RECENT PROGRESS IN SOLVING *A–D–E* LATTICE MODELS

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Abstract

There are many families of solvable *A–D–E* lattice models exhibiting order-disorder transitions. These represent many different universality classes of critical behaviour. Some *A–D–E* models can be solved off-criticality but most can only be solved at criticality. Here we review the methods being developed to solve these models to gain a detailed understanding of their critical behaviour.

1 Introduction

The *A–D–E* lattice models are two-dimensional lattice models on the square lattice. The spins in these lattice models take values on the Dynkin diagrams of the *A–D–E* Lie algebras shown in Figure 1. Adjacent spins on the lattice are restricted by the adjacency of the states on the *A–D–E* adjacency graphs. It is now known [1, 2] that there are many families of exactly solvable *A–D–E* lattice models that undergo order-disorder phase transitions. Potentially, these models provide a wealth of exactly solvable representatives of various universality classes of critical behaviour. Of course, it is one thing to say that a model is solvable and quite another to be able to actually calculate all the quantities of physical interest. At the very least the first goal is to calculate the complete set of critical exponents, or equivalently, scaling dimensions that characterize the critical behaviour of these models. In this article we give a review of the recent progress in this direction.

The layout of the paper is as follows. The *A–D–E* lattice models are described in Section 2. In particular, we define the Temperley Lieb and dilute models which will be the focus of this article. In Section 3 we briefly discuss the Yang-Baxter algebras associated with the *A–D–E* models, namely, the Temperley-Lieb and two-color braid-monoid algebras. Section 4 introduces briefly the notion of intertwiners and then describes the hierarchies of fusion *A–D–E* models and their significance in solving for the central charges, scaling dimensions and critical exponents.

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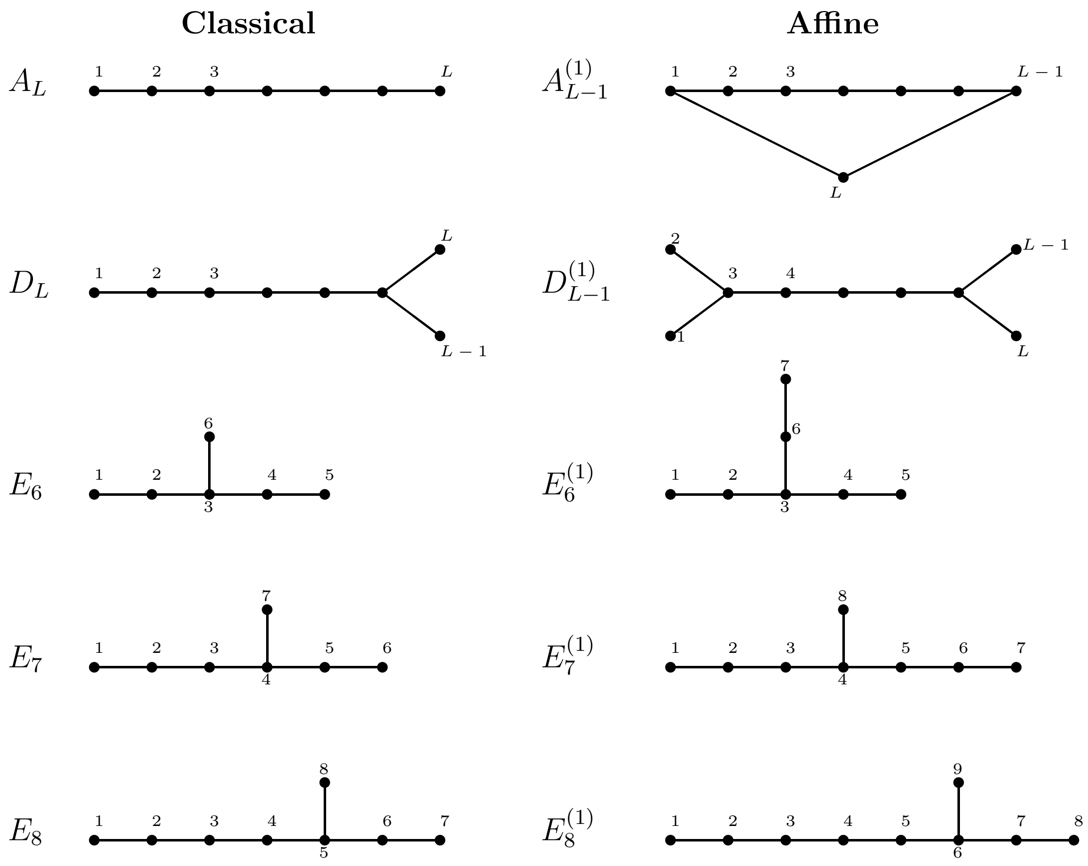


Figure 1. The adjacency graphs of the classical and affine *A*–*D*–*E* models.

A–*D*–*E* **Family**	**Lie Algebra**	**Vertex Model**
Temperley-Lieb	$A_1^{(1)}$	6-Vertex
Dilute	$A_2^{(2)}$	Izergin-Korepin 19-Vertex
. . .	$A_2^{(1)}$	. . .
. . .	$A_3^{(1)}$	. . .
. . .	$A_3^{(2)}$	. . .
. . .	$C_2^{(1)}$	. . .

Table 1. Table of solvable *A*–*D*–*E* families and their associated affine Lie algebras and vertex models.

2 *A–D–E* Lattice Models

It was once commonly believed that exactly solvable lattice models are rare and exceptional and therefore, by their very nature, not representative of critical systems generally. However, it is now known that it is possible to construct solvable two-dimensional lattice models starting with arbitrary graphs. By and large most of these models do not exhibit critical behaviour. It is precisely the critical models that are selected by restricting attention to the *A–D–E* graphs. There are now many solvable families of *A–D–E* lattice models [1, 2]. These are listed in Table 1 along with their associated vertex models and affine Lie algebras [3]. Perhaps the most important and the most studied among these are the Temperley-Lieb *A–D–E* models [1] and the dilute *A–D–E* models [4, 5]. In the sequel we will therefore concentrate on these two examples.

2.1 Temperley-Lieb A–D–E Models

The face weights of the Temperley-Lieb *A–D–E* models are given in terms of the data associated with the *A–D–E* graphs by

$$\begin{aligned} W\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) &= \begin{array}{ccc} d & & c \\ & \boxed{u} & \\ a & & b \end{array} \\ &= \frac{\sin(\lambda-u)}{\sin\lambda}\,\delta_{a,c}A_{a,b}A_{a,d} + \frac{\sin u}{\sin\lambda}\sqrt{\frac{S_aS_c}{S_bS_d}}\,\delta_{b,d}A_{a,b}A_{b,c} \end{aligned} \tag{2.1}$$

The variable u is the spectral parameter and

$$\lambda = \pi/h \tag{2.2}$$

is the crossing parameter with the Coxeter number

$$h = \begin{cases} L+1, & A_L \\ 2L-2, & D_L \\ 12, 18, 30, & E_{6,7,8}. \end{cases} \tag{2.3}$$

The elements of the adjacency matrices A are given by

$$A_{a,b} = \begin{cases} 1, & a,\ b \text{ connected} \\ 0, & \text{otherwise} \end{cases} \tag{2.4}$$

and the nonnegative elements of the Perron-Frobenius eigenvectors S are given by

$$\sum_b A_{a,b}S_b = \begin{cases} 2\cos\lambda\ S_a, & \text{classical} \\ 2S_a, & \text{affine} \end{cases} \tag{2.5}$$

In the affine case the trigonometric functions are replaced by rational functions eg. $\sin x \mapsto x$. Some well known special cases are presented in Figure 2.

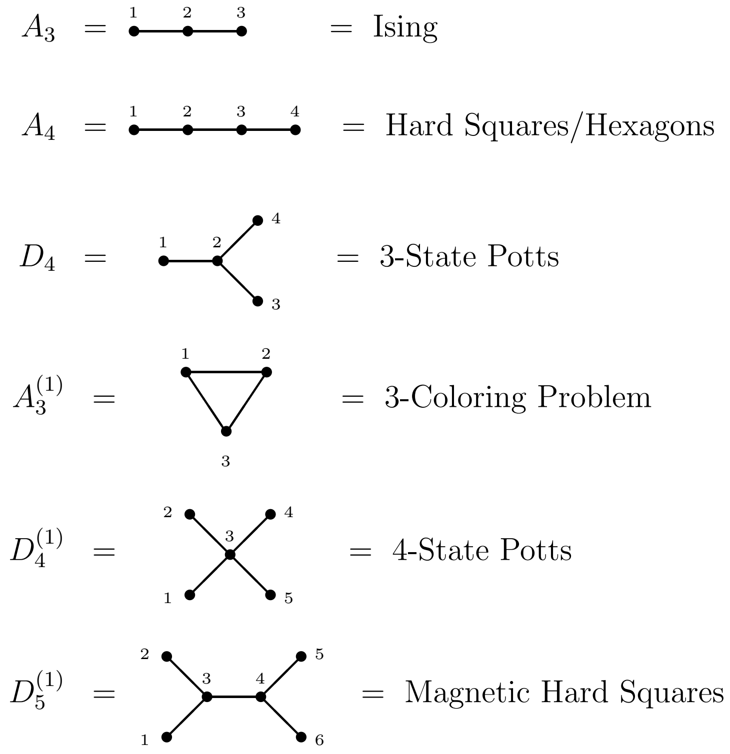


Figure 2. Some well known classical and affine *A–D–E* lattice models.

2.2 Dilute A–D–E Models:

The face weights of the dilute *A–D–E* models [4, 5] are given by

$$\begin{aligned} W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle|\, u\right) &= \rho_1(u)\delta_{a,b,c,d} + \rho_2(u)\delta_{a,b,c}A_{a,d} + \rho_3(u)\delta_{a,c,d}A_{a,b} \\ &+ \sqrt{\frac{S_a}{S_b}}\rho_4(u)\delta_{b,c,d}A_{a,b} + \sqrt{\frac{S_c}{S_a}}\rho_5(u)\delta_{a,b,d}A_{a,c} \\ &+ \rho_6(u)\delta_{a,b}\delta_{c,d}A_{a,c} + \rho_7(u)\delta_{a,d}\delta_{c,b}A_{a,b} \\ &+ \rho_8(u)\delta_{a,c}A_{a,b}A_{a,d} + \sqrt{\frac{S_aS_c}{S_bS_d}}\rho_9(u)\delta_{b,d}A_{a,b}A_{b,c} \end{aligned} \tag{2.6}$$

with the trigonometric functions

$$\rho_1(u) = 1 + \frac{\sin u \sin(3\overline{\lambda} - u)}{\sin(2\overline{\lambda})\sin(3\overline{\lambda})} \tag{2.7}$$

$$\rho_2(u) = \rho_3(u) = \frac{\sin(3\overline{\lambda} - u)}{\sin(3\overline{\lambda})} \tag{2.8}$$

$$\rho_4(u) = \rho_5(u) = \frac{\sin u}{\sin(3\overline{\lambda})} \tag{2.9}$$

$$\rho_6(u) = \rho_7(u) = \frac{\sin u \sin(3\overline{\lambda} - u)}{\sin(2\overline{\lambda})\sin(3\overline{\lambda})} \tag{2.10}$$

$$\rho_8(u) \;=\; \frac{\sin(2\overline{\lambda}-u)\sin(3\overline{\lambda}-u)}{\sin(2\overline{\lambda})\sin(3\overline{\lambda})} \tag{2.11}$$

$$\rho_9(u) \;=\; -\frac{\sin u\sin(\overline{\lambda}-u)}{\sin(2\overline{\lambda})\sin(3\overline{\lambda})} \tag{2.12}$$

and

$$\overline{\lambda} = \frac{(h+1)\pi}{4h}. \tag{2.13}$$

Here we have used the generalized Kronecker delta

$$\delta_{a,b,c,\dots} = \begin{cases} 1, & a=b=c=\dots \\ 0, & \text{otherwise.} \end{cases} \tag{2.14}$$

Again the trigonometric functions need to be replaced by rational functions in some cases.

Notice that the effective adjacency condition for the dilute models is given by $I+A$ where I is the identity matrix and A is the adjacency matrix of the relevant A–D–E graph. This is analogous to adding a loop to each node of the A–D–E graphs to describe the admissible states of nearest neighbor spins.

Some A–D–E lattice models admit an elliptic solution as shown in Table 2. In these cases the models can be solved off-criticality. In particular, in such cases, the free energy can be obtained by solving the inversion relation [6] and the local height probabilities and order parameters can be obtained [7, 8, 9, 10] using corner transfer matrices [11]. It is therefore possible to obtain the associated critical exponents directly. By this means it has been confirmed that the critical behaviours of both the Temperley-Lieb and dilute A–D–E models are described by the unitary minimal series of conformal field theories [12, 13] with central charge

$$c = \begin{cases} 1-\dfrac{6}{h(h-1)}, & \text{classical} \\ 1, & \text{affine.} \end{cases}$$

Strictly speaking, this identification applies only to the critical points separating regimes III and IV ($0<u<\lambda$) for the Temperley-Lieb case and regimes 2^+ and 2^- ($0<u<3\overline{\lambda}$) for the dilute case. For simplicity, we will restrict consideration to these regimes.

	A	D	E
Classical Temperley-Lieb	E	E	T
Affine Temperley-Lieb	E	E	R
Classical Dilute	E	T	T
Affine Dilute	T	T	R

Table 2. Table showing the nature of the exact solution manifold. An E indicates the existence of an elliptic off-critical solution, while a T or an R indicates a trigonometric or rational critical solution.

For the Temperley-Lieb A–D–E models the elliptic nome plays the role of a temperature-like variable t which perturbs away from the critical point. In this case, results for suitable order parameters of the A_L models [7, 14] give

$$R^{(k)} \sim t^{\beta_k}, \qquad \beta_k = \frac{(k+1)^2-1}{8L}, \qquad k = 1, 2, \ldots, L-2 \tag{2.15}$$

which for $L = 3$ and $k = 1$ yields the familiar result for the magnetization of the Ising model

$$m \sim t^{\beta}, \qquad \beta = 1/8. \tag{2.16}$$

In remarkable contrast, at least when L is odd, the elliptic nome plays the role of a symmetry breaking field h for the dilute A_L models. In this case the critical behaviour of the order parameters is found [10] to be

$$\overline{R}^{(k)} \sim h^{1/\delta_k}, \qquad \delta_k = \frac{3L(L+2)}{(k+1)^2-1}, \qquad k = 1, 2, \ldots, L-2. \tag{2.17}$$

The dilute A_3 model lies in the universality class of the Ising model in a magnetic field and is therefore of particular interest. In this case the elliptic nome is identified as the leading magnetic field and, accordingly, the direct calculation of the magnetization [10] leads for $L = 3$ and $k = 1$ to the celebrated critical exponent

$$m \sim h^{1/\delta}, \qquad \delta = 15. \tag{2.18}$$

Previously, this result had only been obtained by invoking scaling laws.

3 Yang-Baxter Algebras

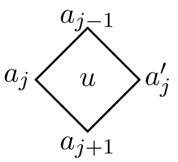
In most cases it is not possible to solve the A–D–E models off-criticality in order to obtain the critical behaviour. Therefore other methods are required to obtain this data from the critical models alone. This can be done. The method is to use the face or Yang-Baxter algebra to construct fusion models. It is then usually possible to find functional equations for the row transfer matrices of the fused models. These in turn can be solved for the finite-size corrections to the eigenvalues which finally yields the central charges, scaling dimensions and critical exponents. In this section we will describe the Yang-Baxter algebras. Fusion will be discussed in Section 4.

3.1 Face Operators

Many aspects of solvable lattice models are best understood in terms of the local algebraic properties of the face weights. This leads us to introduce local face transfer operators whose action is to add a single face to the lattice. In order for a model to be exactly solvable, these face operators must satisfy the Yang-Baxter equations [11]. This invariably leads to consideration of the so-called Yang-Baxter algebras and to remarkable connections with knot theory [15, 16].

If $a = \{a_1, a_2, \ldots\}$ and $a' = \{a'_1, a'_2, \ldots\}$ are two consecutive diagonal rows of spins on the lattice, we define the elements of a local face transfer operator $X_j(u)$ in terms of the face weights by

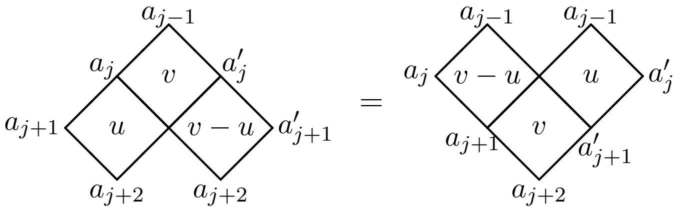
$$\langle a|X_j(u)|a'\rangle \;=\; W\!\left(\begin{matrix} a_{j-1} & a'_j \\ a_j & a_{j+1}\end{matrix}\,\middle|\, u\right)\prod_{k\neq j}\delta(a_k,a'_k) \;= \tag{3.1}$$



The lattice model is then exactly solvable if the face operators satisfy the Yang-Baxter equations

$$X_{j+1}(u)X_j(v)X_{j+1}(v-u) = X_j(v-u)X_{j+1}(v)X_j(u) \tag{3.2}$$

These are illustrated graphically as



(3.3)

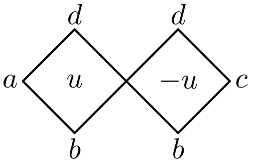
The significance of these equations is that they imply commuting row and corner transfer matrices. By locality, the Yang-Baxter operators also satisfy

$$X_j(u)X_k(v) = X_k(v)X_j(u), \qquad |j-k|\geq 2. \tag{3.4}$$

This relation along with the Yang-Baxter equations are the defining relations of a Yang-Baxter algebra.

For particular models the Yang-Baxter operators will satisfy additional properties. One common property of importance is the inversion relation [6]

$$X_j(u)X_j(-u) = \rho(u)\rho(-u)I \tag{3.5}$$



$$= \;\rho(u)\rho(-u)\delta(a,c) \tag{3.6}$$

In particular, this relation holds for the Temperley-Lieb and dilute *A*–*D*–*E* models with

$$\rho(u) = \begin{cases} \dfrac{\sin(\lambda-u)}{\sin\lambda}, & \text{Temperley-Lieb} \\[2ex] \dfrac{\sin(2\overline{\lambda}-u)\sin(3\overline{\lambda}-u)}{\sin(2\overline{\lambda})\sin(3\overline{\lambda})}, & \text{dilute.} \end{cases} \tag{3.7}$$

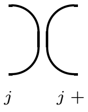
As an immediate consequence we see that the Yang-Baxter operators $X_j(u)$ are invertible except possibly at the singular points

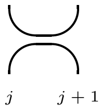
$$u = \begin{cases} \pm\lambda, & \text{Temperley-Lieb} \\ \pm 2\overline{\lambda}, \pm 3\overline{\lambda}, & \text{dilute} \end{cases} \tag{3.8}$$

Consideration of the Yang-Baxter operators at these singular points leads to projectors that are crucial in understanding the structure of the Yang-Baxter algebra.

3.2 Temperley-Lieb Algebra

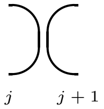
Let us consider the Yang-Baxter algebra of the Temperley-Lieb A–D–E models. Using a convenient graphical notation [17] we can decompose the Yang-Baxter operator as

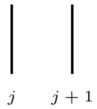
$X_j(u) \;=\; \dfrac{\sin(\lambda-u)}{\sin\lambda}$ 

$+\; \dfrac{\sin u}{\sin\lambda}$ 

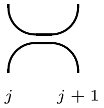
(3.9)

where the diagrams represent the following u-independent operators



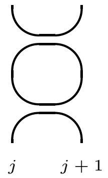
$=$ 

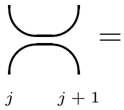
$= \;I\; = \;\text{identity}$ (3.10)



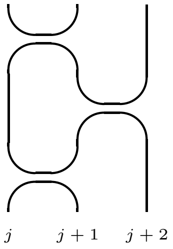
$= \;X_j(\lambda)\; = \;e_j\; = \;\text{monoid}$ (3.11)

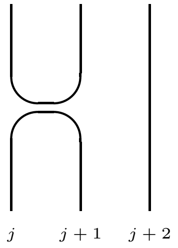
It is then easily verified by direct calculation that the operators e_j satisfy the defining relations of a Temperley-Lieb algebra [18]:

$e_j^2 \;=$ 

$= \;2\cos\lambda$ 

$= 2\cos\lambda\; e_j$ (3.12)

$e_je_{j+1}e_j \;=$ 

$=$ 

$= \;e_j$ (3.13)

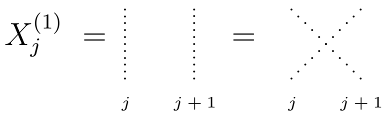
These relations have a natural interpretation in terms of isotopy of strings in a braid. Using them it is straightforward to verify that the Temperley-Lieb A–D–E models satisfy the Yang-Baxter equations.

3.3 Dilute Two-Color Braid-Monoid Algebra

Let us now turn to the Yang-Baxter algebra of the dilute models. In this case we can decompose the face transfer operators as

$$X_j(u) = \sum_{n=1}^{9} \rho_n(u) X_j^{(n)} \tag{3.14}$$

where we have introduced the projectors



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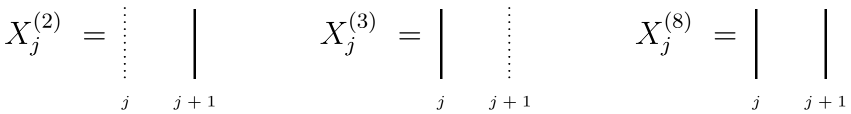
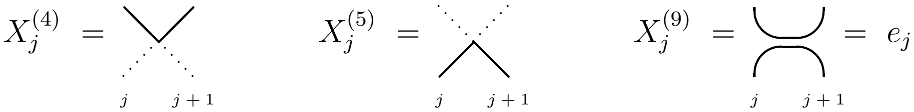


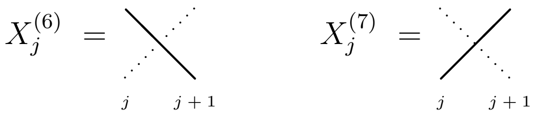
(3.16)

the monoids



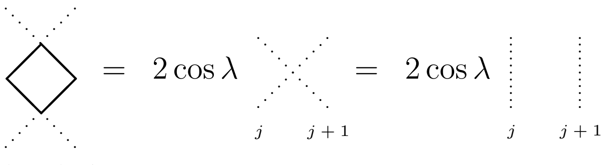
(3.17)

and the braids



(3.18)

The u-independent operators $\{X_j^{(n)}\}$ satisfy many relations as suggested by their graphical representations. Included among these relations are the following:



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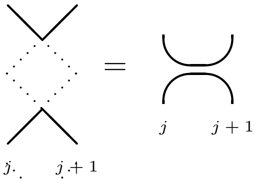
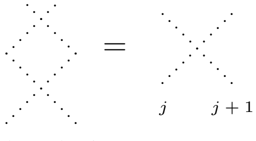


(3.20)



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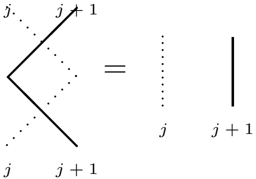

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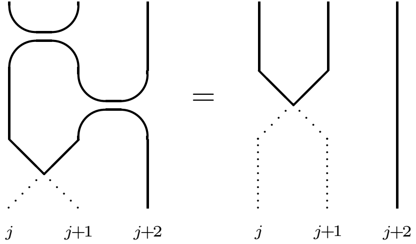

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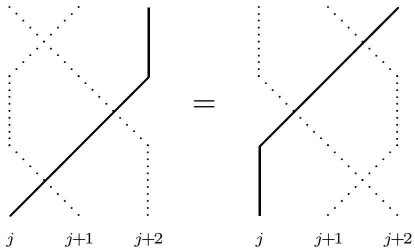
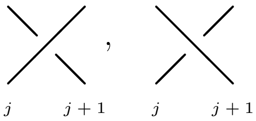


(3.24)

The u independent operators $\{X_j^{(n)}\}$ generate a two-color braid-monoid algebra [19]. The defining relations of this algebra suffice to ensure that the dilute A–D–E models satisfy the Yang-Baxter equations. Notice that the monoid operator $X_j^{(9)}$ is precisely the generator of the previous Temperley-Lieb algebra which is therefore a subalgebra of the two-color braid-monoid algebra. Of course a full discussion of these algebras should take into account the braid operators

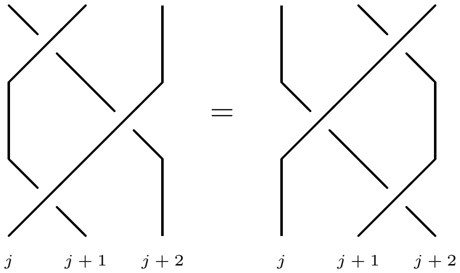
$$b_j^{\pm 1} = I - e^{\pm i\lambda} e_j = \tag{3.25}$$



which satisfy the braid relation

$$b_j b_{j+1} b_j = b_{j+1} b_j b_{j+1}. \tag{3.26}$$

Pictorially this relation becomes



(3.27)

This is done in Grimm and Pearce [19].

4 Intertwiners and Fusion

4.1 Interwiners

In studying A–D–E lattice models most attention is focussed on the A models. This is because the D and E models can be related to the A models by intertwiners. Intertwiners [20, 21, 22, 23, 24] incorporate and extend the more familiar notions of high-low temperature duality, weak-graph duality and orbifold duality [25]. These intertwiners act at three distinct levels: at the level of the adjacency matrices, at the level of the face weights or face algebras and, lastly, at the level of the row transfer matrices. The simplest level is the level of the adjacency matrices. The adjacency matrix C said to intertwine A and $G = D$ or E if

$$AC = CG. \tag{4.1}$$

Likewise there is an intertwiner between the A and G Yang-Baxter algebras if there exists faces or cells C_j, independent of u, that intertwine the face transfer operators

$$X_j^A(u)C_{j+1}C_j = C_{j+1}C_jX_{j+1}^G(u). \tag{4.2}$$

This equation in the analog of the Yang-Baxter equation. If the intertwining cells exist, then it immediately follows that there exists a cell transfer matrix $\mathcal{C}$ that intertwines the A and G row transfer matrices

$$\boldsymbol{T}^A(u)\mathcal{C} = \mathcal{C}\boldsymbol{T}^G(u). \tag{4.3}$$

There are many consequences of the intertwining relations. Most significantly, an intertwining relation implies that there is an overlap of eigenvalues. For the row transfer matrices, in particular, this means that many eigenvalues are exactly in common for a finite size system. It therefore follows that the corresponding central charges, scaling dimensions and critical exponents can be identified. In fact the finite size corrections for the related A and D or E models are obtained by solving precisely the same system of functional equations in the form of a fusion hierarchy. We therefore now turn our attention to fusion.

4.2 Fusion

Given a fundamental solvable A–D–E lattice model, it is possible to construct a hierarchy of solvable models by the process of fusion. Originally, this was carried out for the A models [26, 27]. However, by allowing lattice models with degrees of freedom on the edges of faces in addition to the spin degrees of freedom on the corners of the faces, it is possible to generalize fusion to the D and E models [28]. The adjacency graphs of the fusion models are given by simple fusion rules for the decomposition of the tensor products of representations of spin algebras. Specifically, the fusion rules take the form of matrix recursion relations.

For the Temperley-Lieb A–D–E models the fusion rules are found to be the $su(2)$ rules

$$A^{(\ell)}A^{(1)} \;=\; A^{(\ell-1)} \;+\; A^{(\ell+1)} \tag{4.4}$$

$$A^{(0)} \;=\; I, \qquad A^{(1)} \;=\; A. \tag{4.5}$$

These rules just reflect the usual rules for tensor products of spin angular momentum, namely, if you combine a spin-1/2 with a spin-$\ell/2$ you obtain a spin-$(\ell+1)/2$ and a spin-$(\ell-1)/2$. As an example the fusion graphs of the 3-state Potts model are shown in Figure 3. Fusion can also be carried out for the dilute A–D–E models [29]. Surprisingly, it is the $su(3)$ fusion rules

$$A^{(n,m)}A^{(1,0)} = A^{(n+1,m)}+A^{(n-1,m+1)}+A^{(n-1,m-1)} \tag{4.6}$$

$$A^{(0,0)} = I, \quad A^{(1,0)} = I + A, \quad A^{(n,m)} = A^{(m,n)} \tag{4.7}$$

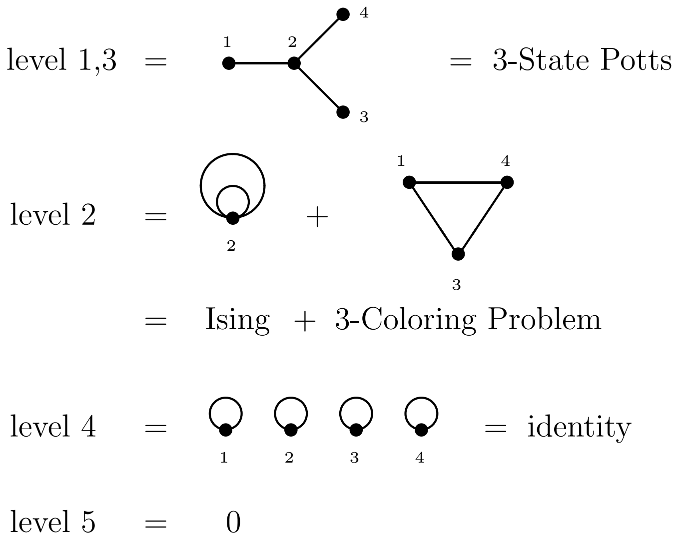


Figure 3. The fusion graphs of the classical Temperley-Lieb D_4 or 3-state Potts model. Remarkably, in this case, the level 2 fusion produces the critical Ising and 3-coloring problems. The first component of the level 2 fusion is in fact the 8-vertex model at the Ising decoupling point.

that are relevant for the dilute models. Here the two indices (n, m) are necessary because the the fusion levels are labelled by representations of $su(3)$. In the dilute case the fusion graphs are more complicated than for the Temperley-Lieb models but in both cases the procedure truncates at a finite level.

The fused face weights can be constructed using the Yang-Baxter face algebras. Their expressions are unwieldy so we will not give them here. On the other hand, the row transfer matrices of the fused models satisfy some remarkable functional equations which are relatively simple to state. For the Temperley-Lieb models this fusion hierarchy takes the form [30, 28]

$$\boldsymbol{T}_0^1\boldsymbol{T}_1^1 = f_{-1}^p f_1^p \boldsymbol{I} + f_0^p \boldsymbol{T}_0^2 \tag{4.8}$$

$$\boldsymbol{T}_0^q\boldsymbol{T}_q^1 = f_q^p \boldsymbol{T}_0^{q-1} + f_{q-1}^p \boldsymbol{T}_0^{q+1} \tag{4.9}$$

where

$$\boldsymbol{T}_k^q = \boldsymbol{T}^{p,q}(u + k\lambda) \tag{4.10}$$

denotes the transfer matrix for $p \times q$ fusion with p fixed and the f_q^p are known combinations of trigonometric functions. An analogous fusion hierarchy has also been found for the dilute models [29], namely,

$$\boldsymbol{T}_0^{(n,0)}\boldsymbol{T}_{n+1}^{(1,0)} = \phi_n \boldsymbol{T}_1^{(n+1,0)} + \phi_{n+2}\boldsymbol{T}_0^{(n-1,1)} \tag{4.11}$$

$$\boldsymbol{T}_0^{(n,0)}\boldsymbol{T}_{n+1}^{(0,1)} = \phi_n \boldsymbol{T}_1^{(n,1)} + \phi_{n+2}\boldsymbol{T}_{-1}^{(n-1,0)} \tag{4.12}$$

$$\boldsymbol{T}_0^{(n,m)}\boldsymbol{T}_{n+m+1}^{(0,1)} + \phi_{n+m+2}\boldsymbol{T}_{-2}^{(n,m-2)} = \boldsymbol{T}_{-1}^{(n,m-1)}\boldsymbol{T}_{n+m+1}^{(1,0)} + \phi_{n+m}\boldsymbol{T}_1^{(n,m+1)} \tag{4.13}$$

where the last equation holds for $m \geq 1$ and again ϕ_n denotes known trigonometric functions.

The fusion hierarchy for the Temperley-Lieb models has been solved [31] for the finite-size corrections and hence the central charges, scaling dimensions and critical exponents. This was achieved by converting the functional equations into an inversion identity hierarchy which is identical in form to the equations of the thermodynamic Bethe ansatz [32]. It should be possible in the near future to solve the fusion hierarchy for the dilute models by similar techniques. In this way it should be possible to completely elucidate the critical behaviour of the dilute models and their fusion hierarchies. Hopefully, the other *A*–*D*–*E* families will also be tractable by these methods.

5 Summary

There are many solvable *A*–*D*–*E* families — most are critical, some extend off-criticality. The order parameters of the off-critical models can be calculated using corner transfer matrices and the critical behaviour extracted directly. The critical behaviour of the models that can only be solved at criticality must be obtained by a more circuitous route. To a large extent it suffices to study the *A* models since intertwiners relate the *D* and *E* models to the *A* models. The face operators of the *A*–*D*–*E* models satisfy Yang-Baxter algebras, ie., the Temperley-Lieb algebra, the multi-color braid-monoid algebras, etc. This algebraic structure can be used to construct a hierarchy of solvable fusion models. The adjacency matrices of the fusion models are given by $su(2)$, $su(3)$ fusion rules etc. The row transfer matrices of the fusion models satisfy special functional equations in the form of a fusion hierarchy. Finally, these functional equations can be solved at criticality for the finite-size corrections, scaling dimensions and hence the critical exponents.

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