YANG-BAXTER ALGEBRAS AND FUSION OF A–D–E LATTICE MODELS

Paul A. Pearce[1] and Yu-kui Zhou[2]

Mathematics Department, University of Melbourne, Parkville, Victoria 3052, Australia

Abstract

There are many families of exactly solvable A–D–E lattice models. Most important among these are the Temperley-Lieb and dilute families of A–D–E models. We review the methods being developed to study the critical behaviour of these models and their fusion hierarchies. In particular, we describe the Yang-Baxter algebras of the A–D–E models and show how fusion leads to transfer matrix functional equations related to the thermodynamic Bethe ansatz. These functional equations can be solved for the central charges and spectra of conformal weights in terms of Rogers dilogarithms and their analytic continuations.

1 Introduction

In recent years the study of solvable two-dimensional lattice models has provided a fertile area for the interaction between physics and mathematics. From a physical viewpoint it is the phase transitions and critical phenomena of two-dimensional lattice models and their associated conformal field theories that are of interest. The study of these physical theories leads to remarkably rich mathematical structures involving Yang-Baxter algebras, Lie algebras, quantum groups, classical q-series and Rogers dilogarithms. In this review we touch on some of these tantalizing connections in the context of solvable A–D–E lattice models.

The A–D–E lattice models are two-dimensional lattice models on the square lattice. The spins assigned to each site take values on the Dynkin diagrams of the A–D–E Lie algebras shown in Figure 1. Adjacent spins on the lattice are restricted by the adjacency of the states on the A–D–E adjacency graphs. There are now many families [1, 2, 3, 4] of exactly solvable A–D–E lattice models that exhibit phase transitions. Of course, it is one thing to say that a model is solvable and quite another to be able to actually calculate the quantities of physical interest such as the critical exponents, or equivalently the conformal weights, that characterize the critical behaviour of these models. In this article we review the algebraic properties of these models that enable the derivation of

[1]Email: pap@mundoe.maths.mu.oz.au

[2]Email: ykzhou@mundoe.maths.mu.oz.au

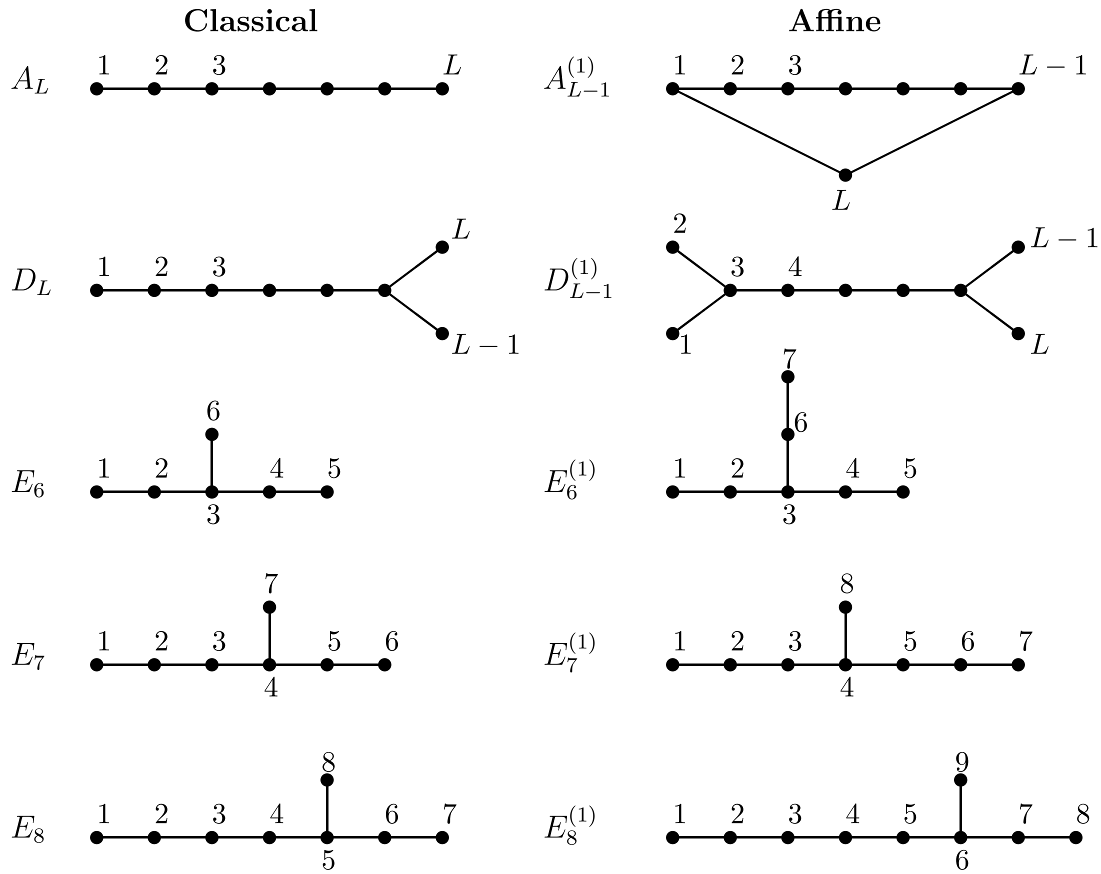


Figure 1. The adjacency graphs of the classical and affine A–D–E models.

functional equations which in turn can be solved for the complete spectra of conformal weights.

The layout of the paper is as follows. The Temperley-Lieb and dilute families of A–D–E lattice models which will be the focus of this article are defined in Section 2. In Section 3 we briefly discuss the Yang-Baxter algebras associated with these A–D–E models, namely, the Temperley-Lieb and two-color or dilute braid-monoid algebras. Finally, in Section 4, we describe the hierarchies of fusion A–D–E models and their significance in deriving functional equations related to the thermodynamic Bethe ansatz. We also indicate how these equations can be solved for the central charges, scaling dimensions and critical exponents with the help of dilogarithm identities.

2 A–D–E Lattice Models

2.1 Temperley-Lieb A–D–E Models

The face weights of the Temperley-Lieb A–D–E models are given, in terms of the data associated with the A–D–E graphs, by

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle|\, u\right) = \frac{\sin(\lambda-u)}{\sin\lambda}\,\delta_{a,c}A_{a,b}A_{a,d} + \frac{\sin u}{\sin\lambda}\sqrt{\frac{S_aS_c}{S_bS_d}}\,\delta_{b,d}A_{a,b}A_{b,c} \tag{2.1}$$

The variable u is the spectral parameter and

$$\lambda = \pi/h \tag{2.2}$$

is the crossing parameter with the Coxeter number

$$h = \begin{cases} L+1, & A_L \\ 2L-2, & D_L \\ 12, 18, 30, & E_{6,7,8}. \end{cases} \tag{2.3}$$

The elements of the adjacency matrices A are given by

$$A_{a,b} = \begin{cases} 1, & a,\, b \text{ connected} \\ 0, & \text{otherwise} \end{cases} \tag{2.4}$$

and the nonnegative elements of the Perron-Frobenius eigenvectors S are given by

$$\sum_b A_{a,b} S_b = \begin{cases} 2\cos\lambda\, S_a, & \text{classical} \\ 2S_a, & \text{affine} \end{cases} \tag{2.5}$$

In the affine case the trigonometric functions are replaced by rational functions eg. $\sin x \mapsto x$.

2.2 Dilute A–D–E Models:

The face weights of the dilute A–D–E models [2, 3] are given by

$$\begin{aligned} W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle|\, u\right) &= \rho_1(u)\delta_{a,b,c,d} + \rho_2(u)\delta_{a,b,c}A_{a,d} + \rho_3(u)\delta_{a,c,d}A_{a,b} \\ &+ \sqrt{\frac{S_a}{S_b}}\rho_4(u)\delta_{b,c,d}A_{a,b} + \sqrt{\frac{S_c}{S_a}}\rho_5(u)\delta_{a,b,d}A_{a,c} + \rho_6(u)\delta_{a,b}\delta_{c,d}A_{a,c} \\ &+ \rho_7(u)\delta_{a,d}\delta_{c,b}A_{a,b} + \rho_8(u)\delta_{a,c}A_{a,b}A_{a,d} + \sqrt{\frac{S_aS_c}{S_bS_d}}\rho_9(u)\delta_{b,d}A_{a,b}A_{b,c} \end{aligned} \tag{2.6}$$

with the trigonometric functions

$$\begin{aligned} \rho_1(u) &= 1 + \frac{\sin u \sin(3\lambda - u)}{\sin(2\lambda)\sin(3\lambda)}, & \rho_2(u) = \rho_3(u) &= \frac{\sin(3\lambda-u)}{\sin(3\lambda)} \\ \rho_4(u) &= \rho_5(u) = \frac{\sin u}{\sin(3\lambda)}, & \rho_6(u) = \rho_7(u) &= \frac{\sin u \sin(3\lambda - u)}{\sin(2\lambda)\sin(3\lambda)} \\ \rho_8(u) &= \frac{\sin(2\lambda - u)\sin(3\lambda-u)}{\sin(2\lambda)\sin(3\lambda)}, & \rho_9(u) &= -\frac{\sin u \sin(\lambda - u)}{\sin(2\lambda)\sin(3\lambda)} \end{aligned} \tag{2.7}$$

and

$$\lambda = \frac{(h+1)\pi}{4h}. \tag{2.8}$$

Here we have used the generalized Kronecker delta

$$\delta_{a,b,c,\ldots} = \begin{cases} 1, & a = b = c = \ldots \\ 0, & \text{otherwise.} \end{cases} \tag{2.9}$$

Again the trigonometric functions need to be replaced by rational functions in the affine cases.

Notice that the effective adjacency condition for the dilute models is given by $I + A$ where I is the identity matrix and A is the adjacency matrix of the relevant A–D–E graph. This is analogous to adding a loop to each node of the A–D–E graphs to describe the admissible states of nearest neighbour spins.

The critical behaviours of both the Temperley-Lieb and dilute families of classical A–D–E models are described by the unitary minimal series of conformal field theories [5, 6] with central charge

$$c = 1 - \frac{6}{h(h-1)}.$$

Strictly speaking, this identification applies only to the critical points separating regimes III and IV ($0 < u < \lambda$) for the Temperley-Lieb case and regimes 2^+ and 2^- ($0 < u < 3\lambda$) for the dilute case. For simplicity, we will restrict consideration to these regimes.

The classical A models admit an elliptic extension so that they can be solved off-criticality. In particular, for these models, the free energy can be obtained by solving the inversion relation [7] and the local height probabilities and order parameters can be obtained [8, 9, 10, 11] using corner transfer matrices [12]. It is therefore possible to obtain the associated critical exponents directly. For the Temperley-Lieb A_L models the elliptic nome plays the role of a temperature-like variable t which perturbs away from the critical point. In particular, results for suitable order parameters of the A_L models [8, 13] give

$$R^{(k)} \sim t^{\beta_k}, \qquad \beta_k = \frac{(k+1)^2 - 1}{8L}, \qquad k = 1, 2, \ldots, L-2 \tag{2.10}$$

which for $L = 3$ and $k = 1$ yields the familiar result for the magnetization of the Ising model

$$m \sim t^{\beta}, \qquad \beta = 1/8. \tag{2.11}$$

In remarkable contrast, at least when L is odd, the elliptic nome plays the role of a symmetry breaking field h for the dilute A_L models. In this case the critical behaviour of the order parameters is found [11] to be

$$\overline{R}^{(k)} \sim h^{1/\delta_k}, \qquad \delta_k = \frac{3L(L+2)}{(k+1)^2 - 1}, \qquad k = 1, 2, \ldots, L-2. \tag{2.12}$$

The dilute A_3 model lies in the universality class of the Ising model in a magnetic field and is therefore of particular interest. In this case the elliptic nome is identified as the leading magnetic field and, accordingly, the direct calculation of the magnetization [11] leads for $L = 3$ and $k = 1$ to the celebrated critical exponent

$$m \sim h^{1/\delta}, \qquad \delta = 15. \tag{2.13}$$

Previously, this result had only been obtained by invoking scaling laws.

3 Yang-Baxter Algebras

In most cases it is not possible to solve the A–D–E models off-criticality in order to obtain the critical behaviour. Therefore other methods are required to obtain this data from the critical models alone. This can be done. The method is to use the face or Yang-Baxter algebra to construct fusion models. Functional equations can then be obtained for the row transfer matrices of the fused models. These in turn can be solved for the finite-size corrections to the eigenvalues which finally yields the central charges, scaling dimensions and critical exponents. In this section we will describe the Yang-Baxter algebras. Fusion will be discussed in Section 4.

3.1 Face Operators

Many aspects of solvable lattice models are best understood in terms of the local algebraic properties of the face weights, that is, in terms of local face transfer operators whose action is to add a single face to the lattice. In order for a model to be exactly solvable, these face operators must satisfy the Yang-Baxter equations [12]. This leads to the consideration of the Yang-Baxter algebras and to unexpected connections with knot theory [14, 15].

If $a = \{a_1, a_2, \ldots\}$ and $a' = \{a'_1, a'_2, \ldots\}$ are two consecutive diagonal rows of spins on the lattice, we define the elements of a local face transfer operator $X_j(u)$ in terms of the face weights by

$$\langle a|X_j(u)|a'\rangle \;=\; W\left(\begin{matrix} a_{j-1} & a'_j \\ a_j & a_{j+1}\end{matrix}\,\middle|\, u\right)\prod_{k\neq j}\delta(a_k, a'_k)\;. \tag{3.1}$$

The lattice model is then exactly solvable if the face operators satisfy the Yang-Baxter equations

$$X_{j+1}(u)X_j(v)X_{j+1}(v-u) = X_j(v-u)X_{j+1}(v)X_j(u) \tag{3.2}$$

The significance of these equations is that they imply commuting row and corner transfer matrices. By locality, the Yang-Baxter operators also satisfy

$$X_j(u)X_k(v) = X_k(v)X_j(u), \qquad |j-k| \geq 2. \tag{3.3}$$

This relation along with the Yang-Baxter equations are the defining relations of a Yang-Baxter algebra.

For particular models the Yang-Baxter operators will satisfy additional properties. One common property of importance is the inversion relation [7]

$$X_j(u)X_j(-u) = \rho(u)\rho(-u)I \tag{3.4}$$

In particular, this relation holds for the Temperley-Lieb and dilute A–D–E models with

$$\rho(u) = \begin{cases} \dfrac{\sin(\lambda-u)}{\sin\lambda}, & \text{Temperley-Lieb} \\[2ex] \dfrac{\sin(2\lambda-u)\sin(3\lambda-u)}{\sin(2\lambda)\sin(3\lambda)}, & \text{dilute.} \end{cases} \tag{3.5}$$

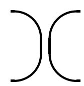
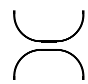
As an immediate consequence we see that the Yang-Baxter operators $X_j(u)$ are invertible except possibly at the singular points

$$u = \begin{cases} \pm\lambda, & \text{Temperley-Lieb} \\ \pm 2\lambda, \pm 3\lambda, & \text{dilute} \end{cases} \tag{3.6}$$

Consideration of the Yang-Baxter operators at these singular points leads to projectors that are crucial in understanding the structure of the Yang-Baxter algebra and the process of fusion.

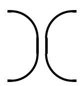
3.2 Temperley-Lieb Algebra

Let us consider the Yang-Baxter algebra of the Temperley-Lieb *A–D–E* models. Using a convenient graphical notation [16] we can decompose the Yang-Baxter operator as

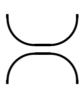
$X_j(u) \;=\; \dfrac{\sin(\lambda-u)}{\sin\lambda}$  $+\; \dfrac{\sin u}{\sin\lambda}$  (3.7)

$j \quad j+1 \qquad\qquad j \quad j+1$

where the diagrams represent the following u-independent operators

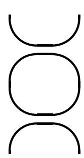
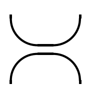
 $= \quad X_j(0) \;=\; I \;=\; \text{identity}$ (3.8)

$j \quad j+1$

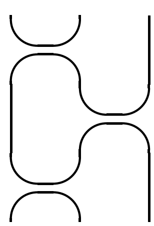
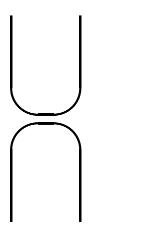
 $= \quad X_j(\lambda) \;=\; e_j \;=\; \text{monoid}$ (3.9)

$j \quad j+1$

It is then easily verified by direct calculation that the operators e_j satisfy the defining relations of a Temperley-Lieb algebra [17]:

$e_j^2 \;=$  $= \quad 2\cos\lambda$  $= 2\cos\lambda\; e_j$ (3.10)

$j \quad j+1 \qquad\qquad j \quad j+1$

$e_j e_{j+1} e_j \;=$  $=$  $= \; e_j$ (3.11)

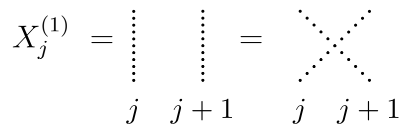
$j \quad j+1\, j+2 \qquad\qquad j \quad j+1\, j+2$

These relations have a natural interpretation in terms of isotopy of strings in a braid. Using these relations and some elementary trigonometric identities it is straightforward to verify that the Temperley-Lieb *A–D–E* models satisfy the Yang-Baxter equations.

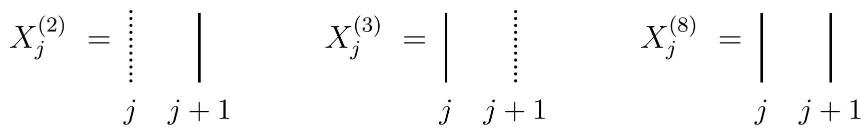
Let us now turn to the Yang-Baxter algebra of the dilute models. In this case we can decompose the face transfer operators as

$$X_j(u) = \sum_{n=1}^{9} \rho_n(u) X_j^{(n)} \tag{3.12}$$

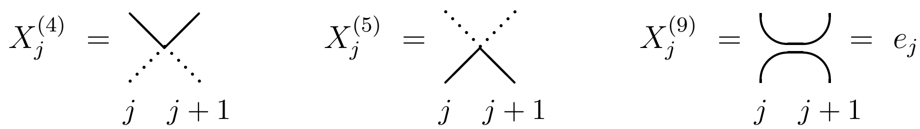
where we have introduced the operators



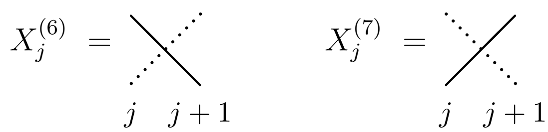
(3.13)



(3.14)



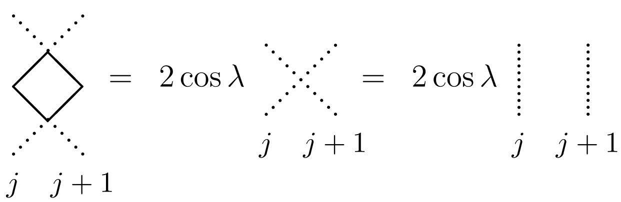
(3.15)



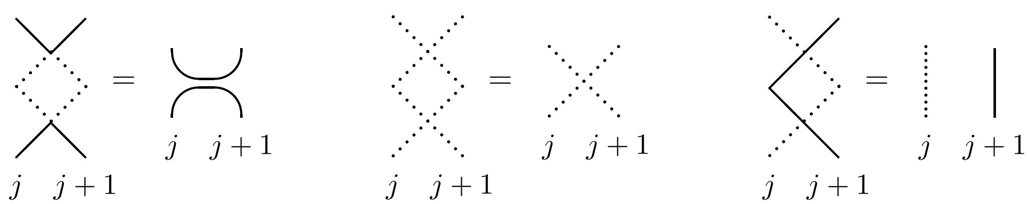
(3.16)

Here (3.13) and (3.14) are projectors, (3.15) are monoids and (3.16) are braids.

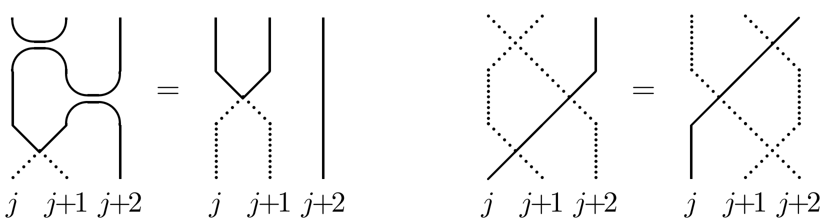
The u-independent operators $\{X_j^{(n)}\}$ satisfy many relations as suggested by their graphical representations. Included among these relations are the following:



(3.17)



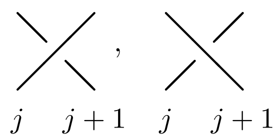
(3.18)



(3.19)

The u independent operators $\{X_j^{(n)}\}$ in fact generate a two-color braid-monoid algebra [18]. The defining relations of this algebra suffice to ensure that the dilute A–D–E models satisfy the Yang-Baxter equations. Notice that the monoid operator $X_j^{(9)}$ is precisely the generator of the previous Temperley-Lieb algebra which is therefore a subalgebra of the two-color braid-monoid algebra. Of course a full discussion of these algebras should take into account the braid operators

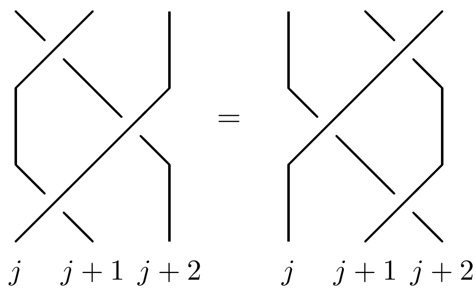
$$b_j^{\pm 1} = I - e^{\pm i\lambda} e_j = \qquad (3.20)$$



which satisfy the braid relation

$$b_j b_{j+1} b_j = b_{j+1} b_j b_{j+1}. \qquad (3.21)$$

Pictorially this relation becomes



(3.22)

This is done in Grimm and Pearce [18].

4 Fusion

4.1 Intertwiners

Given a fundamental solvable A–D–E lattice model, it is possible to construct a hierarchy of solvable models by the process of fusion [19]. Originally, this was carried out for the Temperley-Lieb A models [20, 21]. However, by allowing lattice models with degrees of freedom on the edges of faces in addition to the spin degrees of freedom on the corners of the faces, it is possible to generalize fusion to the D and E models [22]. This can be done because the face weights of the fused D and E models are related to the A models by intertwiners. Intertwiners [23, 24, 25, 26, 27] incorporate and extend the more familiar notions of high-low temperature duality, weak-graph duality and orbifold duality [28]. These intertwiners act at three distinct levels: at the level of the adjacency matrices, at the level of the face weights or face algebras and, lastly, at the level of the row transfer matrices. The same intertwiners automatically intertwine the fusion matrices at each of these levels. The simplest level is the level of the adjacency matrices. The adjacency matrix C is said to intertwine A and $G = D$ or E if

$$AC = CG. \qquad (4.1)$$

Likewise there is an intertwiner between the A and G Yang-Baxter algebras if there exists faces or cells C_j, independent of u, that intertwine the face transfer operators

$$X_j^A(u)C_{j+1}C_j = C_{j+1}C_jX_{j+1}^G(u). \tag{4.2}$$

This equation is the analog of the Yang-Baxter equation. If the intertwining cells exist, then it immediately follows that there exists a cell transfer matrix $\mathcal{C}$ that intertwines the A and G row transfer matrices

$$\boldsymbol{T}^A(u)\mathcal{C} = \mathcal{C}\boldsymbol{T}^G(u). \tag{4.3}$$

There are many consequences of the intertwining relations. Most significantly, an intertwining relation implies that there is an overlap of eigenvalues. For the row transfer matrices, in particular, this means that many eigenvalues are exactly in common for a finite size system. It therefore follows that the corresponding central charges, scaling dimensions and critical exponents can be identified. In fact the finite size corrections for the related A and D or E models are obtained by solving precisely the same system of functional equations in the form of a fusion hierarchy which we now describe.

4.2 Fusion Rules

The adjacency graphs of the fused A–D–E models are given by simple fusion rules for the decomposition of the tensor products of representations of spin algebras. Specifically, the fusion rules take the form of matrix recursion relations.

For the Temperley-Lieb A–D–E models the fusion rules are found to be the $su(2)$ rules

$$A^{(\ell)}A^{(1)} = A^{(\ell-1)} + A^{(\ell+1)} \tag{4.4}$$
$$A^{(0)} = I, \qquad A^{(1)} = A. \tag{4.5}$$

These rules just reflect the usual rules for tensor products of spin angular momentum, namely, if you combine a spin-1/2 with a spin-$\ell/2$ you obtain a spin-$(\ell+1)/2$ and a spin-$(\ell-1)/2$. As an example the fusion graphs of the 3-state Potts model are shown in Figure 2. Fusion can also be carried out for the dilute A–D–E models [29]. Surprisingly, it is the $su(3)$ fusion rules

$$A^{(n,m)}A^{(1,0)} = A^{(n+1,m)} + A^{(n-1,m+1)} + A^{(n,m-1)} \tag{4.6}$$
$$A^{(0,0)} = I, \quad A^{(1,0)} = I + A, \quad A^{(n,m)} = A^{(m,n)} \tag{4.7}$$

that are relevant for the dilute models. Here the two indices (n,m) are necessary because the fusion levels are labelled by representations of $su(3)$. In the dilute case the fusion graphs are more complicated than for the Temperley-Lieb models but in both cases the procedure truncates at a finite level for the classical algebras.

The fused face weights can be constructed using projectors and the Yang-Baxter face algebras. Their expressions are unwieldy so we will not give them here. On the other hand, the row transfer matrices of the fused models satisfy some remarkable functional equations which are simple enough to state.

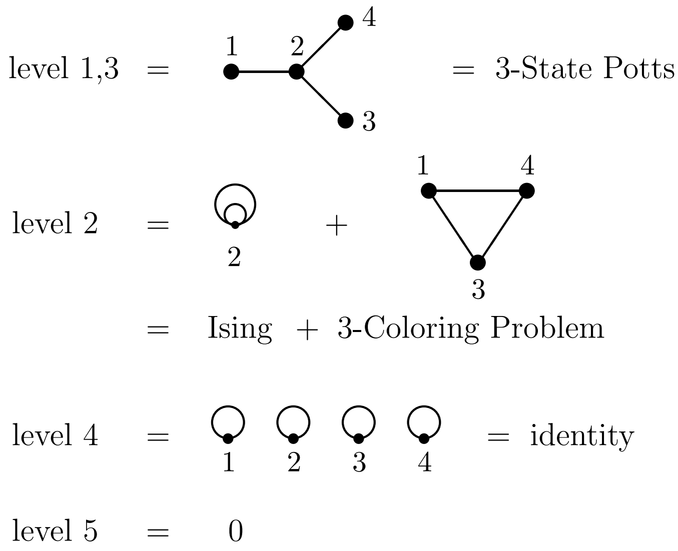


Figure 2. The fusion graphs of the classical Temperley-Lieb D_4 or 3-state Potts model. Remarkably, in this case, the level 2 fusion produces the critical Ising and 3-coloring problems. The first component of the level 2 fusion is in fact the 8-vertex model at the Ising decoupling point. The hierarchy closes at level 5 as does the hierarchy for A_5 with which it intertwines.

4.3 Inversion Identity Hierarchies and the Thermodynamic Bethe Ansatz

The row transfer matrices of the Temperley-Lieb A–D–E models satisfy a fusion hierarchy of equations in the form [30, 31, 22]

$$\boldsymbol{T}_0^q \boldsymbol{T}_q^1 = f_q \boldsymbol{T}_0^{q-1} + f_{q-1} \boldsymbol{T}_0^{q+1} \tag{4.8}$$

where $\boldsymbol{T}_k^q = \boldsymbol{T}^{p,q}(u + k\lambda)$ denotes the N face row transfer matrix for $p \times q$ fusion with p fixed, $\boldsymbol{T}_0^0 = f_{-1}\boldsymbol{I}$ and

$$f_q = \prod_{k=0}^{p-1} \left[\frac{\sin[u + (q-k)\lambda]}{\sin\lambda} \right]^N . \tag{4.9}$$

The functional equation [32]

$$\boldsymbol{T}_0^q \boldsymbol{T}_1^q = f_{-1} f_q \boldsymbol{I} + \boldsymbol{T}_0^{q+1} \boldsymbol{T}_1^{q-1} \tag{4.10}$$

therefore follows by induction. If we further define

$$\boldsymbol{t}_0^q = \frac{\boldsymbol{T}_1^{q-1} \boldsymbol{T}_0^{q+1}}{f_{-1} f_q} \tag{4.11}$$

then these equations can be recast in the form

$$\boldsymbol{t}_0^q \boldsymbol{t}_1^q = [\boldsymbol{I} + \boldsymbol{t}_1^{q-1}][\boldsymbol{I} + \boldsymbol{t}_0^{q+1}] \tag{4.12}$$

which is precisely of the form of the thermodynamic Bethe ansatz [33].

An analogous fusion hierarchy has also been found for the dilute models [29]. Taking N to be even, it can be shown by direct fusion of the face weights that

$$\boldsymbol{T}_0^{(n,0)} \boldsymbol{T}_n^{(1,0)} = \boldsymbol{T}_0^{(n+1,0)} + \boldsymbol{T}_0^{(n-1,1)} \tag{4.13}$$

$$\boldsymbol{T}_0^{(n,0)} \boldsymbol{T}_n^{(0,1)} = \boldsymbol{T}_0^{(n,1)} + f_{n-1} \boldsymbol{T}_0^{(n-1,0)} \tag{4.14}$$

$$\boldsymbol{T}_0^{(n,m)} = \boldsymbol{T}_0^{(n,0)} \boldsymbol{T}_n^{(0,m)} - f_{n-1} \boldsymbol{T}_0^{(n-1,0)} \boldsymbol{T}_{n+1}^{(0,m-1)} \tag{4.15}$$

where $\boldsymbol{T}^{(n,m)} = \boldsymbol{T}^{(n',m'),(n,m)}$ denotes the fused transfer matrix of type $(n',m') \times (n,m)$ with (n',m') fixed, $\boldsymbol{T}_n^{(1,0)} = \boldsymbol{T}(u+2n\lambda)$, $\boldsymbol{T}_n^{(0,0)} = \boldsymbol{I}$ and

$$f_n = s_{n-3/2}s_{n-1}s_{n-1/2}s_{n+1}s_{n+3/2}s_{n+2}, \quad s_n = \left[\frac{\sin(u+2n\lambda)}{\sqrt{\sin 2\lambda \sin 3\lambda}}\right]^N . \tag{4.16}$$

Similarly, it can be shown that the fused transfer matrices satisfy the symmetry

$$\left[\prod_{k=0}^{n-1} s_{k-2}s_{k+1/2}\right]\boldsymbol{T}^{(n,m)}(u) = \left[\prod_{k=0}^{m-1} s_{n+k-3/2}s_{n+k+1}\right]\left[\boldsymbol{T}^{(m,n)}(-u-2(n+m-2)\lambda)\right]^T . \tag{4.17}$$

These equations completely determine the fusion hierarchy which closes at level $n+m=2L$. From these equations it follows that

$$\boldsymbol{T}_0^{(n+1,0)} = \boldsymbol{T}_0^{(n,0)}\boldsymbol{T}_n^{(1,0)} - \boldsymbol{T}_0^{(n-1,0)}\boldsymbol{T}_{n-1}^{(0,1)} + f_{n-2}\boldsymbol{T}_0^{(n-2,0)} \tag{4.18}$$

$$\boldsymbol{T}_0^{(0,n+1)} = \boldsymbol{T}_0^{(0,n)}\boldsymbol{T}_n^{(0,1)} - f_{n-1}\boldsymbol{T}_0^{(0,n-1)}\boldsymbol{T}_n^{(1,0)} + f_{n-1}f_{n-2}\boldsymbol{T}_0^{(0,n-2)} \tag{4.19}$$

and more generally

$$\boldsymbol{T}_0^{(n,m+1)} = \boldsymbol{T}_0^{(n,m)}\boldsymbol{T}_{n+m}^{(0,1)} - f_{n+m-1}\boldsymbol{T}_0^{(n,m-1)}\boldsymbol{T}_{n+m}^{(1,0)} + f_{n+m-1}f_{n+m-2}\boldsymbol{T}_0^{(n,m-2)}. \tag{4.20}$$

Hence it can be shown by induction that

$$\boldsymbol{T}_0^{(n,0)}\boldsymbol{T}_1^{(n,0)} = \boldsymbol{T}_0^{(n+1,0)}\boldsymbol{T}_1^{(n-1,0)} + \boldsymbol{T}_0^{(0,n)} \tag{4.21}$$

$$\boldsymbol{T}_0^{(0,n)}\boldsymbol{T}_1^{(0,n)} = \boldsymbol{T}_0^{(0,n+1)}\boldsymbol{T}_1^{(0,n-1)} + \left(\prod_{k=0}^{n-1} f_k\right)\boldsymbol{T}_1^{(n,0)}. \tag{4.22}$$

If we now define

$$\boldsymbol{t}_0^{(n,0)} = \frac{\boldsymbol{T}_0^{(n+1,0)}\boldsymbol{T}_1^{(n-1,0)}}{\boldsymbol{T}_0^{(0,n)}}, \qquad \boldsymbol{t}_0^{(0,n)} = \frac{\boldsymbol{T}_0^{(0,n+1)}\boldsymbol{T}_1^{(0,n-1)}}{\left(\prod_{k=0}^{n-1} f_k\right)\boldsymbol{T}_1^{(n,0)}} = \boldsymbol{t}_{1/2}^{(n,0)} \tag{4.23}$$

we obtain the thermodynamic Bethe ansatz type equations

$$\frac{\boldsymbol{t}_0^{(n,0)}\boldsymbol{t}_1^{(n,0)}}{\boldsymbol{t}_0^{(0,n)}} = \frac{[\boldsymbol{I}+\boldsymbol{t}_0^{(n+1,0)}][\boldsymbol{I}+\boldsymbol{t}_1^{(n-1,0)}]}{[\boldsymbol{I}+\boldsymbol{t}_0^{(0,n)}]} \tag{4.24}$$

$$\frac{\boldsymbol{t}_0^{(0,n)}\boldsymbol{t}_1^{(0,n)}}{\boldsymbol{t}_1^{(n,0)}} = \frac{[\boldsymbol{I}+\boldsymbol{t}_0^{(0,n+1)}][\boldsymbol{I}+\boldsymbol{t}_1^{(0,n-1)}]}{[\boldsymbol{I}+\boldsymbol{t}_1^{(n,0)}]} \tag{4.25}$$

These are precisely the equations proposed by Kuniba, Nakanishi and Suzuki [34, 35] for the $su(3)$ face models of Jimbo, Miwa and Okado [36]. In the dilute case, however, the equations simplify due to symmetry to give

$$\frac{\boldsymbol{t}_0^n\boldsymbol{t}_1^n}{\boldsymbol{t}_{1/2}^n} = \frac{[\boldsymbol{I}+\boldsymbol{t}_0^{n+1}][\boldsymbol{I}+\boldsymbol{t}_1^{n-1}]}{[\boldsymbol{I}+\boldsymbol{t}_{1/2}^n]} \tag{4.26}$$

for $\boldsymbol{t}_0^n = \boldsymbol{t}_0^{(n,0)}$ or $\boldsymbol{t}_0^{(0,n)}$.

4.4 Central Charges, Scaling Dimensions and Dilogarithms

The fusion hierarchy and thermodynamic Bethe ansatz equations for the Temperley-Lieb models have been solved by Klümper and Pearce [32] for the finite-size corrections and hence the central charges and scaling dimensions. Actually, only the A case has been given in detail but the solutions for the D and E cases can be obtained similarly. Remarkably, it turns out that the central charges and scaling dimensions are all expressed in terms of the Rogers dilogarithm

$$L(x) = -\frac{1}{2}\int_0^x \left[\frac{\log(1-y)}{y} + \frac{\log y}{1-y}\right] dy \tag{4.27}$$

and its analytic continuation. It now appears that this is generally the case for all conformal field theories [38, 39, 40]. Specifically, the central charges of the fused Temperley-Lieb A–D–E models are given by

$$\begin{aligned} c \;=\; & 1+\frac{6}{\pi^2}\left[\sum_{q=1}^{p-1} L\left(\frac{\sin^2\sigma}{\sin^2(q+1)\sigma}\right) + \sum_{q=p+1}^{h-3} L\left(\frac{\sin^2\tau}{\sin^2(q+1-p)\tau}\right)\right. \\ & \left. - \sum_{q=1}^{h-3} L\left(\frac{\sin^2\theta}{\sin^2(q+1)\theta}\right)\right] = \frac{3p}{p+2}\left[1-\frac{2(p+2)}{h(h-p)}\right] \end{aligned} \tag{4.28}$$

where $\theta = \pi/h$, $\sigma = \pi/(p+2)$, $\tau = \pi/(h-p)$ and we have used the identity [37]

$$\frac{6}{\pi^2}\sum_{k=2}^{n-2} L\left(\frac{\sin^2\frac{\pi}{n}}{\sin^2\frac{k\pi}{n}}\right) = \left(2-\frac{6}{n}\right), \quad n \geq 4. \tag{4.29}$$

Similar, although more unwieldy, expressions are obtained for the scaling dimensions in which the angles are replaced by $\theta = s\pi/h$, $\sigma = \overline{s}\pi/(p+2)$ and $\tau = t\pi/(h-p)$ where s, $\overline{s}$ and t are integers. These formulas therefore involve analytic continuations of the dilogarithms and generalized dilogarithm identities.

It should be possible in the near future to solve the fusion hierarchy and thermodynamic Bethe ansatz equations for the dilute A–D–E models by similar techniques. In this way it should be possible to obtain the central charges and scaling dimensions in terms of Rogers dilogarithms and their analytic continuations and hence completely elucidate the critical behaviour of the dilute A–D–E models and their fusion hierarchies.

Acknowledgements

It is a pleasure to acknowledge our many collaborators in work reported here: Uwe Grimm, Andreas Klümper, Bernard Nienhuis, Katherine Seaton and Ole Warnaar. This work is supported by the Australian Research Council.

References

1. V. Pasquier, Nucl. Phys. **B28** (1987) 162; J. Phys. A **20** (1987) L1229, 5707.

2. S. O. Warnaar, B. Nienhuis and K. A. Seaton, Phys. Rev. Lett. **69** (1992) 710.

3. Ph. Roche, Phys. Lett. **B4** (1992) 929.

4. S. O. Warnaar and B. Nienhuis, *Solvable Lattice Models Labelled by Dynkin Diagrams*, preprint (1993).

5. A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. **B241** (1984) 333.

6. D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. **52** (1984) 1575; in "Vertex Operators in Mathematics and Physics", eds. J. Lepowsky, S. Mandelstam and I.M. Singer, Springer, 1984.

7. R. J. Baxter, J. Stat. Phys. **28** (1982) 1.

8. G. E. Andrews, R. J. Baxter and P. J. Forrester, J. Stat. Phys. **35** (1984) 193.

9. P. A. Pearce and K. A. Seaton, Phys. Rev. Lett. **60** (1988) 1347; Ann. Phys. (N.Y) **193** (1989) 326.

10. A. Kuniba and T. Yajima, J. Stat. Phys. **52** (1988) 829.

11. S. O. Warnaar, P. A. Pearce, K. A. Seaton and B. Nienhuis, *Order Parameters of the Dilute A Models*, to appear in J. Stat. Phys. (1993).

12. R. J. Baxter, "Exactly Solved Models in Statistical Mechanics", Academic Press, London, 1982.

13. D. A. Huse, Phys. Rev. B **30** (1984) 3908.

14. M. Wadati, T. Deguchi and Y. Akutsu, Phys. Rep. **180** (1989) 247.

15. F. Y. Wu, Rev. Mod. Phys. **64** (1992) 1099.

16. L. Kauffman, Topology **26** (1987) 395.

17. H. N. V. Temperley and E. H. Lieb, Proc. Roy. Soc. (London) **A322** (1971) 251.

18. U. Grimm and P. A. Pearce, *Multi-Colour Braid-Monoid Algebras*, to appear in J. Phys. A (1993).

19. P. P. Kulish, N. Yu Reshetikhin and Z. K. Sklyanin, Lett. Math. Phys. **5** (1981) 393.

20. E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, Adv. Stud. Pure Math. **16** (1988) 17.

21. T. Takagi, T. Deguchi and M. Wadati, Tokyo University preprint (1992).

22. Y-K Zhou and P. A. Pearce, *Fusion and A–D–E Lattice Models*, preprint (1993).

23. A. Ocneanu, "Quantized Groups, String Algebras and Galois Theory for Algebras in Operator Algebras and Applications", Lond. Math. Soc. Lecture Note Series **136**.

24. V. Pasquier, Commun. Math. Phys. **118** (1988) 335.

25. Ph. Roche, Commun. Math. Phys. **127** (1990) 395.

26. P. Di Francesco and J. B. Zuber, Nucl. Phys. **B338** (1990) 602.

27. P. A. Pearce and Y-K Zhou, *Intertwiners and A–D–E Lattice Models*, Int. J. Mod. Phys. **B7** (1993) to appear.

28. P. Fendley and P. Ginsparg, Nucl. Phys. **B324** (1989) 549.

29. U. Grimm, P. A. Pearce and Y-K. Zhou, *Fusion Hierachies of Izergin-Korepin and Dilute RSOS Models*, in preparation (1993).

30. V. V. Bazhanov and N. Yu Reshetikhin, Int. J. Mod. Phys. **B4** (1989) 115.

31. P. A. Pearce, Int. J. Mod. Phys. **A7**, Suppl. 1B (1992) 791.

32. A. Klümper and P. A. Pearce, Physica A **183** (1992) 304.

33. Al. B. Zamolodchikov, Phys. Lett. **B253** (1991) 391; Nucl. Phys. **B358** (1991) 497.

34. V. V. Bazhanov, N. Yu Reshetikhin, J. Phys. A **23** (1990) 1477.

35. A. Kuniba, T. Nakanishi and J. Suzuki, *Functional Relations in Solvable Lattice Models I and II*, Harvard University preprints HUTP-93/A022, A023 (1993).

36. M. Jimbo, T. Miwa and M. Okado, Commun. Math. Phys. **116** (1988) 507.

37. A. N. Kirillov, N. Yu Reshetikhin, Zap. Nauch. Semin. LOMI **160** (1987) 211.

38. W. Nahm, A. Recknagel and M. Terhoeven, *Dilogarithm Identities in Conformal Field Theory*, Bonn preprint, hep-th/9211034 (1992).

39. A. Kuniba and T. Nakanishi, *Spectra in Conformal Field Theories from Rogers Dilogarithm*,preprint SMS-042-92, to appear in Mod. Phys. Lett. A (1992).

40. A. N. Kirillov, *Dilogarithm Identities and Spectra in Conformal Field Theory*, Cambridge preprint, hep-th/9211137 (1992).