FUSION OF A–D–E LATTICE MODELS

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Abstract

Fusion hierarchies of A–D–E face models are constructed. The fused D and E models yield new solutions of the Yang-Baxter equations with bond variables on the edges of faces in addition to the spin variables on the corners. It is shown directly that the row transfer matrices of the fused models satisfy special functional equations. Intertwiners between the fused A–D–E models are constructed by fusing the cells that intertwine the elementary face weights. As an example, we calculate explicitly the fused 2×2 face weights of the 3-state Potts model associated with the D_4 diagram as well as the fused intertwiner cells for the A_5–D_4 intertwiner. Remarkably, this 2×2 fusion yields the face weights of both the Ising model and 3-state CSOS models.

1 Introduction

The fusion procedure is very useful in studying two-dimensional solvable vertex and face models [1, 2, 3]. Essentially, fusion enables the construction of new solutions to the Yang-Baxter equations from a given fundamental solution. Among A–D–E lattice models [4, 5, 6, 7], much effort has been focused on the fusion of the A models [3, 8]. By contrast, fusion of the D and E models has received no attention. The fusion procedure is important because it plays a key role in the solution of these lattice models. Specifically, it leads to solvable functional equations for the fusion hierarchy of commuting transfer matrices [9, 10]. Indeed, it has been argued [11] that the fusion and inversion hierarchies of functional equations for the D and E models are exactly the same as those for the associated A model related to it by an intertwining relation [12, 13, 14, 15].

Here we extend the fusion procedure to all the A–D–E lattice models. In particular, we establish the fusion and inversion hierarchies directly for the classical D and E models. We also extend the construction of intertwiners to the fusion A–D–E models. In this paper, for simplicity, we focus on the classical A–D–E models although similar arguments apply for the affine and dilute A–D–E models. The paper is organized as follows. In the next section we define the critical classical A–D–E lattice models and modify the

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face weights by an appropriate gauge transformation. The modified face weights satisfy a group of special properties which ensure that they can be taken as the elementary blocks for fusion. In section 3 we give the procedure for constructing the fused A–D–E face weights. This is accomplished by introducing parities for the fusion projectors. In section 4 we derive directly the fusion hierarchies satisfied by the fused A–D–E row transfer matrices. The intertwiners between the fused A and the fused D or E models are presented in section 5. Also, in this section, we find the gauge transformation to obtain the symmetric fused face weights. In section 6, as an example, we give explicitly the fused D_4 face weights and the fused cells that intertwine them with the fused A_5 face weights. Finally, after a brief conclusion, we present in the appendices a comprehensive table of the adjacency diagrams for the classical A–D–E fusion models as well as the parities of the first four fusion levels of the E_6 model.

2 Properties of the Face Weights

The A–D–E lattice models [5, 16, 17] are interaction-round-a-face or IRF models [18] that generalize the restricted solid-on-solid (RSOS) models of Andrews, Baxter and Forrester [4]. At criticality, these models are given by solutions of the Yang-Baxter equation [18] based on the Temperley-Lieb algebra and are associated with the classical and affine A–D–E Dynkin diagrams shown in Figure 1. States at adjacent sites of the square lattice must be adjacent on the Dynkin diagram. The face weights of faces not satisfying this adjacency condition for each pair of adjacent sites around a face vanish.

In this paper we will restrict our attention to the classical A–D–E models. The face weights of the classical A–D–E models at criticality are given by [5]

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = {}^{d}_{a}\boxed{u}{}^{c}_{b} = \frac{\sin(\lambda-u)}{\sin\lambda}\,\delta_{a,c}A_{a,b}A_{a,d} + \frac{\sin u}{\sin\lambda}\sqrt{\frac{S_aS_c}{S_bS_d}}\,\delta_{b,d}A_{a,b}A_{b,c} \tag{2.1}$$

where u is the spectral parameter and $\lambda = \pi/h$ is the crossing parameter. Here

$$h = \begin{cases} L+1, & \text{for } A_L \\ 2L-2, & \text{for } D_L \\ 12, 18, 30, & \text{for } E_L = E_{6,7,8} \end{cases} \tag{2.2}$$

is the Coxeter number and S_a are the elements of the Perron-Frobenius eigenvector S of the adjacency matrix A with elements

$$A_{a,b} = \begin{cases} 1, & (a,b) \text{ adjacent} \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

In analogy to the classical A models, we modify the A–D–E face weights (2.1) by a gauge transformation as follows

$${}^{d}_{a}\boxed{u}{}^{c}_{b} \mapsto \frac{g(d,c)g(c,b)}{g(d,a)g(a,b)}\;{}^{d}_{a}\boxed{u}{}^{c}_{b} = \sqrt{\frac{S_c}{S_a}}\,\frac{f_c}{f_a}\;{}^{d}_{a}\boxed{u}{}^{c}_{b} \tag{2.4}$$

where we set $g(a,b) = g_ag_b$ with $g_a = S_a^{1/4}f_a^{1/2}$ and

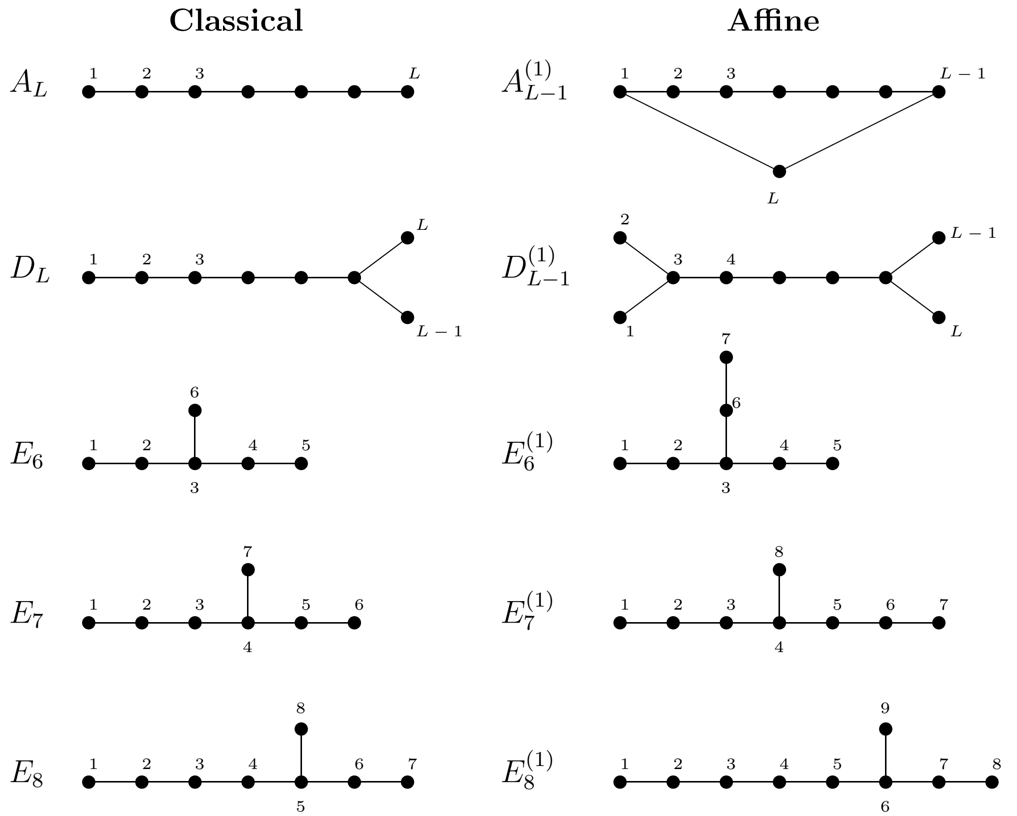


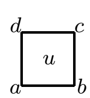
Figure 1: Dynkin diagrams of the classical and affine A–D–E Lie algebras

$$f_a = \quad (-1)^{\frac{a}{2}}, \qquad \text{for } a = 1, 2, \cdots, L \qquad A_L \tag{2.5}$$

$$f_a = \begin{cases} (-1)^{\frac{a}{2}}, & \text{for } a = 1, 2, \cdots, L-1 \\ (-1)^{\frac{L-1}{2}}, & \text{for } a = L \end{cases} \qquad D_L \tag{2.6}$$

$$f_a = \begin{cases} (-1)^{\frac{a}{2}}, & \text{for } a = 1, 2, \cdots, L-3, L-1 \\ (-1)^{\frac{L-4}{2}}, & \text{for } a = L-2 \\ (-1)^{\frac{L-2}{2}}, & \text{for } a = L \end{cases} \qquad E_L = E_{6,7,8} \tag{2.7}$$

In this gauge, the modified face weights are given by

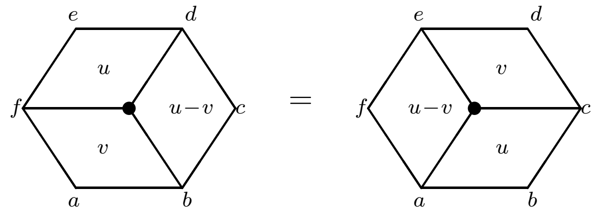


$$= \frac{\sin(\lambda - u)}{\sin \lambda}\, \delta_{a,c} A_{a,b} A_{a,d} + \frac{\sin u}{\sin \lambda} \frac{S_c}{S_b} \epsilon_{a,c}\, \delta_{b,d} A_{a,b} A_{b,c} \tag{2.8}$$

where we have introduced the symmetric sign symbol

$$\epsilon_{a,c} = \epsilon_{c,a} = \frac{f_c}{f_a} = \begin{cases} 1, & a = c \\ 1, & (a,c) = (L-1, L) \text{ or } (L, L-1) \text{ for } D_L \\ 1, & (a,c) = (L-4, L-2) \text{ or } (L-2, L-4) \text{ for } E_L \\ -1, & \text{otherwise.} \end{cases} \tag{2.9}$$

The face weights (2.1) or (2.8) satisfy the Yang-Baxter equations



(2.10)

where the solid circles indicate sums over the central spins.

Each node a of the A–D–E Dynkin diagrams has a coordination number or valence $\mathrm{val}(a) = 1, 2, 3$. Specifically, the valence $\mathrm{val}(a) = 2$ except for the endpoints with $\mathrm{val}(a) = 1$ and branch points with $\mathrm{val}(a) = 3$. In the modified gauge (2.4) the face weights acquire the following properties:

$$W\begin{pmatrix} d & c \\ a & b \end{pmatrix}\Big|\, 0 \;=\; \delta_{a,c} \tag{2.11}$$

$$W\begin{pmatrix} d & c \\ a & b \end{pmatrix}\Big|\, \lambda \;=\; 0, \qquad b \neq d \tag{2.12}$$

$$W\begin{pmatrix} b & c \\ c & b \end{pmatrix}\Big|\, \lambda \;=\; \epsilon_{c,a}\, W\begin{pmatrix} b & c \\ a & b \end{pmatrix}\Big|\, \lambda \;=\; \frac{S_c}{S_b} A_{a,b} A_{b,c}. \tag{2.13}$$

Moreover, at $u = -\lambda$, the face weights also satisfy the properties:

$$W\begin{pmatrix} b & a \\ a & b \end{pmatrix}\Big|\, {-\lambda} \;=\; 0 \qquad \mathrm{val}(b) = 1 \tag{2.14}$$

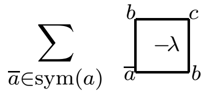
$$W\begin{pmatrix} a & a\pm1 \\ a\pm1 & a \end{pmatrix}\Big|\, {-\lambda} \;=\; W\begin{pmatrix} a & a\mp1 \\ a\pm1 & a \end{pmatrix}\Big|\, {-\lambda} \qquad \mathrm{val}(a) = 2 \tag{2.15}$$

$$W\begin{pmatrix} L-2 & L-3 \\ a & L-2 \end{pmatrix}\Big|\, {-\lambda} \;=\; W\begin{pmatrix} L-2 & L-1 \\ a & L-2 \end{pmatrix}\Big|\, {-\lambda} + W\begin{pmatrix} L-2 & L \\ a & L-2 \end{pmatrix}\Big|\, {-\lambda} \qquad \text{for } D_L \tag{2.16}$$

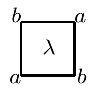
$$W\begin{pmatrix} L-3 & L \\ a & L-3 \end{pmatrix}\Big|\, {-\lambda} \;=\; W\begin{pmatrix} L-3 & L-4 \\ a & L-3 \end{pmatrix}\Big|\, {-\lambda} + W\begin{pmatrix} L-3 & L-2 \\ a & L-3 \end{pmatrix}\Big|\, {-\lambda} \qquad \text{for } E_L \tag{2.17}$$

These properties are useful for constructing the fused face weights. However, to study the fusion hierarchy we also need the additional properties:

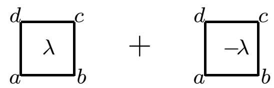
$$\sum_a W\begin{pmatrix} b & c \\ a & b \end{pmatrix}\Big|\, {-\lambda} \;=\; 2\cos\lambda\, A_{b,c} \qquad \mathrm{val}(b) = 2 \tag{2.18}$$



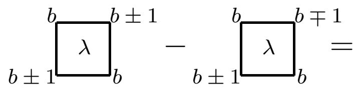
$$= 2\cos\lambda\; A_{b,c}\begin{cases}(\delta_{a,c}+\delta_{c,L-3}) & \text{for } D_L\\ (\delta_{a,c}+\delta_{c,L}) & \text{for } E_L\end{cases} \qquad \mathrm{val}(b)=3 \tag{2.19}$$



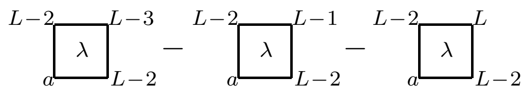
$$= 2\cos\lambda\; A_{a,b} \qquad \mathrm{val}(a)=3,\ \mathrm{val}(b)=1 \tag{2.20}$$



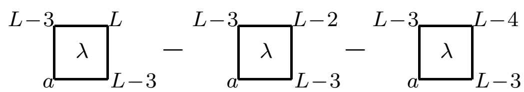
$$= 2\cos\lambda\; A_{b,c}A_{d,c}\delta_{a,c} \tag{2.21}$$



$$= 2\cos\lambda\; A_{b,b\pm1} \qquad \mathrm{val}(b)=2 \tag{2.22}$$



$$= 2\cos\lambda(\delta_{a,L-3}-\delta_{a,L-1}-\delta_{a,L}) \qquad \text{for } D_L \tag{2.23}$$



$$= 2\cos\lambda(\delta_{a,L}-\delta_{a,L-2}-\delta_{a,L-4}) \qquad \text{for } E_L \tag{2.24}$$

where the symmetric sum is over

$$\mathrm{sym}(a) = \begin{cases}\{L-3,L-1\}, & a=L-1\\ \{L-3,L\}, & a=L\end{cases} \qquad \text{for } D_L \tag{2.25}$$

$$\mathrm{sym}(a) = \begin{cases}\{L-4,L\}, & a=L-4\\ \{L-2,L\}, & a=L-2\end{cases} \qquad \text{for } E_L \tag{2.26}$$

We will introduce the corresponding antisymmetric sums in section 3.

3 Elementary Fusion

The Temperley-Lieb A–D–E models are related to the six-vertex model and hence to the spin algebra $su(2)$. The higher-spin representations of this algebra are obtained by taking tensor products of the fundamental representation. The analog of this process for the A–D–E face models is fusion. Starting with a fundamental A, D or E solution of the Yang-Baxter equations it is possible to obtain a hierarchy of "higher-spin" solutions by fusing blocks of faces together. The fused A models have been discussed by a number of authors [2, 3, 9, 10, 8]. In this section, we extend the fusion procedure to the classical D and E models.

3.1 Admissibility

The adjacency matrices $A^{(n)}$ of the level n fused models are determined by the $su(2)$ fusion rules [15]

$$\begin{aligned}
&A^{(n)}A^{(1)} = A^{(n+1)} + A^{(n-1)}, \quad n = 1,2,3,\cdots,h-2 \\
&A^{(0)} = I, \quad A^{(1)} = A, \quad A^{(n)} = 0, \quad n > h-2, \\
&A^{(h-2)} = \begin{cases} I, & \text{for } D_{2L},\ E_7 \text{ and } E_8 \\ Y, & \text{for } A_L\ ,\ D_{2L-1} \text{ and } E_6 \end{cases}
\end{aligned} \tag{3.1}$$

where I is the identity matrix, h is the Coxeter number and Y is the corresponding height reflection operator defined by

$$Y_{a,b} = \delta_{a,r(b)} \tag{3.2}$$

where

$$r(b) \quad = \quad h-b \qquad \text{for } A_L \tag{3.3}$$

$$r(b) \quad = \quad \begin{cases} 6-b & \text{if } b<6 \\ 6 & \text{if } b=6 \end{cases} \qquad \text{for } E_6 \tag{3.4}$$

$$r(b) \quad = \quad \begin{cases} b & \text{if } b<2L-2 \\ 2L-1 & \text{if } b=2L-2 \\ 2L-2 & \text{if } b=2L-1 \end{cases} \qquad \text{for } D_{2L-1} \tag{3.5}$$

Here $A^{(1)} = A$ is the adjacency matrix for the elementary classical A–D–E model. As examples, we draw the adjacency diagrams describing the allowed or admissible states of adjacent sites of the fused D_7 and E_L models in Appendix A. In contrast to fusing the A_L models, the elements of $A^{(n)}$ can in general be nonnegative integers greater than one. In this case we distinguish the edges of the adjacency diagram joining two given sites by bond variables $\alpha,\beta = 1,2,\ldots$ If there is just one edge then the corresponding bond variable is $\alpha = 1$.

3.2 One by two fusion

We implement the elementary fusion of a one by two block of face weights. The properties of this elementary fusion then suffice to establish the fusion of general $m \times n$ blocks of face weights. Notice that in the level 2 fused D and E models, the occurrence of bond variables on the edges of the fused face weights only arises when both adjacent sites are branch points with valence $\text{val}(a) = 3$.

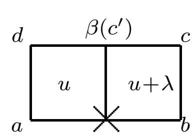


Figure 2: Elementary fusion of two faces. The cross denotes a symmetric sum labelled by $\alpha = 1,2$ as defined in lemma 1. The other spins are fixed. If $\text{val}(c) = \text{val}(d) = 3$ we assume that $c' \neq L-3$ for D_L and $c' \neq L$ for E_L. For clarity both the spin c' and the bond variable β are indicated.

Lemma 1 (Elementary Fusion) *If (a,b) and (d,c) are admissible edges at fusion level two we define the 1×2 fused weights by*

$$W_{12}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) = \sum_{a'} W\left(\begin{matrix} d & c' \\ a & a' \end{matrix}\,\middle|\, u\right) W\left(\begin{matrix} c' & c \\ a' & b \end{matrix}\,\middle|\, u+\lambda\right) \tag{3.6}$$

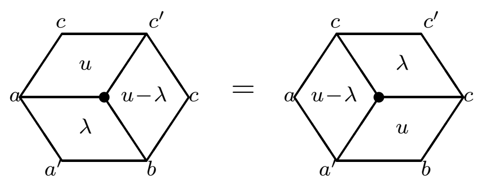
where the sum over a' is over all possible spins (i.e. a normal sum with $\alpha = 1$) if a and b are not both of valence 3. If a and b are both of valence 3, the sum is accomplished in two different ways by summing over $\mathrm{sym}(a')$*. Explicitly, for D_L (resp. E_L) we sum over $L-3$ and $L-1$ (resp. $L-4$ and L) if the bond variable $\alpha = 1$ and over $L-3$ and L (resp. $L-2$ and L) if the bond variable $\alpha = 2$. Then it follows that:*

(i) The RHS is independent of c' except for its dependence on the bond variable

$$\beta(c') = \begin{cases} 2, & c = d = L-2 \text{ and } c' = L; \qquad D_L \\ 2, & c = d = L-3 \text{ and } c' = L-2; \qquad E_L \\ 1, & \text{otherwise.} \end{cases} \tag{3.7}$$

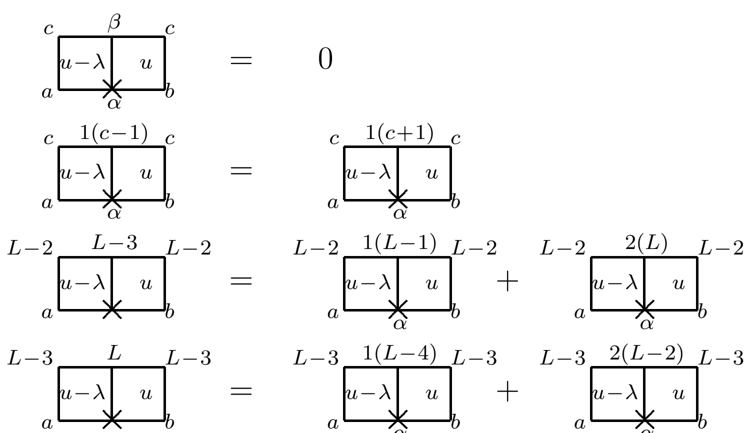
(ii) For all a,b,c,d we have $W_{12}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix}\,\middle|\, 0\right) = 0$.

Proof: To establish (i) it is enough to consider the case $c = d$, otherwise c' is uniquely determined by the adjacency conditions. Setting $v = \lambda$ and $c = e$ in the Yang-Baxter equation (2.10) we have



(3.8)

If $a = b$, then take the special sum over a' in (3.8). Owing to (2.13), the special summation over a' with each fixed c' vanishes in the LHS. Therefore for any (a,b) we always have



val$(c) = 1$ (3.9)

val$(c) = 2$ (3.10)

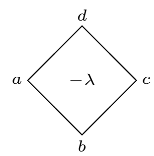
for D_L (3.11)

for E_L (3.12)

These equations imply part (i) of the lemma. Part (ii) follows by (2.11) if $c' \neq a$ and by (2.13) if $c' = a$.

Lemma 1 gives the 1×2 fused face weights incorporating the level two fusion adjacency conditions. A bond variable α has been added between each pair (a,b) of adjacent spins to form edges with states (a,α,b). The adjacency condition for bond variables is that $\alpha=1,2$ if $a=b=L-2$ (resp. $L-3$) for D_L (resp. E_L) and otherwise the bond variable takes the fixed value $\alpha=1$. Similarly, the spin variables are constrained by $|a-b|=0,2$ and $2<|a+b|<2L-4$ (resp. $2L-2$) for D_L (resp. E_L). Observing properties (2.14)–(2.17) we find that this adjacency is completely determined by the operator $P(1,-\lambda)$ with elements

$$P(1,-\lambda)^{d,c,b}_{d,a,b} = \tag{3.13}$$

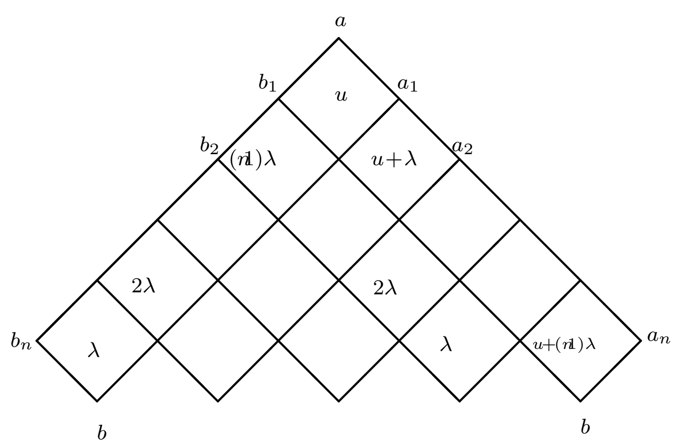


So it can be considered as the projector of level 2 fusion.

3.3 Operator P(n,u)

Let us define graphically

$$P(n,u)^{a,a_1,a_2,\cdots,b}_{a,b_1,b_2,\cdots,b} = \tag{3.14}$$

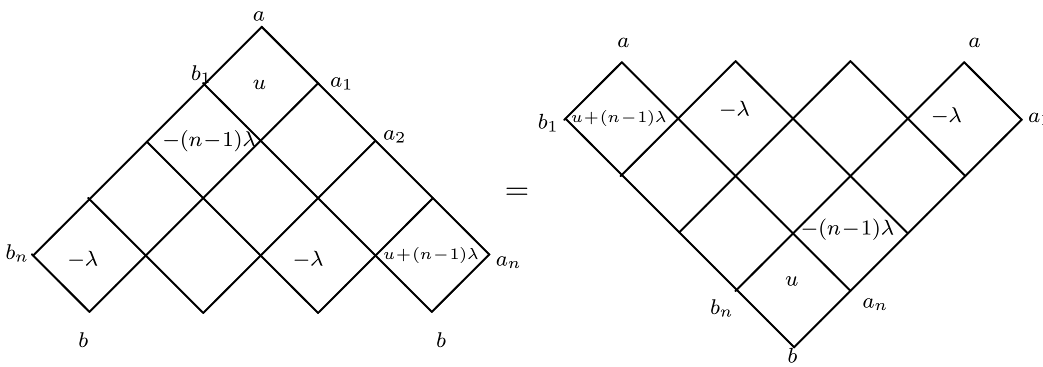


Then the operator $P(n,-n\lambda)$ is the projector of level $n+1$ fusion. For $n=1$ it is the face weight of an elementary block. For $n=2$ it produces the 1 by 2 fusion presented in the last section. This follows from the properties (2.14)–(2.17) and (3.13) we have

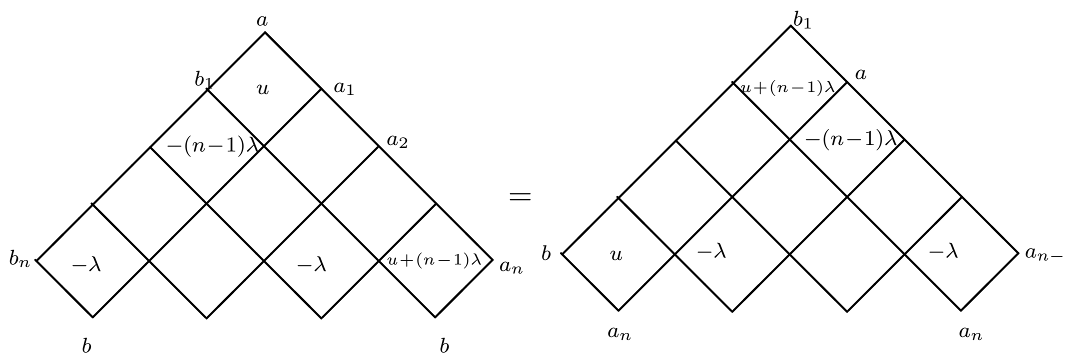
$$P(2,u)^{d,c',c,b}_{d,a,a_1,b} = \begin{cases} P(1,-\lambda)^{a,L-4,b}_{a,a_1,b} W_{12}\left(\begin{matrix} d & \beta & c \\ a & 1 & b\end{matrix}\,\middle|\, u\right) + \\ \qquad P(1,-\lambda)^{a,L-2,b}_{a,a_1,b} W_{12}\left(\begin{matrix} d & \beta & c \\ a & 2 & b\end{matrix}\,\middle|\, u\right) & \text{if } a{=}b{=}L-3 \text{ for } E_L \\ P(1,-\lambda)^{a,L-1,b}_{a,a_1,b} W_{12}\left(\begin{matrix} d & \beta & c \\ a & 1 & b\end{matrix}\,\middle|\, u\right) + \\ \qquad P(1,-\lambda)^{a,L-2,b}_{a,a_1,b} W_{12}\left(\begin{matrix} d & \beta & c \\ a & 2 & b\end{matrix}\,\middle|\, u\right) & \text{if } a{=}b{=}L-2 \text{ for } D_L \\ P(1,-\lambda)^{a,a',b}_{a,a_1,b} W_{12}\left(\begin{matrix} d & \beta & c \\ a & 1 & b\end{matrix}\,\middle|\, u\right) & \text{otherwise} \end{cases} \tag{3.15}$$

where a' is determined by the adjacency condition $A_{a,a'}=A_{a',b}=1$.

We now study the operator $P(n,-n\lambda)$ for level $n+1$ fusion. With the help of Yang-Baxter equation (2.10) we can show that this operator satisfies



(3.16)



(3.17)

These properties will be useful in later sections.

Using the YBE (2.10) and the relations (3.15) it is easy to see that any two adjacent faces with spectral parameters $u+j\lambda$ and $u+(j-1)\lambda$ in (3.14) can be considered as an instance of 1 by 2 fusion. So the properties (3.9)–(3.12) imply

$$P(n,u)^{(a,a_1,\cdots,\cdots,a_{i-1},a_i,a_{i+1},\cdots,\cdots,b)}_{(a,b_1,\cdots,b_{i-2},b_{i-1},b_i,b_{i+1},b_{i+2},\cdots,b)}=0,\quad \text{if val}(b_{i-1})=\text{val}(b_{i+1})=1 \qquad (3.18)$$

$$P(n,u)^{(a,a_1,\cdots,a_{i-2},a_{i-1},a_i,a_{i+1},a_{i+2},\cdots,b)}_{(a,b_1,\cdots,\cdots,b_{i-1},b_i,b_{i+1},\cdots,\cdots,b)}=0,\quad \text{if val}(a_{i-1})=\text{val}(a_{i+1})=1 \qquad (3.19)$$

$$P(n,u)^{(a,a_1,\cdots,L-2,L-3,L-2,\cdots,b)}_{(a,b_1,\cdots,\cdots,b_{i-1},b_i,b_{i+1},\cdots,\cdots,b)}=P(n,u)^{(a,a_1,\cdots,L-2,L-1,L-2,\cdots,b)}_{(a,b_1,\cdots,\cdots,b_{i-1},b_i,b_{i+1},\cdots,\cdots,b)}$$
$$+P(n,u)^{(a,a_1,\cdots,L-2,L,L-2,\cdots,b)}_{(a,b_1,\cdots,\cdots,b_{i-1},b_i,b_{i+1},\cdots,\cdots,b)}\qquad \text{for } D_L \qquad (3.20)$$

$$P(n,u)^{(a,a_1,\cdots,L-3,L,L-3,\cdots,b)}_{(a,b_1,\cdots,\cdots,b_{i-1},b_i,b_{i+1},\cdots,\cdots,b)}=P(n,u)^{(a,a_1,\cdots,L-3,L-4,L-3,\cdots,b)}_{(a,b_1,\cdots,b_{i-1},b_i,b_{i+1},\cdots,b)}$$
$$+P(n,u)^{(a,a_1,\cdots,L-3,L-2,L-3,\cdots,b)}_{(a,b_1,\cdots,\cdots,b_{i-1},b_i,b_{i+1},\cdots,\cdots,b)}\qquad \text{for } E_L \qquad (3.21)$$

$$P(n,u)^{(a,a_1,\cdots,a_{i-1},a_{i-1}-1,a_{i+1},\cdots,b)}_{(a,b_1,\cdots,b_{i-1},b_i,b_{i+1},\cdots,b)}=P(n,u)^{(a,a_1,\cdots,a_{i-1},a_{i-1}+1,a_{i+1},\cdots,b)}_{(a,b_1,\cdots,b_{i-1},b_i,b_{i+1},\cdots,b)}$$
$$\text{for } a_{i-1}=a_{i+1} \text{ and } \text{val}(a_{i-1})=2 \qquad (3.22)$$

Let $p(a,b,n)$ represent the set of all allowed paths of n steps from a to b on the Dynkin diagrams excluding paths, such as in (3.18), which only give zero elements for the projector. Similarly, let $P^{(n)}_{(a,b)}$ be the number of paths in the set $p(a,b,n)$. For convenience let $p(a,b,n)_i$ represent the i-th path in $p(a,b,n)$ and $p(a,b,n)_{i,j}$ be the j-th element of $p(a,b,n)_i$. So we can rewrite the elements of the projector $P(n-1,u)$ to be

$$P(n-1,u)^{p(a,b,n)_i}_{p(a,b,n)_j}$$

The operator $P(n-1,u)$ is a square matrix and can be written in block diagonal form. By the properties (3.20)–(3.22) we may have $|P(n-1,u)^{p(a,b,n)_i}_{p(a,b,n)_k}| = |P(n-1,u)^{p(a,b,n)_j}_{p(a,b,n)_k}|$ or $P(n-1,u)^{p(a,b,n)_i}_{p(a,b,n)_k} = P(n-1,u)^{p(a,b,n)_j}_{p(a,b,n)_k} + P(n-1,u)^{p(a,b,n)_{\overline{j}}}_{p(a,b,n)_k}$ for any path $p(a,b,n)_k$ and suitable j and $\overline{j}$. If so we treat the paths $p(a,b,n)_i$ and $p(a,b,n)_j$ as dependent paths. Otherwise the paths $p(a,b,n)_i$ and $p(a,b,n)_j$ are independent. Suppose there are $m^{(n)}_{(a,b)}$ independent equations deriving from the properties (3.20)–(3.22), then there are $A^{(n)}_{(a,b)} = P^{(n)}_{(a,b)} - m^{(n)}_{(a,b)}$ independent paths in $p(a,b,n)$ where $A^{(n)}_{(a,b)}$ is precisely the element of the fused adjacency matrices given in (3.1). We denote these independent paths by $\alpha(a,b,n)$, $\alpha = 1,2,\cdots,A^{(n)}_{(a,b)}$. There are many ways to choose the independent paths but they all lead to equivalent fused models. The remaining paths should satisfy

$$P(n-1,u)^{p(a,b,n)_i}_{\beta(a,b,n)} = \sum_{\alpha=1}^{A^{(n)}_{(a,b)}} \phi^{(i,\alpha)}_{(a,b,n)}\ P(n-1,u)^{\alpha(a,b,n)}_{\beta(a,b,n)}; \quad i = 1,2,\cdots,m^{(n)}_{(a,b)}. \tag{3.23}$$

$$P(n-1,u)^{a,a'_1,a'_2,\cdots,a'_{n-1},b}_{a,b_1,b_2,\cdots,b_{n-1},b} = 0, \qquad n > h-1 \tag{3.24}$$

The value of $\phi^{(i,\alpha)}_{(a,b,n)}$ is zero if the path $p(a,b,n)_i$ is independent of the path $\alpha(a,b,n)$ and is $+1$ or -1 otherwise. According to (3.23) we can divide $p(a,b,n)$ into $A^{(n)}_{(a,b)}$ independent sets defined by

$$p(n,a,\alpha,b) = \{(p(a,b,n)_i) | \phi^{(i,\alpha)}_{(a,b,n)} \neq 0\}, \quad \alpha = 1,2,\cdots,A^{(n)}_{(a,b)}. \tag{3.25}$$

The first path in $p(n,a,\alpha,b)$ is $\alpha(a,b,n)$, the i-th path is denoted by $p(n,a,\alpha,b)_i$ and $p(n,a,\alpha,b)_{i,j}$ denotes the j-th element of the path $p(n,a,\alpha,b)_i$. We call $\phi^{(i,\alpha)}_{(a,b,n)}$ the parity of the path $p(a,b,n)_i$ relative to the independent path $\alpha(a,b,n)$. By (3.16) it is obvious that

$$\phi^{(\alpha,\alpha)}_{(a,b,n)} = \phi^{(i,i)}_{(a,b,n)} = 1, \tag{3.26}$$

$$\phi^{(i,\alpha)}_{(a,b,n)} = \phi^{(i,\alpha)}_{(b,a,n)}. \tag{3.27}$$

Equation (3.24) holds because all paths in $p(a,b,n)$ with $n > h-1$ are related by (3.22) to $P(n-1,u)^{(a,b_1,\cdots,b_{i-2},b_{i-1},b_i,b_{i+1},b_{i+2},\cdots,b)}_{(a,a_1,\cdots,\cdots,a_{i-1},a_i,a_{i+1},\cdots,\cdots,b)} = 0$ with $\mathrm{val}(b_{i-1}) = \mathrm{val}(b_{i+1}) = 1$.

From (3.20)–(3.22) it follows that the maximum number of terms on the right hand side of (3.23) is two. Let us set $t^\alpha_k = P(n-1,u)^{\alpha(a,b,n)}_{p(a,b,n)_k}$ and $t^i_j = P(n-1,u)^{p(a,b,n)_i}_{p(a,b,n)_j}$. Then in general, by (3.22), we can divide the submatrix $P(n-1,u)^{p(a,b,n)}_{p(a,b,n)}$ of $P(n-1,u)$ into columns

$$\begin{pmatrix} t_1^\beta & \cdots & t_1^\beta & \cdots \\ t_2^\beta & \cdots & t_2^\beta & \cdots \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ t^\beta_{P^{(n)}_{(a,b)}} & \cdots & t^\beta_{P^{(n)}_{(a,b)}} & \cdots \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} t_1^j & \cdots & t_1^j & \cdots \\ t_2^j & \cdots & t_2^j & \cdots \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ t^j_{P^{(n)}_{(a,b)}} & \cdots & t^j_{P^{(n)}_{(a,b)}} & \cdots \end{pmatrix}$$

where $\alpha, \beta = 1, 2, \cdots, A^{(n)}_{(a,b)}$ and t_k^j $(1 \le k \le P^{(n)}_{(a,b)})$ can be expressed as

$$t_k^j = \phi^{(j,\alpha)}_{(a,b,n)} t_k^\alpha + \phi^{(j,\beta)}_{(a,b,n)} t_k^\beta$$

by (3.20)–(3.21). For the A_L models only the first group appears and $A^{(n)}_{(a,b)} = 1$. For the D_L and E_L models the second group is related to first group. It is easy to see that

$$\det P(n-1,u)^{p(a,b,n)}_{p(a,b,n)} = 0 \qquad \text{and} \qquad \det P(n-1,u) = 0$$

This means that the matrix $P(n-1,u)$ or $P(n-1,u)^{p(a,b,n)}_{p(a,b,n)}$ is reducible. The irreducible operator $\wp(n-1,u,a,b)$ is obtained from the reducible one $P(n-1,u)^{p(a,b,n)}_{p(a,b,n)}$ by picking the independent elements as follows

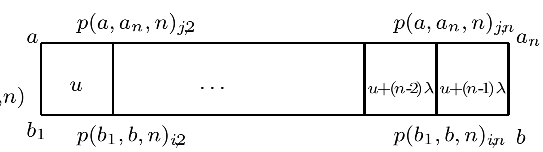
$$\wp(n-1,a,b) = \begin{pmatrix} t_1^1 & t_1^2 & \cdots & t_1^{A^{(n)}_{(a,b)}} \\ t_2^1 & t_2^2 & \cdots & t_2^{A^{(n)}_{(a,b)}} \\ \vdots & \vdots & \ddots & \vdots \\ t^1_{A^{(n)}_{(a,b)}} & t^2_{A^{(n)}_{(a,b)}} & \cdots & t^{A^{(n)}_{(a,b)}}_{A^{(n)}_{(a,b)}} \end{pmatrix} \tag{3.28}$$

where $t^\alpha_\beta = P(n-1,-n\lambda)^{\alpha(a,b,n)}_{\beta(a,b,n)}$. So (3.23) can be written as

$$P(n-1,-n\lambda)^{p(a,b,n)_j}_{\beta(a,b,n)} = \sum_{\alpha=1}^{A^{(n)}_{(a,b)}} \phi^{(j,\alpha)}_{(a,b,n)} \; \wp(n-1,a,b)^{\alpha(a,b,n)}_{\beta(a,b,n)}. \tag{3.29}$$

Finally, using (3.29), the operator (3.14) can be factorized as

$$\sum_{\alpha=1}^{A^{(n)}_{(b_1,a)}} \wp(n-1,b_1,a)^{\alpha(b_1,a,n)}_{\beta(b_1,a,n)} \sum_{i=1}^{L^{(n)}_{(b_1,a)}} \phi^{(i,\alpha)}_{(b_1,a,n)} \tag{3.30}$$



This result implies that the fusion can be carried out if the operator $\wp(n-1,b_1,a)$ is invertible. The existence of the inverse operator $\wp(n-1,b_1,a)^{-1}$ is shown in Section 5.2.

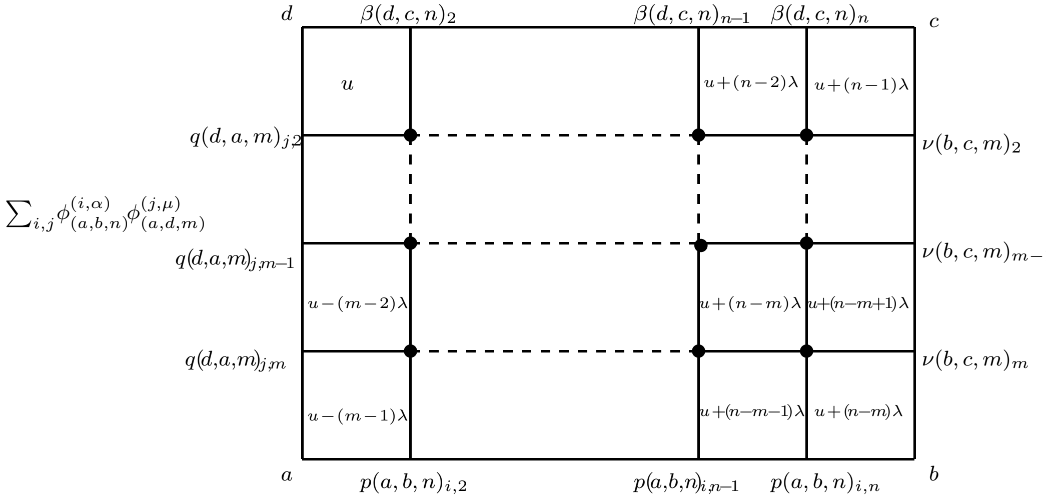
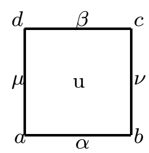


Figure 3: Diagrammatic representation of the face weights of the $m \times n$ fused ADE models. Sites indicated with a solid circle are summed over all possible spin states.

3.4 General Fusion

Let m and n be positive integers and define

$$W_{m\times n}\left(\begin{matrix} d & \beta & c \\ \mu & & \nu \\ a & \alpha & b \end{matrix}\middle|\, u\right) =$$



$$= \sum_{j=1}^{L^{(m)}_{(d,a)}} \phi^{(j,\mu)}_{(a,d,m)} \sum_{\alpha_2,\cdots,\alpha_m} \prod_{k=1}^{m} W_{1\times n}\left(\begin{matrix} p(a,d,m)_{j,k+1} & \alpha_{k+1} & \nu(b,c,m)_{k+1} \\ p(a,d,m)_{j,k} & \alpha_k & \nu(b,c,m)_k \end{matrix}\,\middle|\, u-(m-k)\lambda\right). \qquad (3.31)$$

Here $a = p(a,d,m)_{j,1}$, $b = \nu(b,c,m)_1$, $c = \nu(b,c,m)_{m+1}$, $d = p(a,d,m)_{j,m+1}$, $\alpha = \alpha_1$, $\beta = \alpha_{m+1}$ and the summation over α_k ranges over $\alpha_k = 1,\cdots,A^{(n)}_{(p(a,d,m)_{j,k},\nu(b,c,m)_k)}$. The $1\times n$ fusion in turn is defined by

$$W_{1\times n}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) =$$

$$\sum_{i=1}^{L^{(n)}_{(a,b)}} \phi^{(i,\alpha)}_{(a,b,n)} \prod_{k=1}^{n} W\left(\begin{matrix} \beta(d,c,n)_k & \beta(d,c,n)_{k+1} \\ p(a,b,n)_{i,k} & p(a,b,n)_{i,k+1} \end{matrix}\,\middle|\, u+(k-1)\lambda\right). \qquad (3.32)$$

The fused face weights (3.31) associated with a bond state (a,α,b) are obtained by summing over the dependent paths within the set $p(n,a,\alpha,b)$. A similar idea was first applied for the fusion of the $A^{(1)}_n$ models in [19]. The resulting fused face weights depend on both the spin variables a,b,c,d and the bond variables α,β,μ,ν. For the A_L models these bond variables take only the value 1 whereas they take $A^{(n)}_{(a,b)}$ values for the adjacent spins a,b for the D_L and E_L models. For the A_L models the fused face weights do not change at all if we change the paths $p(m,b,1,c)_1$ to $p(m,b,1,c)_j$ and $p(n,d,1,c)_1$ to $p(n,d,1,c)_i$. But, for the D_L and E_L models, we have the following lemma:

Lemma 2 *If the path $\beta(d,c,n)$ is replaced with its dependent path $p(n,d,\beta,c)_j$ then the fused weight*

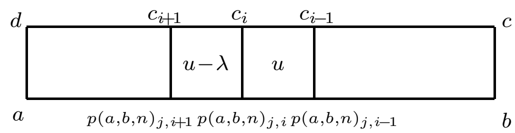
$$W_{m\times n}\left(\begin{array}{ccc|c} d & j & c & \\ \mu & & \nu & u \\ a & \alpha & b & \end{array}\right) = \sum_{\beta'=1}^{A^{(n)}_{(d,c)}} \phi^{(j,\beta')}_{(d,c,n)}\, W_{m\times n}\left(\begin{array}{ccc|c} d & \beta' & c & \\ \mu & & \nu & u \\ a & \alpha & b & \end{array}\right) \tag{3.33}$$

Similarly, if the path $\nu(b,c,m)$ is replaced by its dependent path $p(m,b,\nu,c)_j$ then

$$W_{m\times n}\left(\begin{array}{ccc|c} d & \beta & c & \\ \mu & & j & u \\ a & \alpha & b & \end{array}\right) = \sum_{\nu'=1}^{A^{(m)}_{(b,c)}} \phi^{(j,\nu')}_{(b,c,m)}\, W_{m\times n}\left(\begin{array}{ccc|c} d & \beta & c & \\ \mu & & \nu' & u \\ a & \alpha & b & \end{array}\right). \tag{3.34}$$

Proof: Let us first consider $1\times n$ fusion

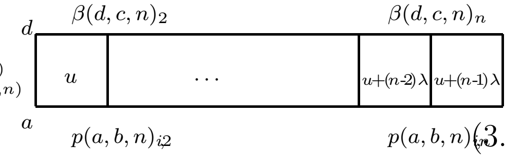
$$\sum_j^{L^{(n)}_{(a,b)}} \phi^{(i,\alpha)}_{(a,b,n)}$$



(3.35)

From (3.30) it follows that the indices (c_{i+1}, c_i, c_{i-1}) of the weight (3.35) satisfy the properties (3.9)–(3.12) (or (3.19)–(3.22)). This means that some of the fused weights in (3.35) are dependent. In total there are $A^{(n)}_{(c,d)}$ independent paths in the set $p(d,c,n)$. Choosing an independent path $c_i = \beta(d,c,n)$ we have the $1\times n$ fused face weight

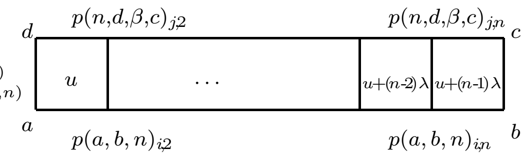
$$W_{1\times n}\left(\begin{array}{ccc|c} d & \beta & c & \\ a & \alpha & b & u \end{array}\right) = \sum_{i}^{P^{(n)}_{(a,b)}} \phi^{(i,\alpha)}_{(a,b,n)} \sum_i^{L^{(n)}_{(a,b)}} \phi^{(i,\alpha)}_{(a,b,n)}$$



(3.36)

where $\alpha = 1, 2, \cdots, A^{(n)}_{(a,b)}$ and $\beta = 1, 2, \cdots, A^{(n)}_{(c,d)}$. These represent the independent fused face weights. The others can be obtained from the independent weights via the relation

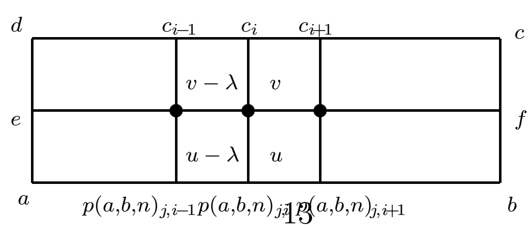
$$W_{1\times n}\left(\begin{array}{ccc|c} d & j & c & \\ a & \alpha & b & u \end{array}\right) = \sum_{i}^{P^{(n)}_{(a,b)}} \phi^{(i,\alpha)}_{(a,b,n)} \sum_i^{L^{(n)}_{(a,b)}} \phi^{(i,\alpha)}_{(a,b,n)}$$



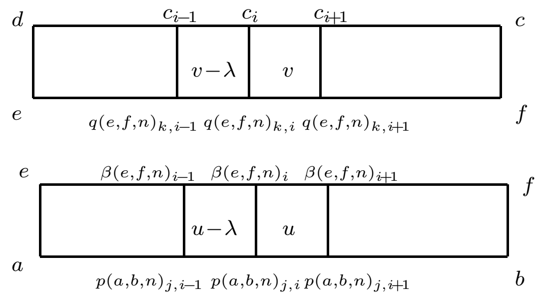
$$= \sum_{\beta'=1}^{A^{(n)}_{(d,c)}} \phi^{(j,\beta')}_{(d,c,n)}\, W_{1\times n}\left(\begin{array}{ccc|c} d & \beta' & c & \\ a & \alpha & b & u \end{array}\right) \tag{3.37}$$

where $\phi^{(j,\beta')}_{(d,c,n)}$ are the parities of the path $p(n,d,\beta,c)_j$ relative to the dependent paths $\beta'(d,c,n)$ with $\beta' = 1, \cdots, \beta, \cdots, A^{(n)}_{(d,c)}$. These properties are exactly the same as (3.29). Furthermore, we have the following push through property from (3.37)

$$\sum_j^{L^{(n)}_{(a,b)}} \phi^{(j,\alpha)}_{(a,b,n)}$$



$$= \sum_{\beta=1}^{A^{(n)}_{(e,f)}} \quad \sum_{k}^{L^{(n)}_{(e,f)}} \phi^{(k,\beta)}_{(e,f,n)} \quad \sum_{j}^{L^{(n)}_{(a,b)}} \phi^{(j,\alpha)}_{(a,b,n)} \tag{3.38}$$



This relation and (3.37) imply (3.33). Moreover, (3.34) follows from (3.33) because of the symmetry $W\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = W\left(\begin{matrix} b & c \\ a & d \end{matrix}\,\middle|\, u\right)$.

By repeated use of (2.10), and with the help of the Lemma 2, we obtain the following theorem:

Theorem 1 *For a triple of positive inters m, n, l, the fused face weights (3.31) satisfy the Yang-Baxter equation*

$$\begin{aligned} &\sum_{(\eta_1,\eta_2,\eta_3)} \sum_{g} W_{l\times n}\left(\begin{matrix} f & \eta_1 & g \\ \rho & & \eta_2 \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) W_{m\times l}\left(\begin{matrix} d & \gamma & c \\ \eta_3 & & \beta \\ g & \eta_2 & b \end{matrix}\,\middle|\, v-u\right) W_{m\times n}\left(\begin{matrix} e & \mu & d \\ \nu & & \eta_3 \\ f & \eta_1 & g \end{matrix}\,\middle|\, v\right) \\ &= \sum_{(\eta_1,\eta_2,\eta_3)} \sum_{g} W_{l\times n}\left(\begin{matrix} e & \mu & d \\ \eta_2 & & \gamma \\ g & \eta_1 & c \end{matrix}\,\middle|\, u\right) W_{m\times l}\left(\begin{matrix} e & \eta_2 & g \\ \nu & & \eta_3 \\ f & \rho & a \end{matrix}\,\middle|\, v-u\right) W_{m\times n}\left(\begin{matrix} g & \eta_1 & c \\ \eta_3 & & \beta \\ a & \alpha & b \end{matrix}\,\middle|\, v\right). \end{aligned} \tag{3.39}$$

By Lemma 1 the weights $W_{m\times n}$ (3.31) have zeros independent of the spins a, b, c, d and bond variables α, β, μ, ν. To remove these zeros we replace the (M, N) fused weight by

$$W_{m\times n}\left(\begin{matrix} d & \beta & c \\ \mu & & \nu \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) \rightarrow W_{m\times n}\left(\begin{matrix} d & \beta & c \\ \mu & & \nu \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) \prod_{k=0}^{n-2} \prod_{j=0}^{m-1} \frac{\sin\lambda}{\sin[u+(k-j)\lambda]}. \tag{3.40}$$

By construction it is obvious that $W_{m\times n}\left(\begin{matrix} d & \beta & c \\ \mu & & \nu \\ a & \alpha & b \end{matrix}\,\middle|\, u\right)$ vanishes unless

$$\begin{aligned} A^{(n)}_{a,b} \neq 0 &\quad \text{and} \quad \alpha = 1, 2, \cdots, A^{(n)}_{a,b} \\ A^{(n)}_{d,c} \neq 0 &\quad \text{and} \quad \beta = 1, 2, \cdots, A^{(n)}_{d,c} \\ A^{(m)}_{d,a} \neq 0 &\quad \text{and} \quad \mu = 1, 2, \cdots, A^{(m)}_{d,a} \\ A^{(m)}_{c,b} \neq 0 &\quad \text{and} \quad \nu = 1, 2, \cdots, A^{(m)}_{c,b} \end{aligned} \tag{3.41}$$

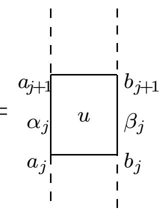
where the fused adjacency matrices are given by (3.1). In particular,

$$W_{m\times n}\left(\begin{array}{ccc|c} d & \beta & c & \\ \mu & & \nu & u \\ a & \alpha & b & \end{array}\right) = 0 \qquad \text{if} \quad n = h-1 \quad \text{or} \quad m = h-1\,. \tag{3.42}$$

4 Row Transfer Matrix Fusion Hierarchy

Suppose that $\boldsymbol{a}(\boldsymbol{\alpha})$ and $\boldsymbol{b}(\boldsymbol{\beta})$ are allowed spin (bond) configurations of two consecutive rows of a lattice with N columns and periodic boundary conditions. The elements of the fused row transfer matrices $\mathbf{T}^{(m,n)}(u)$ of the fused A–D–E models are given by

$$\langle \boldsymbol{a}, \boldsymbol{\alpha}|\mathbf{T}^{m,n}(u)|\boldsymbol{b}, \boldsymbol{\beta}\rangle = \prod_{j=1}^{N}\sum_{\eta_j} W_{m\times n}\left(\begin{array}{ccc|c} a_{j+1} & \eta_{j+1} & b_{j+1} & \\ \alpha_j & & \beta_j & u \\ a_j & \eta_j & b_j & \end{array}\right) = \tag{4.1}$$



where $a_{N+1} = a_1$, $b_{N+1} = b_1$ and $\eta_{N+1} = \eta_1$. Specifically, the Yang-Baxter equations (3.39) imply the commutation relations

$$[\mathbf{T}^{m,n}(u), \mathbf{T}^{m,n'}(v)] = 0. \tag{4.2}$$

Thus if m is held fixed we obtain a hierarchy of commuting families of transfer matrices. These transfer matrices satisfy the following remarkable functional equations:

Theorem 2 (Fusion Hierarchy) *Let us define*

$$\mathbf{T}^{m,n}_k = \mathbf{T}^{m,n}(u+k\lambda), \quad \mathbf{T}^{m,0}_0 = f^m_{-1}\mathbf{I}, \quad f^m_n = [s^m_n]^N \tag{4.3}$$

and

$$s^n_k = \prod_{j=0}^{n-1} \frac{\sin[u+(k-j)\lambda]}{\sin\lambda}. \tag{4.4}$$

Then

$$\mathbf{T}^{m,n}_0 \mathbf{T}^{m,1}_n = f^m_n \mathbf{T}^{m,n-1}_0 + f^m_{n-1}\mathbf{T}^{m,n+1}_0 \tag{4.5}$$

where the hierarchy closes at fusion level $h-1$ with

$$\mathbf{T}^{p,h-1} = 0. \tag{4.6}$$

Theorem 3 (TBA Hierarchy) *If we further define*

$$\mathbf{t}^{m,n}_0 = \frac{\mathbf{T}^{m,n+1}_0 \mathbf{T}^{m,n-1}_1}{f^m_{-1} f^m_n}. \tag{4.7}$$

Then the thermodynamic Bethe ansatz equations

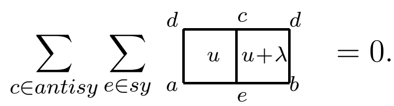
$$\mathbf{t}^{m,n}_0 \mathbf{t}^{m,n}_1 = (\mathbf{I} + \mathbf{t}^{m,n+1}_0)(\mathbf{I} + \mathbf{t}^{m,n-1}_1) \tag{4.8}$$

hold where

$$\mathbf{t}^{m,0}_0 = \mathbf{t}^{m,h-2}_0 = 0. \tag{4.9}$$

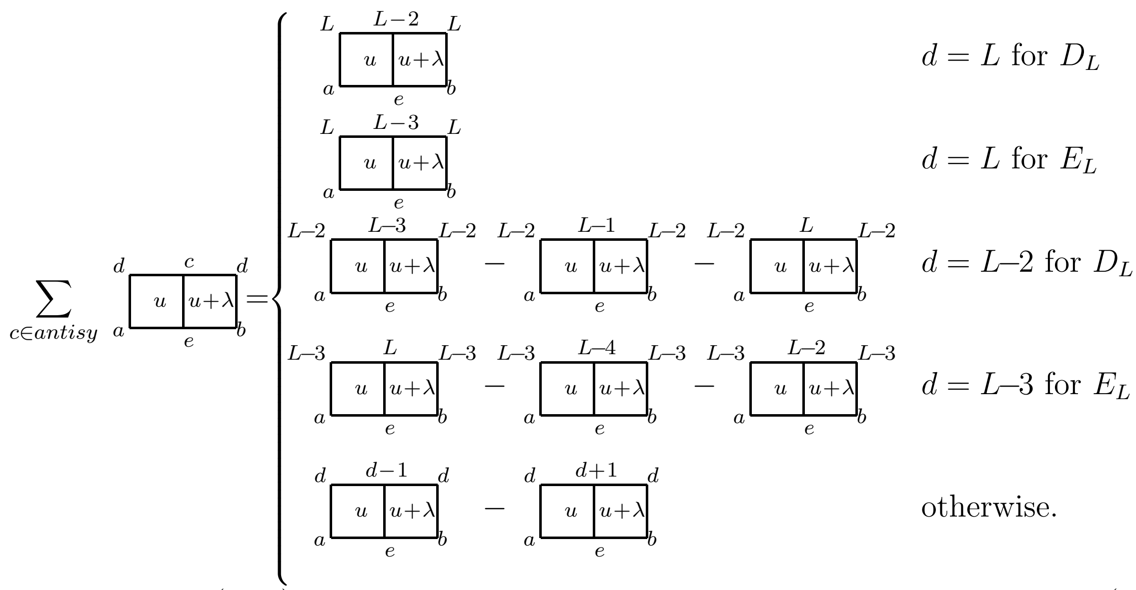
The main purpose of this section is to prove these theorems. Clearly, the functional equations for the D_L and E_L models are the same as those for the A_L models. In the A_L case the fusion hierarchy of functional equations was obtained by Bazhanov and Reshetikhin [9]. Although intertwiners can be constructed [15] between the row transfer matrices of the D or E models and an associated A model, these intertwiners do not relate all eigenvalues. Rather, only a subset of common eigenvalues are intertwined. As a consequence, the functional relations of the D_L and E_L models cannot be obtained from those of the A_L models using intertwiners alone. Instead it is necessary to prove these functional equations directly for the D_L and E_L models as is done here.

In Section 3 we described fusion of the A–D–E models corresponding to the symmetric representation of the tensor products of n elementary blocks. To prove the theorems we need the fusion procedure corresponding to antisymmetric representations. We therefore now describe the antisymmetric fusion of the tensor product of 2 elementary blocks. The symmetric and antisymmetric fusion procedures are orthogonal to each other in the sense that

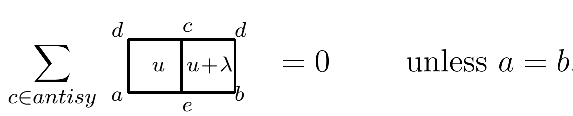


(4.10)

From (3.10)–(3.12) we can indeed see that (4.10) holds where the antisymmetric sum is defined by

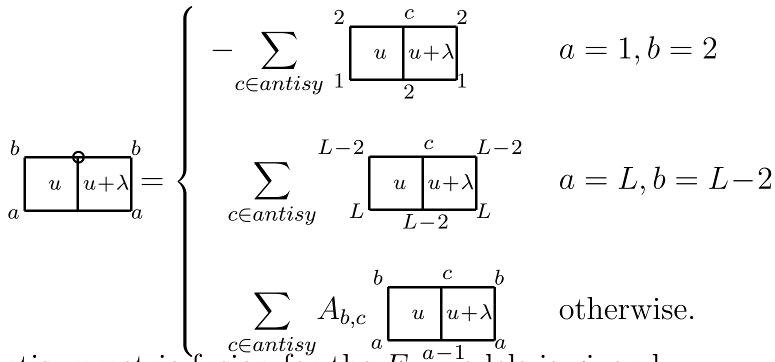


Furthermore (4.10) imples that (4.11)



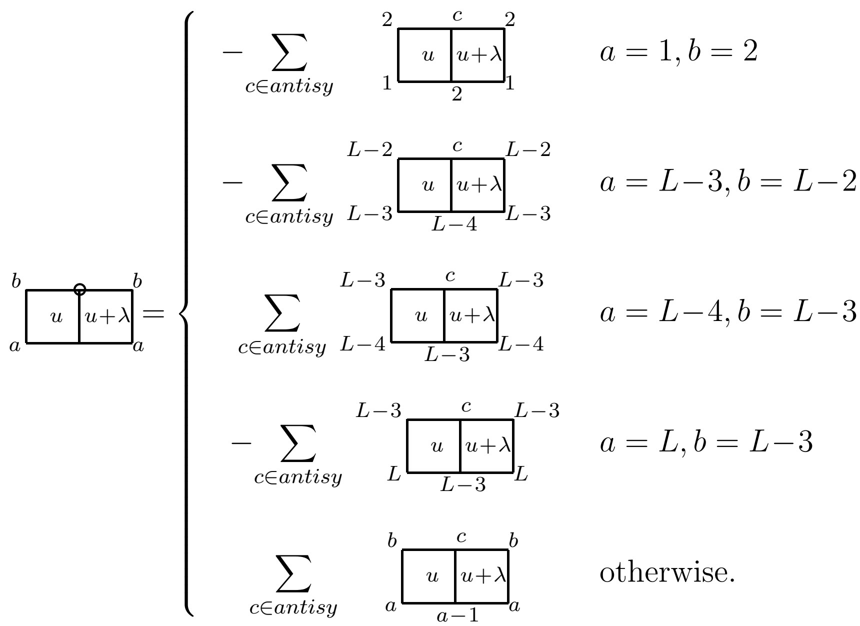
(4.12)

Hence, for the D_L models, we can construct the antisymmetric fusion by



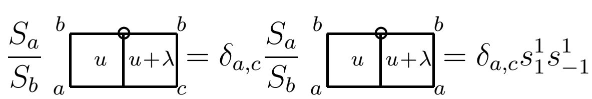
(4.13)

Similarly, the antisymmetric fusion for the E_L models is given by



(4.14)

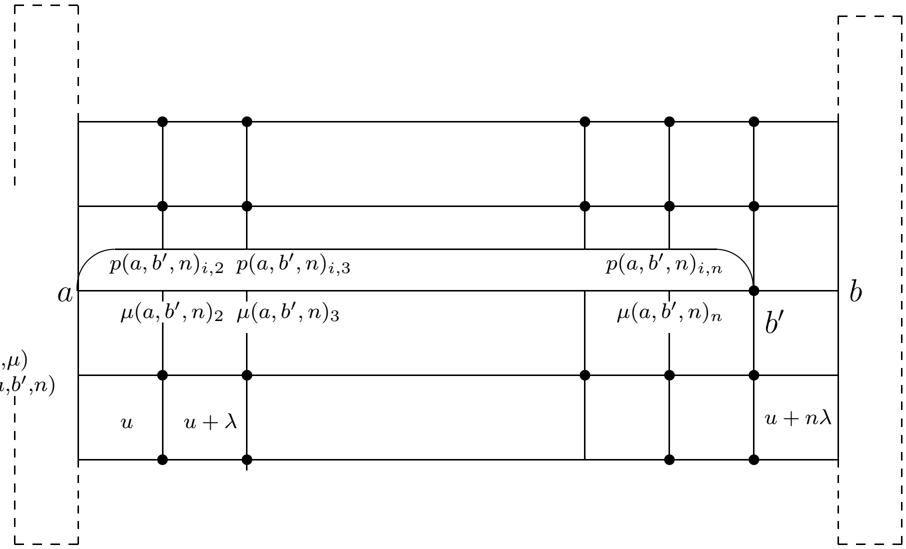
By direct calculation we have



(4.15)

Proof of Theorem 2: For simplicity we prove the functional equations only for the case of $m=1$. The general case can be proved similarly. Representing $\mathbf{T}_0^{1,n}\mathbf{T}_n^{1,1}$ graphically as

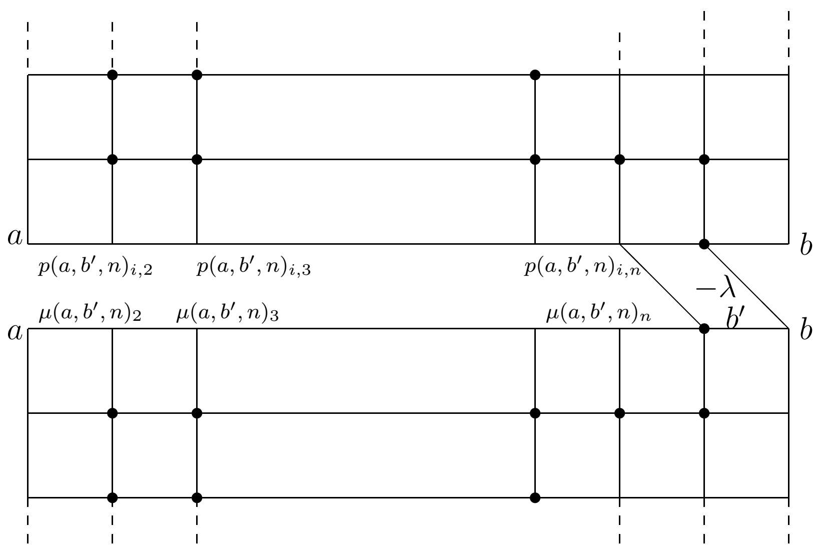
$$\sum_{i=1}^{L_{(a,b')}^{(n)}} \sum_{\mu=1}^{A_{(a,b')}^{(n)}} \times \phi_{(a,b',n)}^{(i,\mu)}$$



(4.16)

and inserting (2.21) we obtain the sum of two terms:

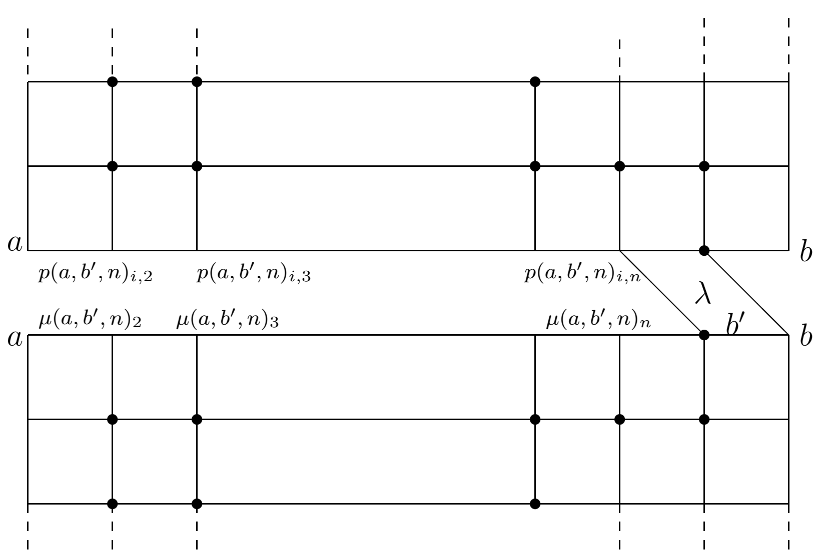
$$\sum_{i=1}^{L_{(a,b')}^{(n)}} \sum_{\mu=1}^{A_{(a,b')}^{(n)}} \phi_{(a,b',n)}^{(i,\mu)} (2\cos\lambda)^{-1}$$



(4.17)

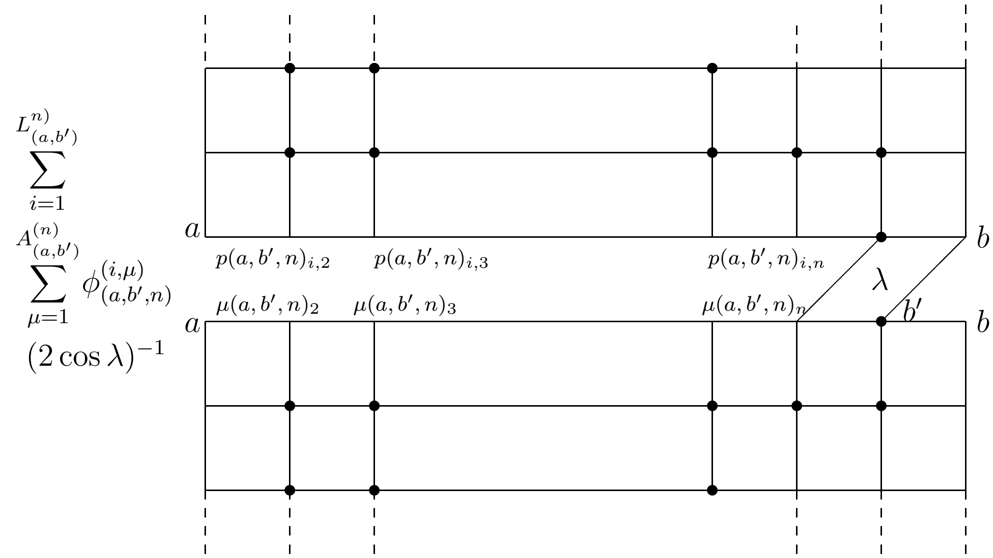
and

$$\sum_{i=1}^{L_{(a,b')}^{n)}} \sum_{\mu=1}^{A_{(a,b')}^{(n)}} \phi_{(a,b',n)}^{(i,\mu)} (2\cos\lambda)^{-1}$$



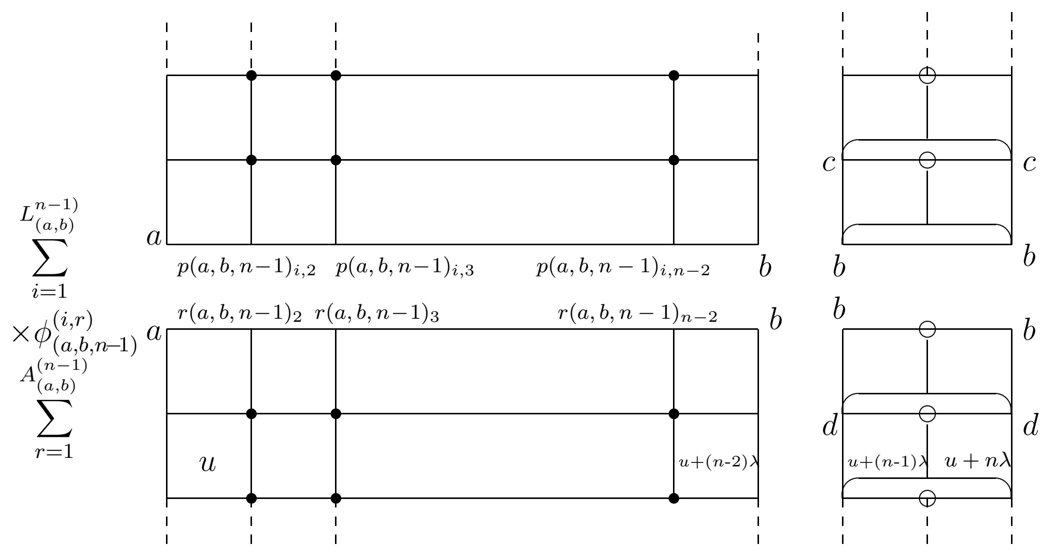
(4.18)

But now, by (2.12), the second term (4.18) vanishes unless $p(a,b',n)_{i,n} = b$. In this case we can choose an independent path with $\mu(a,b',n)_n = b$ so (4.18) becomes



(4.19)

Using (2.12)–(2.13), (2.20)–(2.23) and (4.13)–(4.14), this can be reduced to



(4.20)

By virtue of (4.15) this gives the first term $f_n^1 \mathbf{T}_0^{1,n-1}$ in the fusion hierarchy. From the push through property (3.38) and (3.15) of the 1×2 fusion we can see that the path of 3 steps $(\mu(a,b',n)_n, b', b)$ in (4.17) satisfies the properties (3.9)–(3.12). This together with the push through property (3.38) ensures that the path of $n+1$ steps from a to b to satisfies (3.19)–(3.22). Applying the push through property (3.38) to the $n+1$ blocks we obtain the level $n+1$ fusion transfer matrix given by the second term $f_{n-1}^1 \mathbf{T}_0^{1,n+1}$.

Proof of Theorem 3: Following Klümper and Pearce [10] the functional equations

$$\mathbf{T}_0^{m,n} \mathbf{T}_1^{m,n} = f_{-1}^m f_n^m \mathbf{I} + \mathbf{T}_0^{m,n+1} \mathbf{T}_1^{m,n-1} \tag{4.21}$$

are derived by substituting the fusion hierarchy (4.5) into the identity

$$\mathbf{T}_0^{m,n} (\mathbf{T}_1^{m,n-1} \mathbf{T}_n^{m,1}) = (\mathbf{T}_0^{m,n} \mathbf{T}_n^{m,1}) \mathbf{T}_1^{m,n-1} \ . \tag{4.22}$$

This then yields

$$\begin{aligned}
\mathbf{t}_0^{m,n}\mathbf{t}_1^{m,n} &= \frac{(\mathbf{T}_1^{m,n-1}\mathbf{T}_2^{m,n-1})(\mathbf{T}_0^{m,n+1}\mathbf{T}_1^{m,n+1})}{f_0^m f_n^m f_{-1}^m f_{n+1}^m} \\
&= \left(\mathbf{I}+\frac{\mathbf{T}_1^{m,n}\mathbf{T}_2^{m,n-2}}{f_0^m f_n^m}\right)\left(\mathbf{I}+\frac{\mathbf{T}_0^{m,n+2}\mathbf{T}_1^{m,n}}{f_{-1}^m f_{n+1}^m}\right) \\
&= (\mathbf{I}+\mathbf{t}_1^{m,n-1})(\mathbf{I}+\mathbf{t}_0^{m,n+1}). \qquad (4.23)
\end{aligned}$$

The functional equations (4.8) are identical in form to the equations of the thermodynamic Bethe ansatz [20, 21, 22, 23]. The fusion hierarchy for the A_L has been solved [10] for the finite-size corrections and hence the central charges, scaling dimensions and critical exponents. A similar analysis can be carried out for the D_L and E_L models.

5 Intertwiners and Symmetric Fused Weights

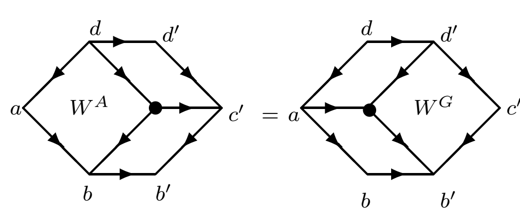
Here we extend the A–D–E intertwiners constructed in [15] to the fused A–D–E models. We build symmetric fused face weights and generalize the intertwining relation to apply directly to the symmetric face weights. We also construct the intertwiners between the row transfer matrices of the fused A–D–E models.

5.1 Intertwiners

Let A and G be adjacency matrices of an A and a D or E model respectively. These are square matrices with nonnegative integer elements. Then the adjacency matrix C is said to intertwine A and G if

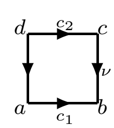
$$AC = CG. \qquad (5.1)$$

In general C is a rectangular matrix with nonegative integer elements. Similarly, there is an intertwining relation between the symmetric face weights W^A of A model and the symmetric face weights W^G of the D or E models if [15],



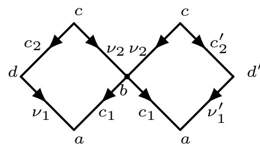
(5.2)

where

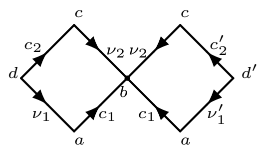


(5.3)

is a family of cells labelled by four bond variables. Here the cells vanish unless the spins d, a are adjacent sites of A, the spins c, b are adjacent sites of G and the spins a, b and d, c are adjacent sites of the intertwining graph C. The bond variables $c_1(c_2) = 1, 2, \cdots, C_{a,b}(C_{d,c})$, $\nu = 1, 2, \cdots, G_{c,b}$. These cells satisfy two unitarity conditions which can be written in the form

$$\sum_{(b,c_1,\nu_2)}$$ 

$$= \delta_{d,d'}\delta_{\nu_1,\nu_1'}\delta_{c_2,c_2'} \tag{5.4}$$

$$\sum_{(b,c_1,\nu_2)}$$ 

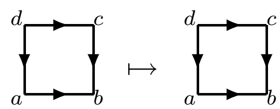
$$\frac{S_bS_d}{S_cS_a} = \delta_{d,d'}\delta_{\nu_1,\nu_1'}\delta_{c_2,c_2'} \; . \tag{5.5}$$

Using the adjacency intertwining relation (5.1) and the fusion rules (3.1) it follows that the same intertwining relations hold between the fused adjacency matrices, that is,

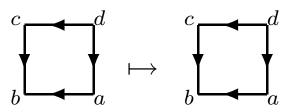
$$A^{(n)}C = CG^{(n)}. \tag{5.6}$$

We therefore expect to find fused cells that intertwine between the fused face weights.

Let us perform the following gauge transformations for the cells

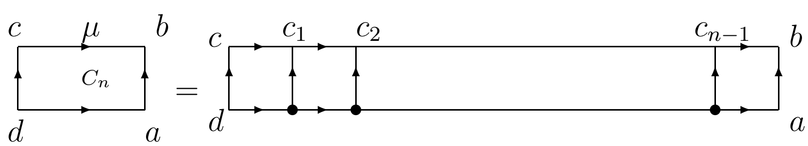


$$\sqrt{\frac{S_c^G}{S_a^A}\frac{f_c^G}{f_a^A}} \tag{5.7}$$



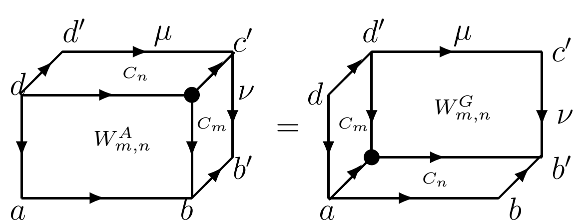
$$\sqrt{\frac{S_d^A}{S_b^G}\frac{f_d^A}{f_a^G}} \tag{5.8}$$

Here we do not need the bond variables because they take the value 1 for unfused face weights. The transformed cells can be fused in the same way as the A models. The level n fusion of the transformed cells (5.7) is given by



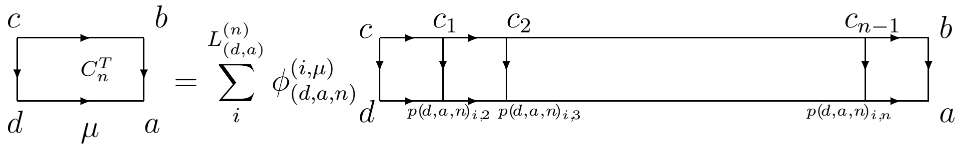
$$\tag{5.9}$$

where the solid circles indicate a summation over all possible paths $p(d,a,n)$ of the A model. The fused cell satisfies the same properties with respect to the path $p(c,b,n)$ of the G model as does the operator P presented in (3.19)–(3.22). We can therefore restrict our attention to the independent paths from c to b of the G model with $c_i = \mu(c,b,n)_{i+1}$. Applying the intertwining relation (5.2) to the $m \times n$ blocks we therefore obtain the intertwining relation between the fused weights $W^A_{m,n}$ and $W^G_{m,n}$ given by (3.31)



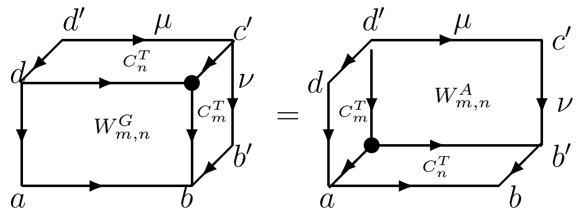
$$\tag{5.10}$$

Here summation is implied over each of the inner bond and spin variables. Alternatively, we can fuse the transformed cells (5.8) giving



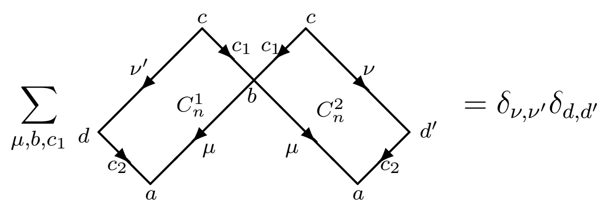
(5.11)

The path $p(c,b,n)$ in the fused cell is satisfies the same properties as the operator P of the A model presented in (3.19) and (3.22). That is, it is independent of $c_1, c_2, \cdots, c_{n-1}$ if the fused cell is nonzero. We thus have another intertwining relation for $W^A_{m,n}$ and $W^G_{m,n}$

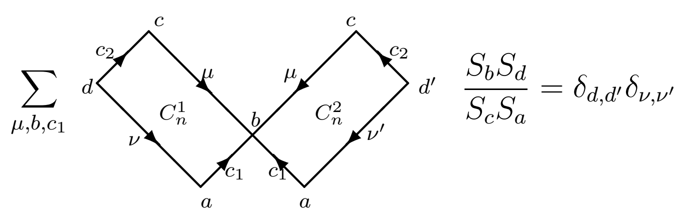


(5.12)

where again summations are implied over inner bond and spin variables. From the unitarity conditions (5.4)–(5.5) it is easy to check the unitarity conditions for the fused cells



(5.13)



(5.14)

where (C^1_n, C^2_n) is (C_n, C^T_n) or (C^T_n, C_n). The bond variables $\nu = \nu' = 1$ for $(C^1_n, C^2_n) = (C_n, C^T_n)$ because there is only one independent path between two spins of the fused A models. In such cases we discard the bond variable between adjacent spins.

The fused cells are given by both (5.9) and (5.11). They give the intertwining relations (5.10) and (5.12) respectively, either of which can be taken as the intertwining relation between the fused face weights of the A and D or E models. However, the fused cells (5.9) and (5.11) are independent. Since we need both the fused cells and their conjugates, the fused weights of the D or E models cannot be obtained from those of the A model and the fused cells (5.9) alone.

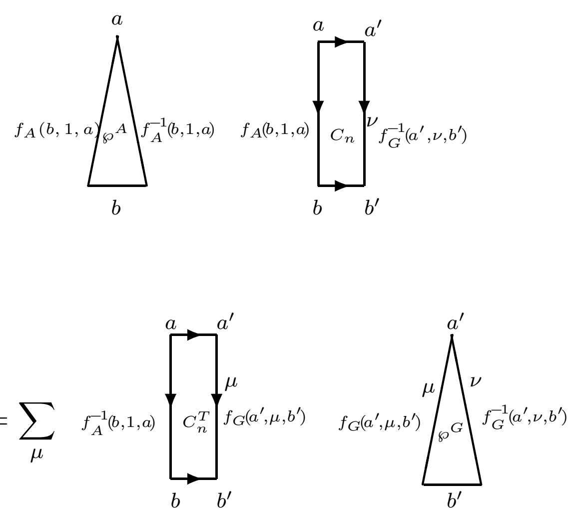
5.2 Symmetric weights

The fused face weights given by (3.31) are not symmetric, that is,

$$W^s_{m\times n}\left(\begin{matrix} d & \beta & c \\ \mu & & \nu \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) \neq W^s_{n\times m}\left(\begin{matrix} d & \mu & a \\ \beta & & \alpha \\ c & \nu & b \end{matrix}\,\middle|\, u\right). \tag{5.15}$$

To symmetrize the fused face weights we need to apply a gauge transformation.

Although the operator $P(n-1,u,a,b)$ does not have an inverse matrix, the $A^{(n)}_{(a,b)} \times A^{(n)}_{(a,b)}$ matrix $\wp(n-1,u,a,b)$ is nonsingular. This can be shown using intertwiners. Specifically, from (5.2), (5.9)–(5.11) and the properties (3.19)–(3.22) we have an intertwining relation between the operators $\wp^A$ and the $\wp^G$,

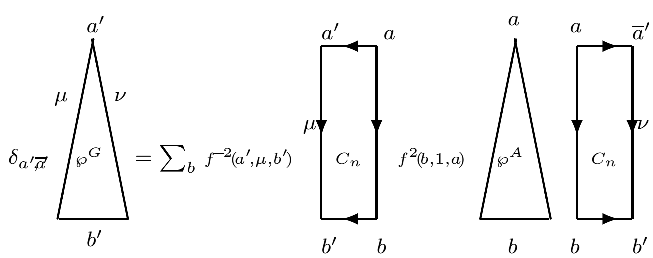


(5.16)

Here we have expressed the operator $\wp(n-1,-(1-n)\lambda,a,b)^{\nu(a,b,n)}_{\mu(a,b,n)}$ graphically as a triangle with

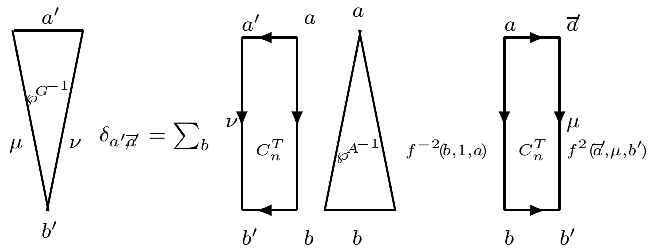
$$f_A(a,\mu,b) = \prod_{i=1}^{n} \sqrt{S^A_{\mu(a,b,n)_i}}\, f^A_{\mu(a,b,n)_i},$$
$$f_G(a,\mu,b) = \prod_{i=1}^{n} \sqrt{S^G_{\mu(a,b,n)_i}}\, f^G_{\mu(a,b,n)_i}.$$

From these equations, and with the help of (5.13)–(5.14), we can easily obtain



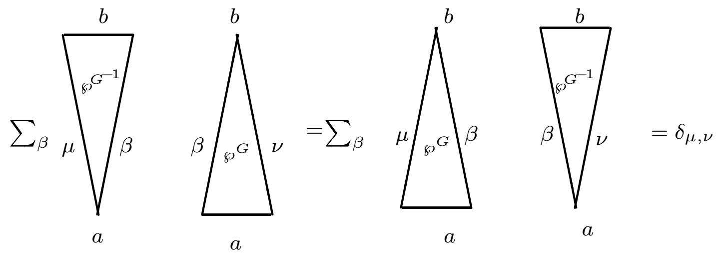
(5.17)

and the inverse of the operator $\wp$



(5.18)

As a result we have shown that $\wp(n-1,-(n-1)\lambda,a,b)$ is nonsingular



(5.19)

We use the square root of this operator to build the symmetric face weights from the unsymmetric ones.

Theorem 4 *Define the* $A^{(n)}_{(a,b)} \times A^{(n)}_{(a,b)}$ *matrix*

$$G(a,b,n) = \sqrt{F(a,b,n)\wp(n-1,-(n-1)\lambda,a,b)F(a,b,n)}, \tag{5.20}$$

where $F(a,b,n)$ *is the diagonal matrix*

$$F(a,b,n) = Diag\,[\; f(a,1,b),\cdots,f(a,A^{(n)}_{(a,b)},b)\;]. \tag{5.21}$$

Then the symmetric weights

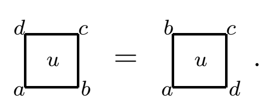
$$W^s_{m\times n}\left(\begin{array}{ccc|c} d & \beta & c & \\ \mu & & \nu & u \\ a & \alpha & b & \end{array}\right) = \sum_{\alpha',\nu',\beta',\mu'} \frac{G(d,a,n)_{\mu,\mu'}G(a,b,n)_{\alpha,\alpha'}}{G(c,b,n)_{\nu',\nu}G(d,c,n)_{\beta',\beta}}\, W^s_{m\times n}\left(\begin{array}{ccc|c} d & \beta' & c & \\ \mu' & & \nu' & u \\ a & \alpha' & b & \end{array}\right) \tag{5.22}$$

satisfy

$$W^s_{m\times n}\left(\begin{array}{ccc|c} d & \beta & c & \\ \mu & & \nu & u \\ a & \alpha & b & \end{array}\right) = W^s_{n\times m}\left(\begin{array}{ccc|c} d & \mu & a & \\ \beta & & \alpha & u \\ c & \nu & b & \end{array}\right) \tag{5.23}$$

$$= W^s_{n\times m}\left(\begin{array}{ccc|c} b & \nu & c & \\ \alpha & & \beta & u-(n-m)\lambda \\ a & \mu & d & \end{array}\right). \tag{5.24}$$

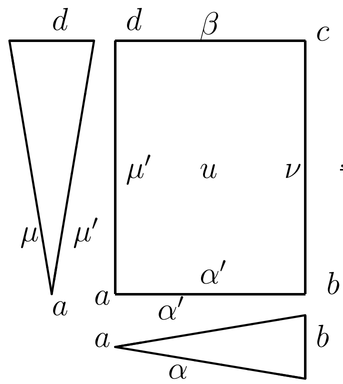
Proof: The symmetry (5.24) is implied by the symmetry of the elementary face



(5.25)

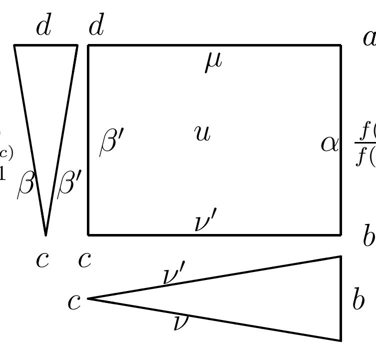
The symmetry (5.23) follows from the equality

$$\sum_{\alpha'=1}^{A^{(n)}_{(a,b)}} \sum_{\mu'=1}^{A^{(m)}_{(d,a)}}$$



$$\frac{f(d,\mu,a)f(a,\alpha,b)}{f(d,\beta,c)f(c,\nu,b)}$$

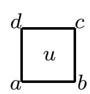
$$= \sum_{\nu'=1}^{A^{(m)}_{(c,b)}} \sum_{\beta'=1}^{A^{(n)}_{(d,c)}}$$



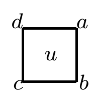
$$\frac{f(d,\beta,c)f(c,\nu,b)}{f(d,\mu,a)f(a,\alpha,b)}$$

(5.26)

which follows from the Yang-Baxter equation (2.10) and



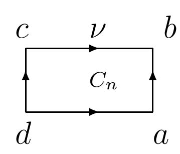
$$= \frac{S_c f_c}{S_a f_a}$$



. (5.27)

It should be noted that this gauge transformation is different from that used by Date et al [3] for the fused A models. Obviously, the symmetric fused weights $W^{s,A}_{m\times n}$ and $W^{s,G}_{m\times n}$ satisfy the intertwining relations (5.10) and (5.12) with the cells replaced by

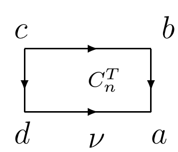
$$\sum_{\nu} \frac{G^A(d,a,n)}{G^G(c,b,n)_{\nu,\mu}}$$



(5.28)

and

$$\sum_{\nu} \frac{G^G(d,a,n)_{\mu,\nu}}{G^A(a,b,n)}$$



(5.29)

5.3 Row Transfer Matrix Intertwiners

In this section we study intertwiners relating the row transfer matrices of the fused A–D–E models.

It is easy to show [15] that the adjacency intertwining relation (5.6)

$$A^{(n)} \overset{C}{\sim} G^{(n)} \tag{5.30}$$

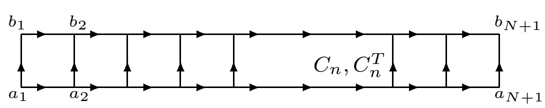
is an equivalence relation among symmetric matrices. The existence of an intertwiner reflects a symmetry relating the two graphs associated with $A^{(n)}$ and $G^{(n)}$. In particular, we observe that the intertwining relation implies that

$$[CC^T, A^{(n)}] = [C^T C, G^{(n)}] = 0 \tag{5.31}$$

so that the symmetry operators CC^T and C^TC commute with $A^{(n)}$ and $G^{(n)}$ respectively and their eigenvalues can be used as quantum numbers labelling the eigenvectors of $A^{(n)}$ and $G^{(n)}$.

The above properties of intertwiners at the adjacency matrix level carry over to those at the row transfer matrix level. Let us introduce cell row transfer matrices with fused cells C_n, C_n^T

$$\langle \boldsymbol{a}|\mathcal{C}_{(n)}|\boldsymbol{b}\rangle, \langle \boldsymbol{a}|\mathcal{C}^T_{(n)}|\boldsymbol{b}\rangle \ =$$

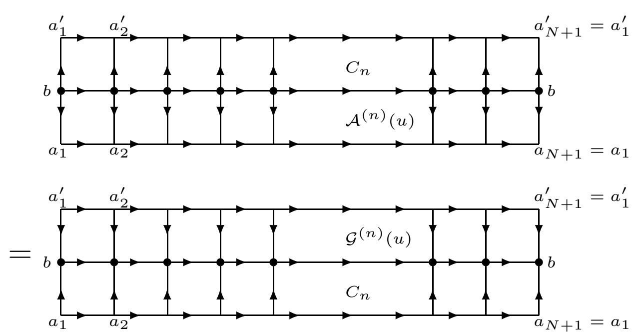


(5.32)

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are allowed row configurations of the graphs $A^{(n)}$ and $G^{(n)}$ with periodic boundary conditions $a_{N+1} = a_1$ and $b_{N+1} = b_1$. In general, the row intertwiner is a rectangular matrix. Using the cell intertwiner relations (5.10) and (5.12) it can be shown that

$$\mathcal{A}^{(n)}(u)\mathcal{C}_{(n)} = \mathcal{C}_{(n)}\mathcal{G}^{(n)}(u) \ \text{ or } \ \mathcal{A}^{(n)}(u) \overset{\mathcal{C}_{(n)}}{\sim} \mathcal{G}^{(n)}(u), \tag{5.33}$$

where $\mathcal{A}^{(n)} = \mathbf{T}_A^{(n,m)}(u)$ and $\mathcal{G}^{(n)} = \mathbf{T}_G^{(n,m)}(u)$ are the row transfer matrices of two fused models. This interwining relation can be pictured as follows



(5.34)

where a solid circle indicates a summation over the corresponding spin.

This intertwining relation is (i) reflexive, (ii) symmetric and (iii) transitive, that is, the intertwining relation is again an equivalence relation

$$\begin{aligned}
&\text{(i)}\quad \mathcal{A}^{(n)}(u) \stackrel{I}{\sim} \mathcal{A}^{(n)}(u) \\
&\text{(ii)}\quad \mathcal{A}^{(n)}(u) \stackrel{\mathcal{C}_{(n)}}{\sim} \mathcal{G}^{(n)}(u) \quad \text{implies} \quad \mathcal{G}^{(n)}(u) \stackrel{\mathcal{C}^T_{(n)}}{\sim} \mathcal{A}^{(n)}(u) \\
&\text{(iii)}\quad \mathcal{A}^{(n)}(u) \stackrel{\mathcal{C}_{(n)}}{\sim} \mathcal{B}^{(n)}(u) \quad \text{and} \quad \mathcal{B}^{(n)}(u) \stackrel{\mathcal{C}'_{(n)}}{\sim} \mathcal{G}^{(n)}(u) \\
&\qquad\qquad \text{implies} \quad \mathcal{A}^{(n)}(u) \stackrel{\mathcal{C}_{(n)}\mathcal{C}'_{(n)}}{\sim} \mathcal{G}^{(n)}(u).
\end{aligned} \tag{5.35}$$

and moreover

$$[\mathcal{C}_{(n)}\mathcal{C}^T_{(n)}\ ,\ \mathcal{A}^{(n)}(u)] = [\mathcal{C}^T_{(n)}\mathcal{C}_{(n)}\ ,\ \mathcal{G}^{(n)}(u)] = 0. \tag{5.36}$$

Hence the symmetry operators $\mathcal{C}_{(n)}\mathcal{C}^T_{(n)}$ and $\mathcal{C}^T_{(n)}\mathcal{C}_{(n)}$ and the row transfer matrices $\mathcal{A}^{(n)}(u)$ and $\mathcal{G}^{(n)}(u)$, respectively, have the same eigenvectors and can be simultaneously diagonalised. The eigenvectors that are not annihilated by the symmetry operators give the eigenvalues that are intertwined and are common to $\mathcal{A}^{(n)}(u)$ and $\mathcal{G}^{(n)}(u)$. Since

$$\mathcal{C}_{(n)}\mathcal{C}^T_{(n)} \stackrel{\mathcal{C}_{(n)}}{\sim} \mathcal{C}^T_{(n)}\mathcal{C}_{(n)} \tag{5.37}$$

it is precisely the nonzero eigenvalues of these symmetry operators that are in common.

Let us now consider the A_L–$D_{(L+3)/2}$ fused models, with L odd, and define the height reversal operators $\mathcal{R}_A$ and $\mathcal{R}_D$ for these models by the elements

$$\langle \boldsymbol{a}|\mathcal{R}_A|\boldsymbol{b}\rangle = \prod_{j=1}^{N} \delta_{a_j,r(b_j)}, \qquad\qquad \langle \boldsymbol{a}|\mathcal{R}_D|\boldsymbol{b}\rangle = \prod_{j=1}^{N} \delta_{a_j,r(b_j)} \tag{5.38}$$

where for the A models $r(b) = h - b$ and for the D models

$$r(b) = \begin{cases} b, & \text{for } b = 1, 2, \cdots, (L-1)/2 \\ (L+3)/2, & \text{for } b = (L+1)/2 \\ (L+1)/2, & \text{for } b = (L+3)/2. \end{cases} \tag{5.39}$$

These matrix operators implement the $\mathbb{Z}_2$ symmetry of the models. It is easy to show that the fused cell row transfer matrices satisfy

$$\mathcal{C}_{(n)}\mathcal{C}^T_{(n)} = I + \mathcal{R}_A, \qquad\qquad \mathcal{C}^T_{(n)}\mathcal{C}_{(n)} = I + \mathcal{R}_D. \tag{5.40}$$

The row transfer matrices of the fused A and D models commute with the corresponding height reversal operators. An immediate consequence of this is that the eigenvalues of $\mathcal{A}(u)^{(n)} = \mathbf{T}_A^{(n,n)}(u)$ and $\mathcal{D}(u)^{(n)} = \mathbf{T}_D^{(n,n)}(u)$ are in common if and only if the corresponding eigenvectors are even under the $\mathbb{Z}_2$ symmetry. In particular, since the largest eigenvalue has an even eigenvector, the largest eigenvalue is in common and hence the intertwined models have the same central charge

$$c = \frac{3n}{n+2}\left(1 - \frac{2(n+2)}{h(h-n)}\right). \tag{5.41}$$

Similarly, following Klümper and Pearce [10], it can be shown that the conformal weights

$$(\Delta_{r,s}, \overline{\Delta}_{r,s}) \tag{5.42}$$

of the excited states are given by

$$\Delta_{r,s} = \frac{[ht-(h-n)s]^2 - n^2}{4nh(h-n)} + \frac{(s_0-1)(n-s_0+1)}{2n(n+2)} \tag{5.43}$$

where s and r label the rows and columns of the Kac table and s_0 is the unique integer determined by $1 \le s_0 \le n+1$ and $s_0 - 1 = \pm(t-s) \bmod 2n$. However, in contrast to the case of the A models, nondiagonal terms with $\Delta_{r,s} \ne \overline{\Delta}_{r,s}$ occur for the D models.

6 An example: D_4

In this section we find the 2×2 fused face weights of D_4 model and construct explicitly the intertwining relation between the A_5 and D_4 models. The D_4 model is an interesting example because it corresponds to the three-state Potts model.

The adjacency matrices for the fused A_5 and D_4 models are given by the fusion rules (3.1). The adjacency graphs are thus as shown in Figure 4.

The adjacency graphs decompose into two groups for level 2 fusion. The symmetric 2×2 fused face weights of the A_5 model are [3]

$$\begin{aligned}
W^A_{2\times2}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} u\Big) &= \frac{\cos(u+\lambda)\cos u}{\sin\lambda}, \\
W^A_{2\times2}\begin{pmatrix} 3 & 3 \\ 1 & 3 \end{pmatrix} u\Big) &= \frac{\sin(2u)}{\sqrt{2}\sin\lambda}, \\
W^A_{2\times2}\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} u\Big) &= \frac{\sin(u+2\lambda)\sin(u+\lambda)}{\sin^2\lambda}, \\
W^A_{2\times2}\begin{pmatrix} 3 & 3 \\ 3 & 1 \end{pmatrix} u\Big) &= \frac{\cos(u+2\lambda)\cos(u+\lambda)}{\sin^2\lambda}, \\
W^A_{2\times2}\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} u\Big) &= \frac{\cos\lambda}{\sin\lambda}, \\
W^A_{2\times2}\begin{pmatrix} 3 & 5 \\ 1 & 3 \end{pmatrix} u\Big) &= \frac{\sin(u+\lambda)\sin u}{\sin\lambda}, \\
W^A_{2\times2}\begin{pmatrix} 5 & 3 \\ 3 & 1 \end{pmatrix} u\Big) &= \frac{\sin(u-\lambda)\sin(u-2\lambda)}{\sin^2\lambda}
\end{aligned} \tag{6.1}$$

for group 1 and

$$W^A_{2\times2}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} u\Big) = \frac{\cos(u-\lambda)\cos u}{2\sin^2\lambda},$$

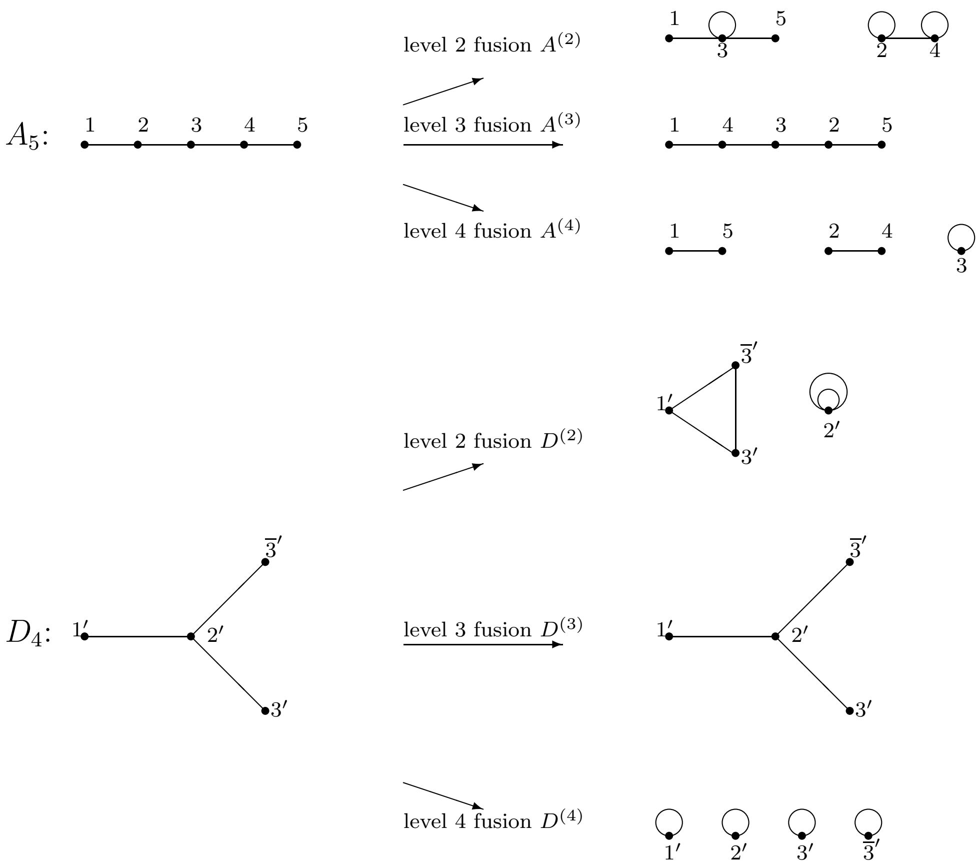


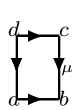
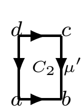
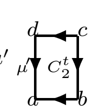
Figure 4: The adjacency graphs of the fused A_5 and D_4 models.

$$\begin{aligned} W^A_{2\times2}\begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}\bigg| u\bigg) &= \frac{\sqrt{3}\sin(u-\lambda)\cos u}{2\sin^2\lambda}, \\ W^A_{2\times2}\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}\bigg| u\bigg) &= \frac{\sin(u-\lambda)\sin u}{2\sin^2\lambda}, \\ W^A_{2\times2}\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}\bigg| u\bigg) &= \frac{\cos(u-\lambda)\cos u}{2\sin^2\lambda}; \\ W^A_{2\times2}\begin{pmatrix} 2 & 4 \\ 2 & 2 \end{pmatrix}\bigg| u\bigg) &= -\frac{2\sqrt{3}\sin(u+2\lambda)\sin u}{\sin^2\lambda} \end{aligned} \tag{6.2}$$

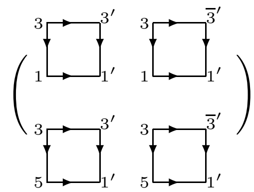
for group 2. These weights satisfy the following symmetries

$$W^A_{2\times2}\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = W^A_{2\times2}\left(\begin{matrix} d & a \\ c & b \end{matrix}\,\middle|\, u\right) = W^A_{2\times2}\left(\begin{matrix} 6-d & 6-c \\ 6-a & 6-b \end{matrix}\,\middle|\, u\right). \tag{6.3}$$

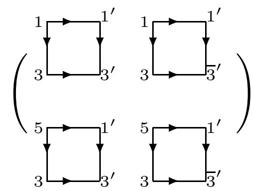
To obtain the fused cells we find a gauge transformation $g^A(a,b)$ for the A paths and $g^D(a,b)$ for the D paths such that

 $= \sum_{\mu'} g^A(d,a)$  $g^D(c,b)^{-1}_{\mu',\mu} = \sum_{\mu'} g^D(d,a)_{\mu,\mu'}$  $g^A(c,b)^{-1}.$ (6.4)

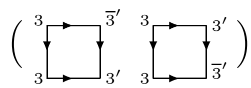
The unitary conditions then take the form given by (5.4) and (5.5). Dividing the fused cells into two groups, we find



$$= \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

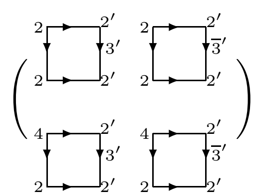


$$= \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \qquad (6.5)$$

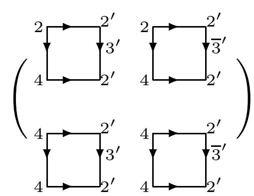


$$= \begin{pmatrix} 1 & 1 \end{pmatrix}$$

for the first group and



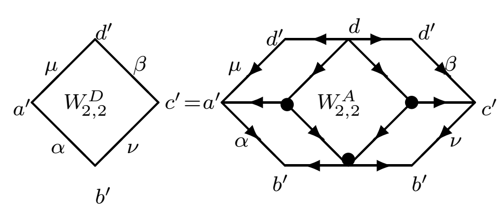
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad (6.6)$$

for the second group.

These cells satisfy the unitary conditions (5.4) and (5.5). Hence, from the intertwining relation the 2×2 fused D face weights must be given [13] in terms of the A face weights by



(6.7)

independent of the spin d. Inserting the fused weights W^A_{22} and fused cells given above we find the fused face weights of the D_4 model. Explicitly, for the first group the nonzero weights read

$$W^D_{2\times 2}\begin{pmatrix} b & a \\ a & b \end{pmatrix} u \Big) = \frac{\cos\lambda}{\sin\lambda} \tag{6.8}$$

$$W^D_{2\times 2}\begin{pmatrix} b & c \\ a & b \end{pmatrix} u \Big) = \frac{\sin(2\lambda-2u)}{\sin\lambda} \tag{6.9}$$

$$W^D_{2\times 2}\begin{pmatrix} c & a \\ a & b \end{pmatrix} u \Big) = \frac{\sin 2u}{\sin\lambda}. \tag{6.10}$$

where $a, b, c, d = 1', 3', \overline{3}'$ are distinct. For the second group the face weights are

$$W^D_{2\times 2}\left(\begin{matrix} 2' & \mu & 2' \\ \mu & & \mu \\ 2' & \mu & 2' \end{matrix} \;\middle|\; u\right) = \sin 2\lambda \left(1 - \frac{\sin(2u-\lambda)}{\sin\lambda}\right) \tag{6.11}$$

$$W^D_{2\times 2}\left(\begin{matrix} 2' & \mu & 2' \\ \nu & & \nu \\ 2' & \mu & 2' \end{matrix} \;\middle|\; u\right) = \sin(2\lambda) \left(1 + \frac{\sin(2u-\lambda)}{\sin\lambda}\right) \tag{6.12}$$

$$W^D_{2\times 2}\left(\begin{matrix} 2' & \nu & 2' \\ \nu & & \mu \\ 2' & \mu & 2' \end{matrix} \;\middle|\; u\right) = \frac{\cos u\cos(u-\lambda)}{\sin\lambda} \tag{6.13}$$

$$W^D_{2\times 2}\left(\begin{matrix} 2' & \nu & 2' \\ \mu & & \nu \\ 2' & \mu & 2' \end{matrix} \;\middle|\; u\right) = \frac{\sin u\sin(\lambda-u)}{\sin\lambda} \tag{6.14}$$

where the bond variables $\mu \neq \nu = 3', \overline{3}'$. It can be directly verified that these fused weights satisfy the Yang-Baxter equation. In fact, the first group gives precisely the face weights of the critical 3-state CSOS model [7]. The second group gives the weights of the 8-vertex model at the Ising decoupling point.

The 2×2 fused face weights of the D_4 model have been obtained here via the intertwining relation. However, precisely the same results are obtained by following the fusion procedure presented in Sections 3 and 5. Although we have concentrated in this article on fusion of the classical *A–D–E* models, the affine *A–D–E* and dilute *A–D–E* models [24, 25] can also be fused using these methods. Similarly, the methods are easily extended to fuse the elliptic off-critical *D* models.

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A Appendix: Fused A–D–E Adjacency Graphs

A.1 Adjacency graphs of the fused D_7 models

A.2 Adjacency graphs of the fused E_6 models

A.3 Adjacency graphs of the fused E_7 models

A.4 Adjacency graphs of the fused E_8 models

B Appendix: Parities ϕ of the E_6

Fusion level 2:

$p(2,2,2)_i$	$\alpha(2,2,2)$
	$\alpha=1$ (2,1,2)
(2,1,2)	1
(2,3,2)	1

$p(4,4,2)_i$	$\alpha(4,4,2)$
	$\alpha=1$ (4,3,4)
(4,3,4)	1
(4,5,4)	1

$p(3,3,2)_i$	$\alpha(3,3,2)$	
	$\alpha=1$ (3,2,3)	$\alpha=2$ (3,4,3)
(3,2,3)	1	0
(3,4,3)	0	1
(3,6,3)	1	1

$\phi^{(i,\alpha)}_{(a,b,2)} = \phi^{(1,1)}_{(a,b,2)} = 1$ for other paths because $L^{(2)}_{(a,b)} = 1$.

Fusion level 3:

$p(2,3,3)_i$	$\alpha(2,3,3)$	
	$\alpha=1$ (2,3,2,3)	$\alpha=2$ (2,3,4,3)
(2,1,2,3)	1	0
(2,3,2,3)	1	0
(2,3,4,3)	0	1
(2,3,6,3)	1	1

$p(3,6,3)_i$	$\alpha(3,6,3)$
	$\alpha=1$ (3,2,3,6)
(3,2,3,6)	1
(3,4,3,6)	-1

$p(4,3,3)_i$	$\alpha(4,3,3)$	
	$\alpha=1$ (4,3,2,3)	$\alpha=2$ (4,3,4,3)
(4,3,2,3)	1	0
(4,3,4,3)	0	1
(4,3,6,3)	1	1

$\phi^{(i,\alpha)}_{(3,2,3)} = \phi^{(i,\alpha)}_{(2,3,3)}$, $\phi^{(i,\alpha)}_{(3,4,3)} = \phi^{(i,\alpha)}_{(4,3,3)}$, $\phi^{(i,\alpha)}_{(a,b,3)} = \phi^{(1,1)}_{(a,b,3)} = 1$ for other paths because $L^{(3)}_{(a,b)} = 1$.

Fusion level 4:

$p(3,3,4)_i$	$\alpha(3,3,4)$		
	$\alpha=1$ (3,2,3,2,3)	$\alpha=2$ (3,2,3,4,3)	$\alpha=3$ (3,4,3,2,3)
(3,2,1,2,3)	1	0	0
(3,2,3,2,3)	1	0	0
(3,2,3,6,3)	1	0	0
(3,6,3,2,3)	1	0	0
(3,2,3,4,3)	0	1	0
(3,2,3,6,3)	0	1	0
(3,6,3,4,3)	0	1	0
(3,4,3,2,3)	0	0	1
(3,4,3,6,3)	0	0	1
(3,6,3,2,3)	0	0	1
(3,4,5,4,3)	-1	-1	-1
(3,4,3,4,3)	-1	-1	-1
(3,4,3,6,3)	-1	-1	-1
(3,6,3,4,3)	-1	-1	-1

$p(1,3,4)_i$	$\alpha(1,3,4)$ $\alpha=1$ (1,2,3,4,3)
(1,2,3,4,3)	1
(1,2,3,6,3)	1

$p(5,3,4)_i$	$\alpha(5,3,4)$ $\alpha=1$ (5,4,3,2,3)
(5,4,3,2,3)	1
(5,4,3,6,3)	1

$p(2,2,4)_i$	$\alpha(2,2,4)$ $\alpha=1$ (2,3,4,3,2)
(2,3,4,3,2)	1
(2,3,6,3,2)	1

$p(4,4,4)_i$	$\alpha(4,4,4)$ $\alpha=1$ (4,3,2,3,4)
(4,3,2,3,4)	1
(4,3,6,3,4)	1

$p(2,4,4)_i$	$\alpha(2,4,4)$ $\alpha=1$ (2,3,2,3,4)
(2,1,2,3,4)	1
(2,3,2,3,4)	1
(2,3,6,3,4)	1

$p(2,6,4)_i$	$\alpha(2,6,4)$ $\alpha=1$ (2,3,2,3,6)
(2,1,2,3,6)	1
(2,3,2,3,6)	1
(2,3,4,3,6)	-1

$p(4,6,4)_i$	$\alpha(4,6,4)$ $\alpha=1$ (4,3,2,3,6)
(4,3,2,3,6)	1
(4,3,4,3,6)	-1
(4,5,4,3,6)	-1

$p(6,6,4)_i$	$\alpha(6,6,4)$ $\alpha=1$ (6,3,2,3,6)
(6,3,2,3,6)	1
(6,3,4,3,6)	-1

The others are given by $\phi^{(i,\alpha)}_{(a,b,4)} = \phi^{(i,\alpha)}_{(b,a,4)}$

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