

THE 3-STATE POTTS MODEL, HARD HEXAGONS AND UNIVERSALITY

Paul A. Pearce¹

*Mathematics Department, University of Melbourne,
Parkville, Victoria 3052, Australia*

Abstract

The concept of universality class is central in the modern theory of critical phenomena. In essence, the universality hypothesis asserts that two critical systems will possess the same critical exponents if they have the same dimensionality and their order parameters exhibit the same symmetry. In the case of the 3-state Potts and hard hexagon models a very strong form of universality can be demonstrated by direct calculations on the lattice models, namely, that the operator content and modular invariant partition functions are identical.

1 Introduction

The universality hypothesis is a key ingredient in classifying the critical behaviours of statistical systems. In its simplest form Kadanoff [1] explained it thus:

All phase transition problems can be divided into a number of different classes depending upon the dimensionality of the system and the symmetries of the ordered state. Within each class all phase transitions have identical behaviour in the critical region. Only the names of the thermodynamic variables are changed.

Since the critical behaviour is characterized by critical exponents, two systems belonging to the same universality class will possess the same set of critical exponents $\alpha, \beta, \gamma, \delta, \dots$. This general principle has found wide application in the study of critical phenomena. Among other things it means that the correct critical behaviour can be obtained by studying simplified models provided that the essential physics, that is, the dimensionality and symmetries of the system are preserved. In two dimensions, however, it is expected that the modular invariant partition functions of critical systems should also be universal. This allows us to make the following stronger and more precise assertion:

In two dimensions, two systems lie in the same universality class if and only if they have the same modular invariant partition function.

¹Email: pap@mundoe.maths.mu.oz.au

In the last few years new methods have emerged to allow the direct analytic calculation of the operator content and modular invariant partition functions of critical lattice models. It is therefore possible to show directly that two exactly solvable models lie in the same universality class. In this paper we describe how this program is carried out for the 3-state Potts and hard hexagon models. It was first argued by Alexander [2] in 1975 that these models are in the same universality class on simple symmetry arguments. We discuss the 3-state Potts and hard hexagon models because the current results are most complete for these models. It is expected that similar demonstrations will be possible for many other models in the near future.

The layout of the paper is as follows. The 3-state Potts and hard hexagon models are described in Sections 2 and 3. In these sections we also review the progress in studying the critical behaviour of these models prior to the advent of conformal and modular invariance. In Section 4 we describe the predictions of conformal and modular invariance and how these have been confirmed by direct calculations starting with the critical lattice models.

2 The 3-State Potts Model

The partition function of the anisotropic Q -state Potts model on the square lattice [3] is

$$Z_N = \sum_{\text{spins}} \exp \left[(J/kT) \sum_{\langle ij \rangle} \delta(\sigma_i, \sigma_j) + (K/kT) \sum'_{\langle ij \rangle} \delta(\sigma_i, \sigma_j) \right] \quad (2.1)$$

where the sums in the exponential are over nearest-neighbour horizontal and vertical pairs of sites $\langle ij \rangle$, J and K are positive interaction strengths, k is Boltzmann's constant, T is temperature and the outer sum is over all configurations of the N spins $\sigma_i = 1, 2, \dots, Q$. The Kronecker delta is given by

$$\delta(\sigma, \sigma') = \begin{cases} 1, & \text{if } \sigma = \sigma' \\ 0, & \text{if } \sigma \neq \sigma'. \end{cases} \quad (2.2)$$

The fundamental problem to be solved is to obtain the statistical properties of this model in the thermodynamic limit $N \rightarrow \infty$.

The $Q = 2$ Potts model is the Ising model. It was solved in 1944 by Onsager [4]. The Potts model with $Q > 2$, however, has not been solved in general and it remains one of the outstanding problems in statistical physics. Nevertheless, in 1973, Baxter [5] was able to solve the Q -state Potts model precisely at the critical temperature given by

$$[\exp(J/kT) - 1][\exp(K/kT) - 1] = Q. \quad (2.3)$$

In particular, he showed that the transition is critical for $Q \leq 4$ and first-order for $Q > 4$.

The 3-state Potts model, which is of primary interest here, exhibits a conventional order-disorder transition. To make the symmetries manifest it is convenient to define a complex order parameter

$$R = \rho_1 + \omega \rho_2 + \omega^2 \rho_3 \quad (2.4)$$

where $\omega = \exp(2\pi i/3)$ and

$$\rho_k = \langle \delta(\sigma_i, k) \rangle = \text{fraction of spins in state } k = 1, 2, \dots, Q. \quad (2.5)$$

Here the angle brackets denote canonical ensemble averages in the thermodynamic limit. Since $\rho_1 + \rho_2 + \rho_3 = 1$, the order parameter is restricted to the triangular shaded region shown in Figure 1. Clearly, since the labelling of any two states can be interchanged,

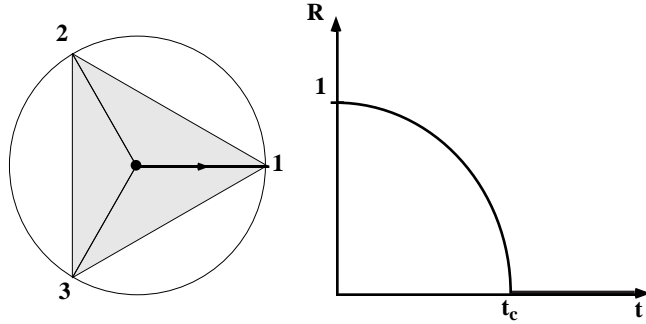


Figure 1: The complex order parameter R takes values in the triangular shaded region of the complex plane shown on the left. The variation in the magnitude of the order parameter along the positive real axis is shown on the right. It vanishes at the critical point with a power-law behaviour. The temperature-like variable $t = T$ for the 3-state Potts model and $t = 1/z$, where z is the activity, for the hard hexagon model.

the symmetry of the order parameter is given by the symmetric group S_3 . At high temperatures the symmetry between the 3 states is maintained, the fractions of each spin are equal and so the order parameter vanishes. As the temperature is lowered through the critical temperature T_c , however, this symmetry is spontaneously broken, the spins prefer to align in one of the three states (say state 1) and the order parameter assumes nonzero values as shown in Figure 1. As the critical point is approached from the ordered phase the order parameter vanishes with a power-law behaviour

$$R = (3\rho - 1)/2 \sim (T - T_c)^\beta \quad (2.6)$$

where β is a critical exponent characterizing the phase transition.

Other critical exponents of interest are defined by

$$C \sim |T - T_c|^{-\alpha}, \quad h = 0; \quad R \sim h^{1/\delta}, \quad T = T_c \quad (2.7)$$

where C is the specific heat and h is a symmetry breaking field. The first predictions for the value of these exponents for the general Q -state Potts model were obtained by renormalization arguments involving mapping the Potts model onto the critical line of the eight-vertex model which was solved earlier by Baxter [6] in 1971. Specifically, in 1979 and 1980, den Nijs [7], Nienhuis, Riedel and Schick [8] and Pearson [9] conjectured that

$$\alpha = \frac{2 - 4y}{3 - 3y}, \quad \beta = \frac{1 + y}{12} \quad (2.8)$$

where the renormalization group exponent y is given by

$$y = \frac{2}{\pi} \arccos \frac{\sqrt{Q}}{2}. \quad (2.9)$$

From the scaling relation $2 - \alpha = \beta(1 + \delta)$ it follows that

$$\delta = \frac{15 - 8y + y^2}{1 - y^2}. \quad (2.10)$$

Hence, for the 3-state Potts model, the conjecture yields $y = 1/3$ and

$$(\alpha, \beta, \delta) = (1/3, 1/9, 14). \quad (2.11)$$

3 The Hard Hexagon Model

The Potts model was proposed on theoretical grounds by generalizing the spin-reversal symmetry of the Ising model. By contrast the hard hexagon model, which appeared in early numerical work by Runnels and Coombs [10] and Gaunt [11], is strongly associated with a physical system realizable in a laboratory. This system is Helium adsorbed on a graphite surface. The graphite substrate is composed of hexagonal cells formed by six carbon atoms with an interatom distance of 2.46 Å. Energetically, the adsorbed Helium atoms prefer to sit in the potential well at the center of the hexagonal cells. The diameter of the Helium atom however is 2.56 Å which precludes the simultaneous occupation of neighbouring cells by excluded volume effects as shown in Figure 2. Some beautiful

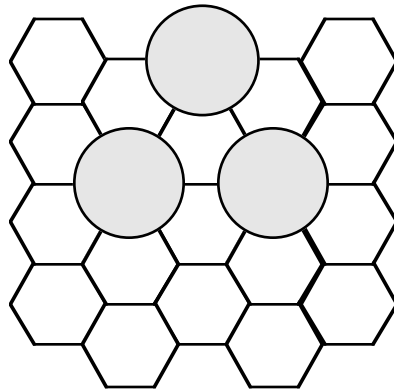


Figure 2: Hexagonal cells of graphite showing some allowed positions of Helium atoms.

experiments carried out by Bretz [12] in 1977 indicate that this system undergoes a phase transition. Indeed, Bretz took precise measurements of the specific heat as the temperature, or equivalently, the activity z is varied and obtained a symmetric power-law divergence at the critical point

$$C \sim |z - z_c|^\alpha, \quad \alpha \approx 0.36 \quad (3.1)$$

with critical exponent α close to $1/3$. Of course one does not actually see divergences experimentally. Rather, it is the presence of dramatic peaks in the specific heat that are the hallmarks of a second order transition.

The centers of the hexagonal graphite cells form a triangular lattice so the physical system of Helium adsorbed on graphite is naturally described by a triangular lattice gas with nearest neighbour exclusion. This model is equivalent to placing non-overlapping

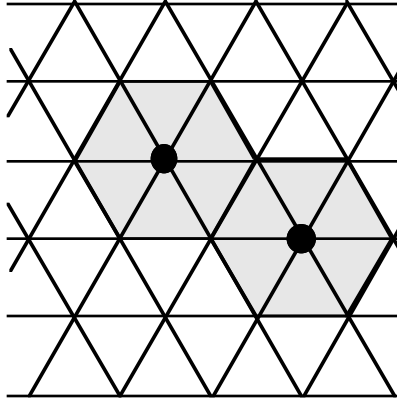


Figure 3: The hard core triangular lattice gas showing non-overlapping hexagonal tiles.

hexagonal tiles on the triangular lattice as shown in Figure 3. Hence the name hard hexagons. In the close-packed limit the particles fully occupy one of the three independent sublattices of the triangular lattice labelled by $k = 1, 2, 3$.

The partition function of hard hexagons is

$$Z_N = \sum_{\text{spins } \langle ij \rangle} \prod z^{(\sigma_i + \sigma_j)/6} (1 - \sigma_i \sigma_j) \quad (3.2)$$

where the product is over all bonds $\langle ij \rangle$ of the triangular lattice, z is the activity and the sum is over all configurations of the N spins or occupation numbers $\sigma_i = 0, 1$. The exponent of z arises because the activity is shared out between the 6 bonds incident at each site. The remaining term ensures that neighbouring sites are not occupied simultaneously by excluding such terms from the sum. Again the fundamental problem is to obtain the statistical properties of this model such as the sublattice densities

$$\rho_k = \langle \sigma_k \rangle = \text{fraction of spins sitting on sublattice } k = 1, 2, 3 \quad (3.3)$$

in the thermodynamic limit $N \rightarrow \infty$.

Remarkably, the hard hexagon model was solved exactly by Baxter [13] in 1980 for arbitrary activity z . It undergoes a conventional order-disorder transition at

$$z_c = \left(\frac{1 + \sqrt{5}}{2} \right)^5. \quad (3.4)$$

Analogously to the 3-state Potts model, we can define a complex order parameter R by (2.4). Since the value of the density ρ satisfies

$$\rho = (\rho_1 + \rho_2 + \rho_3)/3 \leq 1/3 \quad (3.5)$$

the order parameter is again restricted to the shaded region in Figure 1 and exhibits S_3 symmetry. For small values of the activity there are only a few particles scattered randomly so the sublattice symmetry is preserved and $R = 0$. However, for larger values of the activity and higher densities, one of the three sublattices must be preferentially occupied so the sublattice symmetry is spontaneously broken. Assuming sublattice $k = 1$

is preferentially occupied, the behaviour of the order parameter as z varies is as shown in Figure 1 with

$$R = \rho_1 - \rho_2 \sim (z - z_c)^\beta, \quad \beta = 1/9. \quad (3.6)$$

Notice that the symmetry between sublattices $k = 2$ and 3 is not broken.

The exact critical exponent $\beta = 1/9$ was obtained by Baxter using corner transfer matrices. The precise result for the order parameter R is tantalizing and representative of Baxter's other results so it is worth stating here. Baxter showed that the order parameter R is given implicitly as a function of z by

$$R = \frac{Q(x)Q(x^5)}{Q(x^3)^2}, \quad z = \frac{G(x)^5}{xH(x)^5} \quad (3.7)$$

where $0 \leq x \leq 1$ for $z \geq z_c$ and the elliptic and Rogers-Ramanujan functions are

$$Q(x) = \prod_{n=1}^{\infty} (1 - x^n) \quad (3.8)$$

$$G(x) = \prod_{n=1}^{\infty} \frac{1}{(1 - x^{5n-4})(1 - x^{5n-1})}, \quad H(x) = \prod_{n=1}^{\infty} \frac{1}{(1 - x^{5n-3})(1 - x^{5n-2})} \quad (3.9)$$

Although it is far from obvious, because $x \rightarrow 1$ is a subtle limit, it indeed follows that $\beta = 1/9$. Similar results in elliptology yield

$$C \sim |z - z_c|^\alpha, \quad \alpha = \frac{1}{3}. \quad (3.10)$$

If we assume scaling, we therefore obtain

$$(\alpha, \beta, \delta) = (1/3, 1/9, 14) \quad (3.11)$$

as conjectured for the 3-state Potts model.

Actually, Rodney Baxter did not solve the hard hexagon model directly. Instead he solved a generalized hard hexagon model which is a model of anisotropically interacting hard squares. The anisotropy introduces an additional parameter u which is called the spectral parameter. The original hard hexagon model is obtained in the limit $u \rightarrow -\pi/5$. In 1982, Baxter and Pearce [14] showed that the normalized row transfer matrix $T(u)$ of the generalized hard hexagon model satisfies the remarkably simple functional equation

$$T(u)T(u + \lambda) = I + T(u - 2\lambda) \quad (3.12)$$

where $\lambda = \pi/5$. Since $T(u)$ is a commuting family of matrices this equation can be solved for the eigenvalues of $T(u)$ and hence the correlation length ξ and interfacial tension σ . The associated critical exponents are given by

$$\xi \sim (z - z_c)^{-\nu}, \quad \sigma \sim (z - z_c)^{-\mu} \quad (3.13)$$

where it was found that $\nu = \mu = 5/6$. This confirms the hyperscaling relation

$$2 - \alpha = d\nu \quad (3.14)$$

where $d = 2$ is the lattice dimension. Notice that the functional equation (3.12) is an exact equation for a finite system with N faces in a row. This is significant because this equation contains additional information on finite-size corrections.

4 Operator Content and Modular Invariant Partition Functions

The study of the critical behaviour of two-dimensional statistical systems changed dramatically in 1984 when it was realized [15] that the critical behaviour of such systems could be described by the corresponding conformal field theory. Each conformal field theory is characterized by a central charge c . Moreover, if $c < 1$, the central charge is restricted to the discrete unitary minimal series

$$c = 1 - \frac{6}{h(h-1)}, \quad h = 4, 5, 6, \dots \quad (4.1)$$

The critical exponents are then related to the scaling dimensions

$$x = \Delta + \bar{\Delta} \quad (4.2)$$

where the conformal weights Δ and $\bar{\Delta}$ take values in the associated Kac table. The set of values $(\Delta, \bar{\Delta})$ which actually appear in a theory is called the operator content.

Dotsenko [16] showed that the 3-state Potts model is to be identified with the unitary minimal model with $h = 6$ and $c = 4/5$. The Kac table of conformal weights for $h = 6$ is shown in Table 1. In particular, Dotsenko showed that the thermal and magnetic scaling

s					
5	3	7/5	2/5	0	
4	13/8	21/40	1/40	1/8	
3	2/3	1/15	1/15	2/3	
2	1/8	1/40	21/40	13/8	
1	0	2/5	7/5	3	
	1	2	3	4	r

Table 1: Kac table of conformal weights $\Delta_{r,s}$ for $h = 6$. Only the conformal weights in odd rows $s = 1, 3, 5$ appear in the 3-state Potts and hard hexagon models.

dimensions are given by

$$x_\epsilon = \frac{2}{5} + \frac{2}{5} = \frac{4}{5}, \quad x_\sigma = \frac{1}{15} + \frac{1}{15} = \frac{2}{15}. \quad (4.3)$$

The values of the associated critical exponents are thus obtained from the relations

$$x_\epsilon = 2 - \frac{2}{2-\alpha}, \quad x_\sigma = \frac{2\beta}{2-\alpha} \quad (4.4)$$

giving the now familiar results $\alpha = 1/3$ and $\beta = 1/9$.

In addition to being conformally invariant, a critical lattice model on an $\ell \times \ell'$ periodic lattice or torus exhibits modular invariance [17] in the thermodynamic limit. Specifically, the partition function on a finite lattice can be written as

$$Z_{\ell, \ell'} \sim \exp(-\ell \ell' f) Z(q), \quad \ell, \ell' \rightarrow \infty \quad (4.5)$$

where f is the bulk free energy and $Z(q)$ is a *universal term* describing the leading finite-size corrections in the limit $\ell, \ell' \rightarrow \infty$ with the aspect ratio $\delta = \ell'/\ell$ fixed. The argument $q = \exp(-2\pi\delta)$ is the modular parameter and the universal term $Z(q)$ is called the modular invariant partition function. As a consequence of modular invariance, Cardy [17] predicted that the modular invariant partition function of the 3-state Potts model should be given by

$$Z(q) = |\chi_0(q) + \chi_3(q)|^2 + |\chi_{2/5}(q) + \chi_{7/5}(q)|^2 + 2|\chi_{1/15}(q)|^2 + 2|\chi_{2/3}(q)|^2 \quad (4.6)$$

where the known functions

$$\chi_{\Delta}(q) = q^{\Delta - c/24} \sum_{k=0}^{\infty} d_{\Delta}(k) q^k \quad (4.7)$$

are Virasoro characters.

If $\ell' = 1$ and $\ell = N$ then the partition function is just the trace of the row transfer matrix T

$$Z_{N,1} = \text{Tr } T = \sum_n \Lambda_n, \quad \delta = 1/N, \quad q = \exp(-2\pi/N) \quad (4.8)$$

and the modular invariant partition function completely predicts the finite-size $1/N$ corrections to the eigenenergies $E_n = -\log \Lambda_n$. More precisely, looking just at the $1/N$ corrections, you should see a tower of energy levels in the spectrum with spacing equal to $2\pi/N$ above each of the primary levels with $x = 0, 2/15, 4/5, 4/3$. As you go up the towers the degeneracies are given by the coefficients in the series expansions of the Virasoro characters in the four terms in the modular invariant partition function.

The operator content

$$\{(\Delta, \bar{\Delta})\} = \{(0, 0), 2(1/15, 1/15), (2/5, 2/5), 2(2/3, 2/3), (2/5, 7/5), (7/5, 2/5), (7/5, 7/5), (0, 3), (3, 0), (3, 3)\} \quad (4.9)$$

was confirmed numerically for relevant operators in 1988 by Kim [18]. The same operator content was obtained analytically in 1991 by Pearce and Klümper [19, 20]. This was achieved by solving the functional equation (3.12) for the finite-size $1/N$ corrections to the eigenvalues at criticality. The correction to the largest eigenvalue yields the central charge

$$c = \frac{12}{\pi^2} L \left(\frac{3 - \sqrt{5}}{2} \right) = \frac{4}{5} \quad (4.10)$$

where the Rogers dilogarithm is

$$L(x) = -\frac{1}{2} \int_0^x \left[\frac{\log(1-y)}{y} + \frac{\log y}{1-y} \right] dy. \quad (4.11)$$

Corrections to the other eigenvalues involve analytic continuations of the Rogers dilogarithms [21] and lead to precisely the same operator content (4.9) with towers above each primary level.

A similar analysis can be applied to the anisotropic 3-state Potts model. At criticality, this model can be parametrized in terms of trigonometric functions of the spectral parameter u which is related to the relative strengths of the interactions J and K . In the isotropic case $u = \pi/12$. In 1992, Pearce [22] showed that the row transfer matrices $V(u)$ of the critical 3-state Potts model satisfy the functional equations

$$V_0 V_1 V_2 = f_0 f_2 V_0 + f_1 f_5 V_2 + f_0 f_1 V_4 \quad (4.12)$$

$$f_3 V_0 V_1 = f_1 f_3 f_5 - f_0 f_2 f_4 + f_0 V_3 V_4 \quad (4.13)$$

where

$$V_n = V(u + n\lambda), \quad f_n = \left[\frac{\sin(u + n\lambda)}{\sin \lambda} \right]^N, \quad \lambda = \frac{\pi}{6}. \quad (4.14)$$

It follows that the fused models [23] satisfy an inversion identity hierarchy so the methods of Klümper and Pearce [21] can be used to obtain the finite-size corrections. This time the central charge is given by

$$c = \frac{6}{\pi^2} [L(1) + S(5) - S(6)] = 4/5 \quad (4.15)$$

where

$$S(n) = \sum_{q=2}^{n-2} L \left(\frac{\sin^2(\pi/n)}{\sin^2(q\pi/n)} \right) = \left(2 - \frac{6}{n} \right) \frac{\pi^2}{6} \quad (4.16)$$

and $L(1) = \pi^2/6$. Again the corrections to the other eigenvalues yield the conformal weights and scaling dimensions and, although the expressions are very different, the same operator content (4.9) emerges.

Solving functional equations gives the energy levels and conformal weights, however, it gives no information on the degeneracies that are encoded in the Virasoro characters. Recently the Stony Brook group [24], starting with the Bethe ansatz, have obtained empirical counting rules that give the degeneracies of each level. In this way the complete modular invariant partition functions for the 3-state Potts and hard hexagon models can be derived leading to new fermionic representations of the Virasoro characters [25].

In conclusion, it is now possible to show directly that the operator content and modular invariant partition functions of the critical 3-state Potts and hard hexagon models are precisely the same. This represents a very strong demonstration of universality.

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