were able to calculate directly the central charges

$$c = 1 - \frac{6}{h(h-1)}, \quad h = 4, 5, 6, \ldots \tag{5.1}$$

and conformal weights Δ, $\overline{\Delta}$ given by the Kac formula

$$\Delta = \frac{[(ht-(h-1)s]-1}{4h(h-1)}, \quad 1 \le t \le h-2, 1 \le s \le h-1, s \le t. \tag{5.2}$$

A crucial input into these calculations is the asymptotic values

$$T = 2\cos(m_j/h) \tag{5.3}$$

of the eigenvalues as $\mathrm{Im}\ u \to \pm i\infty$ in the physical analyticity strip where h is the Coxeter number and m_j are the Coxeter exponents. For the A_L models, it turns out that the limits in the upper and lower half planes are equal. As a consequence, we find that $\Delta = \overline{\Delta}$ and only spinless operators appear in the modular invariant partition function of the critical A_L models. For the D_L and $E_{6,7,8}$ models, on the other hand, the asymptotic values which occur are a subset of the allowed asymptotic values of the A_L models, however, the two limits need not be equal. Consequently, the conformal weights Δ for these models will be a subset of the Kac table (5.2) and, since Δ need not be equal to $\overline{\Delta}$, some operators with spin will occur as dictated by the nondiagonal modular invariant partition functions.

It is hoped that similar conclusions will be able to be drawn in the future for the affine *A–D–E* models. The infinite inversion identity hierarchy relevant to these models is presently under investigation. Similarly, it would be interesting to extend the results for the classical *A–D–E* models to the general case $\lambda = s\pi/h$. Such lattice models have been considered by Forrester and Baxter [32] and correspond to nonunitary conformal field theories. Finally, it is perhaps worthwhile to point out that there exist fusion models corresponding to each of the fusion hierarchies constructed in this paper. These solvable lattice models are of interest in their own right. For A_L these fused models are already known [33,34] and the fused $A^{(1)}_{L-1}$ models can be constructed similarly [35]. In particular, the fused face weights in these cases can be written down quite explicitly and involve modified adjacency conditions. On the other hand, it appears that the fused D and E models are new. Expressions for the face weights of the fused D and E models can also be given, but since they are cumbersome and not needed here they are not given in this paper. Unlike the fused A models, however, the fused D and E models are not described by simple adjacency conditions. Rather, in these cases, it is necessary to introduce extra states associated with the edges of the faces of the fused models. Again it is hoped that these matters will be taken up in more detail elsewhere.

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