# STATISTICAL MECHANICS: FROM ISING MODELS TO UNITARY MINIMAL MODELS

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ABSTRACT

A review is given of solvable two-dimensional lattice models constructed from $A$–$D$–$E$ graphs. At criticality, these models realize the unitary minimal models of conformal field theory. They provide integrable representatives of many universality classes of critical behaviour including the critical and tricritical Ising model, the critical and tricritical 3-state Potts model and higher order generic multicritical points. The consequences of conformal invariance, modular invariance and local symmetries are discussed. In particular, lattice realizations based on $A$–$D$–$E$ models are presented for the complete $A$–$D$–$E$ classification of modular invariant partition functions found by Capelli, Itzykson and Zuber.

## 1. Introduction

Bert Green was born in 1920 and Angas Hurst in 1923. This was precisely the time that Lenz[1] was formulating a simple lattice spin model of a ferromagnet with his Ph.D. student Ising[2]. This period gave birth to the subject of exactly solvable lattice models. The Ising model has played a central role in the development of this subject and Green and Hurst have made many significant contributions along the way. Of course, the major impetus to the study of lattice spin models came in 1944 when Onsager[3] solved the Ising model on the square lattice. He showed rigorously that the model exhibits a phase transition and studied in some detail its critical behaviour thus leading to the whole modern study of phase transitions and critical phenomena. Today we can solve infinite families of lattice spin models and much progress is being made on the problem of classifying all possible critical behaviours of two-dimensional statistical systems. This is the subject of this review.

All in all, we have much to celebrate this year. Bert Green and Angas Hurst are pursuing research in their seventies with the same interest and vigour that has been characteristic throughout their careers. It is exactly fifty years since Onsager

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solved the two-dimensional Ising model and thirty years since the appearance of the classic book[4] by Bert Green and Angas Hurst entitled "Order-Disorder Phenomena" which so wonderfully encapsulates the development of the subject up till that time.

The layout of this paper is as follows. In the next section I review the history of the Ising model and describe its critical behaviour. In Section 3, I summarize the main results from conformal field theory concentrating on the consequences of conformal and modular invariance. This section culminates with the $A$–$D$–$E$ classification of modular invariant partition functions by Capelli, Itzykson and Zuber[5]. In Section 4, I discuss the solvable $A$–$D$–$E$ lattice models of Pasquier[6] and their dilute couterparts due to Warnaar, Nienhuis and Seaton[7] and Roche[8]. The latter models provide lattice realizations of the complete $A$–$D$–$E$ classification of unitary minimal modular invariant partition functions.

## 2. Two-Dimensional Ising Model

The face weights of the two-dimensional Ising model on the square lattice are given by

$$W\begin{pmatrix} d & c \\ a & b \end{pmatrix} = \begin{array}{c} d \quad\;\; c \\ \boxed{\phantom{xx}} \\ a \quad\;\; b \end{array} = e^{\beta[Jac+Kbd+h(a+b+c+d)/4]} \tag{2.1}$$

where $J, K$ are interaction strengths, $h$ is the magnetic field, $k$ is Boltzmann's constant, $\beta = 1/kT$ is the inverse temperature and the spins $a, b, c, d = \pm 1$. More generally, one can consider any interaction-round-a-face model [9] on the square lattice where the face weights depend on spins $a, b, c, d$ which are restricted to values in a finite set.

The problem of statistical mechanics is to calculate the partition function

$$Z_N = \sum_{\text{spins}} \prod_{\text{faces}} W\begin{pmatrix} d & c \\ a & b \end{pmatrix} \tag{2.2}$$

for a lattice of $N$ sites and the free energy $\psi$ in the thermodynamic limit given by

$$-\beta\psi = \lim_{N\to\infty} \frac{1}{N} \log Z_N. \tag{2.3}$$

For the Ising model another quantity of interest is the magnetization

$$m = -\frac{\partial\psi}{\partial h} = \langle\sigma\rangle \tag{2.4}$$

where $\sigma = \pm 1$ is a spin in the center of the lattice. This one point function gives a measure of the alignment of the spins with the external magnetic field and hence a measure of the order in the system. More complicated systems can have many such order parameters $R_k$, $k = 1, 2, \ldots$

As shown by Onsager[3], the two-dimensional Ising model exhibits a phase transition in zero magnetic field. At high temperatures the magnetization vanishes as would be expected from spin-reversal symmetry. At temperatures below the critical temperature $T_c$, however, the spin-reversal symmetry is spontaneously broken and the magnetization does not vanish. This is an example of an order-disorder transition. The critical point $T = T_c$, $h = 0$ is a singular point of the free energy. Indeed, Onsager showed that the specific heat in zero field

$$C_0 = -T\frac{\partial^2\psi}{\partial T^2} \tag{2.5}$$

diverges logarithmically. More typically, such quantities display power law behaviours near critical points. The critical behaviour can therefore be quantified in terms of critical exponents such as $\alpha$, $\beta$, $\gamma$ and $\delta$ defined for magnetic systems as follows:

$$\begin{array}{lclll} \psi & \sim & (T-T_c)^{2-\alpha}, & T \to T_c, & h = 0 \\ R & \sim & |T-T_c|^{\beta}, & T \to T_c-, & h = 0 \\ \chi & \sim & |T-T_c|^{-\gamma}, & T \to T_c, & h = 0 \\ R & \sim & h^{1/\delta}, & T = T_c, & h \to 0. \end{array} \tag{2.6}$$

Here $R = m$ is the magnetization and $\chi = \frac{\partial R}{\partial h}$ is the susceptibility. From general principles of scaling and the renormalization group it is known that only two of these critical exponents are independent. Generally, it is expected that these exponents will satisfy the scaling relations

$$\begin{array}{rcl} \alpha + 2\beta + \gamma & = & 2 \\ \alpha + \beta(1+\delta) & = & 2. \end{array} \tag{2.7}$$

Onsager's result for the specific heat implies that the critical exponent $\alpha = 0$ for the two-dimensional Ising model. Further to this, Yang[10] calculated the zero-field magnetization in 1952. In particular, Yang showed that as $T$ approaches the critical value $T_c$ from below the magnetization vanishes with a power law

$$m \sim (T_c - T)^{1/8}. \tag{2.8}$$

It follows from the scaling relations that the critical exponents of the two-dimensional Ising model take the values

$$\alpha = 0, \quad \beta = 1/8, \quad \gamma = 7/4, \quad \delta = 15. \tag{2.9}$$

Although Yang's calculation is very complicated it is now possible to calculate the order parameters of solvable lattice models more straightforwardly using corner transfer matrices[9]. For the Ising model it is in fact possible to calculate many correlation functions. Perhaps the most elegant and efficient way to do this is to use the method of Pfaffians introduced by Hurst and Green[11]. Even to this day, very little progress has been made on the problem of calculating general correlation functions for other solvable lattice models. Green and Hurst summarized the situation in 1964 as follows:

> "The Ising problem is almost the only nontrivial problem of statistical mechanics for which exact solutions have been obtained."

The situation today is in stark contrast. Not only can we solve infinite families of lattice spin models, we also have a very good understanding of their critical behaviour from conformal field theory.

## 3. Two-Dimensional Conformal Field Theory

The modern study of two dimensional statistical systems began precisely ten years ago in 1984. In this year Belavin, Polyakov and Zamolodchikov[12] introduced the minimal series of conformally invariant field theories and showed that conformal invariance plays a key role in the study of such theories. In this same year Andrews, Baxter and Forrester[13] solved the first infinite hierarchy of solvable lattice models. By a remarkable coincidence, these models actually realize[14] the unitary members[15] of the minimal series.

### 3.1. *Conformal Invariance*

The notion of conformal invariance extends the notion of scale invariance to local scale invariance. Conformal transformations preserve angles. In three or higher dimensions the conformal group is finite dimensional but in two dimensions this group is infinite dimensional. This symmetry therefore places very strong constraints on two dimensional conformally invariant systems. In two dimensions the conformal transformations can be realized as analytic maps in the complex plane.

Conformal field theories are characterized in part by a number $c$ called the central charge. Belavin, Polyakov and Zamolodchikov showed that for $c < 1$ the central charge is restricted to the discrete minimal series

$$c = 1 - \frac{6(p-p')^2}{pp'} \tag{3.1}$$

where $p$ and $p'$ are coprime positive integers. The conformal weights of the minimal series are given by the Kac formula

$$\Delta = \Delta_{r,s}^{(p,p')} = \frac{(rp'-sp)^2-(p'-p)^2}{4pp'} \tag{3.2}$$

with

$$1 \le r \le p-1, \quad 1 \le s \le p'-1. \tag{3.3}$$

Moreover, Friedan, Qiu and Shenker[15] showed that if the theory is unitary, which is required for the theory to be physical, then the central charge is further restricted by $|p-p'| = 1$, that is,

$$c = 1 - \frac{6}{h(h-1)}, \qquad h = 4, 5, 6, \ldots \tag{3.4}$$

where $h = \max(p, p')$. The grids of conformal weights for $h = 4, 5$ and 6 are shown in Figure 1. These are identified with the Ising model ($c = 1/2$), tricritical Ising model ($c = 7/10$) and tetracritical Ising model ($c = 4/5$) respectively. The critical

$h = 4$

| $s$ | | |
|---|---|---|
| 3 | 1/2 | 0 |
| 2 | 1/16 | 1/16 |
| 1 | 0 | 1/2 |
| | 1 | 2 $r$ |

$h = 5$

| $s$ | | | |
|---|---|---|---|
| 4 | 3/2 | 7/16 | 0 |
| 3 | 3/5 | 3/80 | 1/10 |
| 2 | 1/10 | 3/80 | 3/5 |
| 1 | 0 | 7/16 | 3/2 |
| | 1 | 2 | 3 $r$ |

$h = 6$

| $s$ | | | | |
|---|---|---|---|---|
| 5 | 3 | 7/5 | 2/5 | 0 |
| 4 | 13/8 | 21/40 | 1/40 | 1/8 |
| 3 | 2/3 | 1/15 | 1/15 | 2/3 |
| 2 | 1/8 | 1/40 | 21/40 | 13/8 |
| 1 | 0 | 2/5 | 7/5 | 3 |
| | 1 | 2 | 3 | 4 $r$ |

Fig. 1. Conformal grids of conformal weights for the unitary minimal models with $h = 4, 5, 6$. The table with $h = 3$ is identified with the Ising model, $h = 4$ is identified with the tricritical Ising model and $h = 5$ with the tetracritical Ising model. The odd rows of the $h = 5$ Kac table give the critical exponents of the 3-state Potts model.

exponents of the 3-state Potts model are also related to the conformal weights in the odd rows of the $h = 6$ Kac table.

The critical exponents are easily extracted from the conformal weights once the central charge and certain other identifications have been established. First, the scaling dimensions are given by

$$x = \Delta + \overline{\Delta} \tag{3.5}$$

where $\Delta$ and $\overline{\Delta}$ are any two conformal weights in the appropriate Kac table such that $\Delta - \overline{\Delta}$ is an integer. The set of pairs $(\Delta, \overline{\Delta})$ which actually occur in a particular theory is called the operator content. More specifically, each such pair is identified with an operator such as the identity, the energy (thermal operator) or an ordering field (magnetic operator). The critical exponent $\alpha$ is then given in terms of the thermal scaling dimension $x_\epsilon$ by

$$2 - \alpha = \frac{2}{2 - x_\epsilon}. \tag{3.6}$$

Similarly, a magnetic critical exponent $\beta$ is calculated from the corresponding magnetic scaling dimension $x_\sigma$ by

$$\frac{2\beta}{2 - \alpha} = x_\sigma. \tag{3.7}$$

In the case of the Ising model

$$x_\epsilon = \frac{1}{2} + \frac{1}{2} = 1, \qquad x_\sigma = \frac{1}{16} + \frac{1}{16} = \frac{1}{8} \tag{3.8}$$

and so, as expected, we obtain

$$\alpha = 0, \qquad \beta = \frac{1}{8}. \tag{3.9}$$

### 3.2. *Modular Invariance*

In addition to being conformally invariant, a critical system on a periodic lattice or torus also exhibits modular invariance[16]. Let us consider a conformal field theory or a critical lattice model on a finite $\ell\times\ell'$ periodic lattice or torus as shown in Figure 2 and suppose that the lattice spacing is given by $1/\ell$. Then in the continuum limit

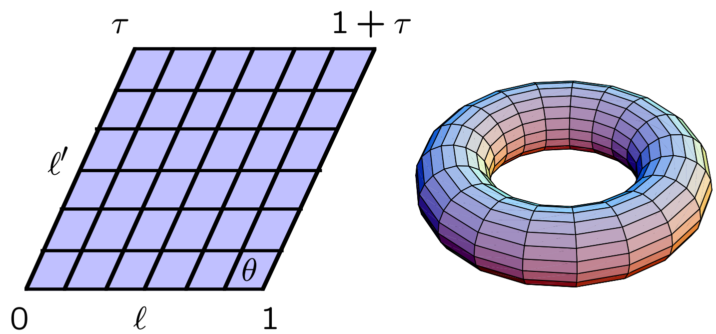



Fig. 2. Distorted square lattice of $\ell\times\ell'$ sites drawn in the complex plane. The spacing between sites is $1/\ell$. With periodic boundary conditions the lattice can be wrapped on a torus. The modular parameter $q=\exp(2\pi i\tau)$ is related to the anisotropy angle $\theta$ by $\tau=(\ell'/\ell)\exp[i(\pi-\theta)]$. For an isotropic lattice $\theta=\pi/2$ and $\tau=i\ell'/\ell$ is just the aspect ratio. The anisotropy angle $\theta$ is simply related to the spectral parameter $u$ of the solvable lattice models.

the torus partition function can be written as

$$Z_{\ell\ell'}\sim\exp(-\ell\ell'\beta\psi)Z(q) \tag{3.10}$$

where $\psi$ is the bulk free energy and $Z(q)$ is a universal term describing the leading finite-size corrections in the limit of $\ell,\ell'$ large with $\ell'/\ell$ fixed. More precisely, the limit

$$Z(q)=\lim_{\substack{\ell,\ell'\to\infty\\ \ell'/\ell\ \text{fixed}}}\exp(\ell\ell'\beta\psi)Z_{\ell\ell'} \tag{3.11}$$

exists and depends only on the modular parameter

$$q=\exp(2\pi i\tau),\qquad \tau=(\ell'/\ell)\exp[i(\pi-\theta)] \tag{3.12}$$

where $\theta$ is the anisotropy angle. Moreover, $Z(q)$ is invariant under the action of the modular group generated by

$$\begin{aligned} T:&\quad \tau\mapsto 1+\tau\\ S:&\quad \tau\mapsto -1/\tau. \end{aligned} \tag{3.13}$$

For this reason $Z(q)$ is usually called the modular invariant partition function.

Modular invariance implies further constraints on the theory. The requirement of modular invariance is strong enough to fix the operator content. In fact, Capelli, Itzykson and Zuber have obtained a complete classification of minimal modular invariant partition functions. Remarkably they obtain two series in one-to-one correspondence with the $A$–$D$–$E$ classical Lie algebras.

### 3.3. *A–D–E* Classification

The $A$–$D$–$E$ classification of minimal modular invariant partition functions of Capelli, Itzykson and Zuber is shown in Table 1. The Virasoro characters are defined by

$$\begin{aligned}\chi_{r,s}(q) &= \text{Virasoro character of } G' \\ &= \frac{q^{-c/24+\Delta^{(p,p')}_{r,s}}}{Q(q)} \sum_{n=-\infty}^{\infty} \left\{ q^{n(npp'+rp'-sp)} - q^{(np'+s)(np+r)} \right\}\end{aligned} \tag{3.14}$$

where $q$ is the modular parameter and

$$Q(q) = \prod_{n=1}^{\infty} (1-q^n). \tag{3.15}$$

Table 1. $A$–$D$–$E$ classification of minimal modular invariant partition functions. The central charges are $c = c(G') = 1 - \frac{6(p-p')^2}{pp'}$, $\chi_{r,s} = \chi_{r,s}(q)$ are Virasoro characters and bars denote complex conjugates. In this series $r, s$ are Coxeter exponents of $(A, G')$ and $p' > p$. There is a second series where $r, s$ are Coxeter exponents of $(G, A')$. The unitary minimal models have $|p - p'| = 1$.

| $(G, G')$ | Modular Invariant Partition Function |
|---|---|
| $(A_{p-1}, A_{p'-1})$ | $Z = \frac{1}{2} \sum_{r=1}^{p-1} \sum_{s=1}^{p'-1} \lvert\chi_{r,s}\rvert^2$ |
| $(A_{p-1}, D_{2\rho+2})$ $p'=4\rho+2\geq 6$ | $Z = \frac{1}{2} \sum_{r=1}^{p-1} \left\{ \sum_{\substack{s=1 \\ s \text{ odd}}}^{2\rho-1} \lvert\chi_{r,s} + \chi_{r,4\rho+2-s}\rvert^2 + 2\lvert\chi_{r,2\rho+1}\rvert^2 \right\}$ |
| $(A_{p-1}, D_{2\rho+1})$ $p'=4\rho\geq 8$ | $Z = \frac{1}{2} \sum_{r=1}^{p-1} \left\{ \sum_{\substack{s=1 \\ s \text{ odd}}}^{4\rho-1} \lvert\chi_{r,s}\rvert^2 + \lvert\chi_{r,2\rho}\rvert^2 + \sum_{\substack{s=2 \\ s \text{ even}}}^{2\rho-2} (\chi_{r,s}\bar{\chi}_{r,4\rho-s} + \bar{\chi}_{r,s}\chi_{r,4\rho-s}) \right\}$ |
| $(A_{p-1}, E_6)$ $p'=12$ | $Z = \frac{1}{2} \sum_{r=1}^{p-1} \left\{ \lvert\chi_{r,1} + \chi_{r,7}\rvert^2 + \lvert\chi_{r,4} + \chi_{r,8}\rvert^2 + \lvert\chi_{r,5} + \chi_{r,11}\rvert^2 \right\}$ |
| $(A_{p-1}, E_7)$ $p'=18$ | $Z = \frac{1}{2} \sum_{r=1}^{p-1} \Big\{ \lvert\chi_{r,1} + \chi_{r,17}\rvert^2 + \lvert\chi_{r,5} + \chi_{r,13}\rvert^2 + \lvert\chi_{r,7} + \chi_{r,11}\rvert^2$ |
| | $+ \lvert\chi_{r,9}\rvert^2 + [(\chi_{r,3} + \chi_{r,15})\bar{\chi}_{r,9} + (\bar{\chi}_{r,3} + \bar{\chi}_{r,15})\chi_{r,9}] \Big\}$ |
| $(A_{p-1}, E_8)$ $p'=30$ | $Z = \frac{1}{2} \sum_{r=1}^{p-1} \Big\{ \lvert\chi_{r,1} + \chi_{r,11} + \chi_{r,19} + \chi_{r,29}\rvert^2$ |
| | $+ \lvert\chi_{r,7} + \chi_{r,13} + \chi_{r,17} + \chi_{r,23}\rvert^2 \Big\}$ |

Table 2. The Coxeter number $h$ and Coxeter exponents $s$ of the classical $A$–$D$–$E$ Lie algebras.

| $G$ | $h$ | $s$ |
|---|---|---|
| $A_L$ | $L+1$ | $1, 2, 3, \ldots, L$ |
| $D_L$ | $2L-2$ | $L-1, 1, 3, 5, \ldots, 2L-3$ |
| $E_6$ | 12 | $1, 4, 5, 7, 8, 11$ |
| $E_7$ | 18 | $1, 5, 7, 9, 11, 13, 17$ |
| $E_8$ | 30 | $1, 7, 11, 13, 17, 19, 23, 29$ |

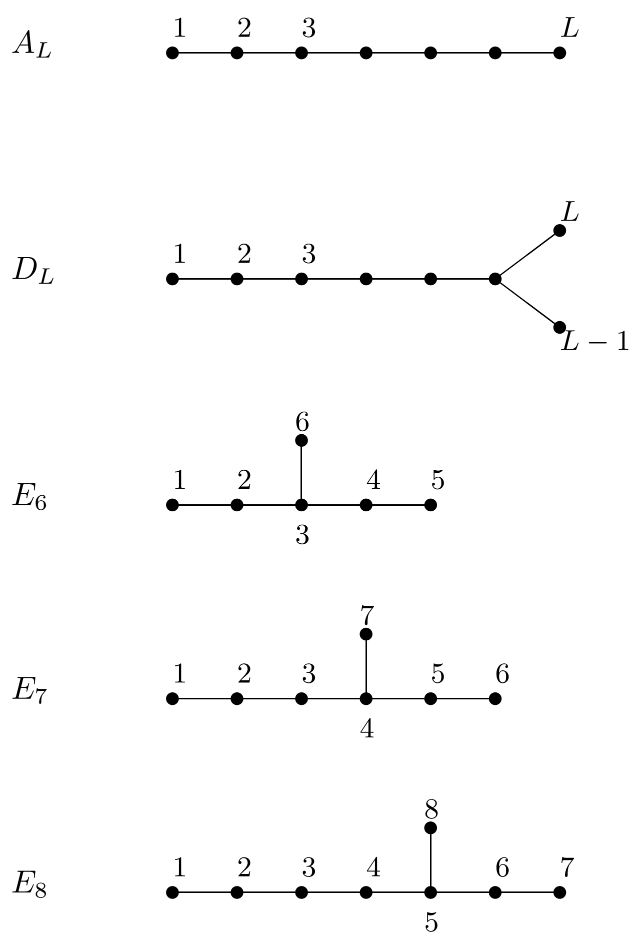



Fig. 3. The Dynkin diagrams of the classical $A$–$D$–$E$ Lie algebras. The $A$–$D$–$E$ graphs classify all graphs whose associated adjacency matrices have eigenvalues strictly less than 2. The eigenvalues of the adjacency matrices are in fact given by $2\cos(s\pi/h)$ where $s$ ranges over the Coxeter exponents.

Here we are primarily interested in the unitary minimal models with $p'-p=\pm1$. In this case there are two $A$–$D$–$E$ series where

$$(r,s) = \text{Coxeter exponents of } (G,G'), \tag{3.16}$$

$$(A,G') = \begin{cases} (A_{h-2},A_{h-1}) \\ (A_{h-2},D_{(h+2)/2}) \\ (A_{10},E_6) \\ (A_{16},E_7) \\ (A_{28},E_8) \end{cases} \qquad (G,A') = \begin{cases} (A_{h-2},A_{h-1}) \\ (D_{(h+1)/2},A_{h-1}) \\ (E_6,A_{12}) \\ (E_7,A_{18}) \\ (E_8,A_{30}) \end{cases} \tag{3.17}$$

and

$$c = c(G') = 1 - \frac{6}{h(h-1)}. \tag{3.18}$$

The Coxeter number $h=\max(p,p')$ and the Coxeter exponents $s$ of the classical $A$–$D$–$E$ Lie algebras are shown in Table 2. The Dynkin diagrams are shown in Figure 3. Some members of these series are identified as follows:

$$\begin{array}{lll} (A_2,A_3) = \text{critical Ising} & & c=1/2 \\ (A_4,D_4) = \text{critical 3-state Potts} & & c=4/5 \\ (A_3,A_4) = \text{tricritical Ising} & & c=7/10 \\ (D_4,A_6) = \text{tricritical 3-state Potts} & & c=6/7 \end{array} \tag{3.19}$$

For this reason I will refer to the $(A,G')$ series as the critical series and the $(G,A')$ series as the tricritical series. Note that the $(A,A')$ theories appear in both series.

At first sight it is very surprising to see conformal field theories being classified by the $A$–$D$–$E$ Lie algebras. On the other hand, perhaps we should not be surprised at all since Lie algebras arise whenever integrability and local symmetries are involved. The classic example is the classification of regular convex polyhedra shown in Figure 4. In this situation there are also two series depending on whether faces or vertices are equivalent. Other examples of $A$–$D$–$E$ classifications occur in the classification of critical points in catastrophe theory and in the classification of the subgroups of $su(2)$.

## 4. Solvable $A$–$D$–$E$ Lattice Models

It is now well established that the critical behaviour of two-dimensional lattice models is described by conformal field theory. The $A$–$D$–$E$ classification of unitary minimal conformal field theories gives an exhaustive list of theories with $c<1$. In other words, this is a complete list of universality classes giving all possible critical behaviours for two-dimensional statistical systems with $c<1$. It is therefore natural to ask whether a solvable lattice model can be found to represent each universality class allowed by the $A$–$D$–$E$ classification.

In 1984 Andrews, Baxter and Forrester[13] solved the first infinite hierarchy of lattice models. The spins in these models take values on the $A_L$ Dynkin diagram and are subject to the constraint that the states of adjacent spins on the square lattice must be adjacent on the $A_L$ diagram. Huse[14] showed that the critical behaviour of

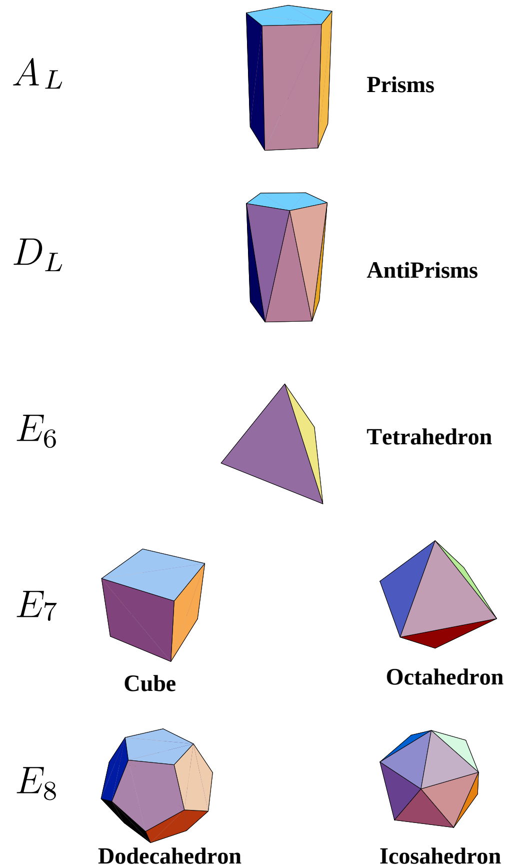



Fig. 4. The $A$–$D$–$E$ classification of regular polyhedra. The polyhedra are convex and regular in the sense that each vertex is equivalent. A similar classification holds for convex polyhedra with equivalent faces. The two series are related by a face-vertex duality.

these $L$ height RSOS models is indeed described by the unitary minimal series. It turns out[16] that the modular invariant partition functions of the ABF RSOS models give the $(A_{L-1}, A_L)$ series with $L = 3, 4, 5, \ldots$ The critical series was completed in 1987 by Pasquier[6] who constructed solvable models whose states take values on the $A$–$D$–$E$ graphs. The $A_L$ models are just the ABF models. Although the $A$ and $D$ models admit off-critical elliptic extensions, the exceptional $E$ models can only be solved at criticality.

### 4.1. *Pasquier's* $A$–$D$–$E$ *Models*

The face weights of Pasquier's critical $A$–$D$–$E$ models are given by

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = \begin{array}{ccc} d & & c \\ & \boxed{u} & \\ a & & b \end{array} = \frac{\sin(\lambda-u)}{\sin\lambda}\delta_{a,c}A_{a,b}A_{a,d} + \frac{\sin u}{\sin\lambda}\sqrt{\frac{S_aS_c}{S_bS_d}}\delta_{b,d}A_{a,b}A_{b,c} \quad (4.1)$$

where the spins $a, b, c, d$ take values on a given $A$–$D$–$E$ graph. The parameter $u$ is called the spectral parameter. In the regime of interest here the spectral parameter lies in the interval $0 < u < \lambda$. The adjacency matrices are given by

$$A_{a,b} = \begin{cases} 1, & a,\, b \text{ connected} \\ 0, & \text{otherwise.} \end{cases} \quad (4.2)$$

The components $S_a \geq 0$ of the Perron-Frobenius eigenvector are determined by

$$\sum_b A_{a,b} S_b = 2\cos\lambda\; S_a \tag{4.3}$$

where $2\cos\lambda$ is the largest eigenvalue of the adjacency matrix and

$$\lambda = \pi/h. \tag{4.4}$$

The Coxeter number $h$ is given in Table 2.

Pasquier's $A$–$D$–$E$ models include some much studied models in statistical mechanics. Some prototypes are shown in Figure 5. The modular invariant partition functions[6] of Pasquier's critical $A$–$D$–$E$ models precisely realize the $(A, G')$ series of Capelli, Itzykson and Zuber. This is a very satisfying situation. However, for many years lattice realizations of the $(G, A')$ series have been missing.

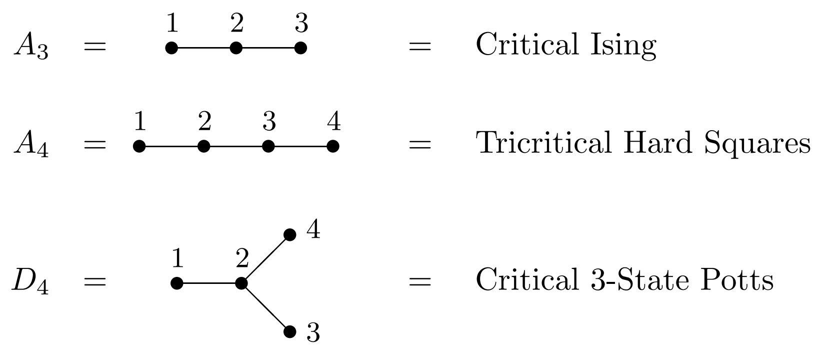



Fig. 5. Some prototype classical $A$–$D$–$E$ lattice models.

## 4.2. *Dilute* $A$–$D$–$E$ *Models*

In 1992 Warnaar, Nienhuis and Seaton[7] and Roche[8] independently obtained a second series of solvable lattice models whose states take values on the $A$–$D$–$E$ graphs. These lattice models are called the dilute $A$–$D$–$E$ models. The face weights of the dilute $A$–$D$–$E$ lattice models at criticality are given by

$$\begin{aligned} W\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) &= \rho_1(u)\delta_{a,b,c,d} + \rho_2(u)\delta_{a,b,c}A_{a,d} + \rho_3(u)\delta_{a,c,d}A_{a,b} \\ &+ \sqrt{\frac{S_a}{S_b}}\rho_4(u)\delta_{b,c,d}A_{a,b} + \sqrt{\frac{S_c}{S_a}}\rho_5(u)\delta_{a,b,d}A_{a,c} + \rho_6(u)\delta_{a,b}\delta_{c,d}A_{a,c} \\ &+ \rho_7(u)\delta_{a,d}\delta_{c,b}A_{a,b} + \rho_8(u)\delta_{a,c}A_{a,b}A_{a,d} + \sqrt{\frac{S_aS_c}{S_bS_d}}\rho_9(u)\delta_{b,d}A_{a,b}A_{b,c} \end{aligned} \tag{4.5}$$

where, as before, the adjacency matrix is

$$A_{a,b} = \begin{cases} 1, & a, b \text{ adjacent} \\ 0, & \text{otherwise} \end{cases} \tag{4.6}$$

and the Perron-Frobenius vector is given by

$$\sum_b A_{a,b}\, S_b = 2\cos\left(\frac{\pi}{h}\right)\, S_a. \tag{4.7}$$

The generalized Kronecker delta is

$$\delta_{a,b,c,\ldots} = \begin{cases} 1, & a=b=c=\ldots \\ 0, & \text{otherwise} \end{cases} \tag{4.8}$$

and the trigonometric functions are

$$\begin{aligned}
\rho_1(u) &= 1 + \frac{\sin u \sin(3\overline{\lambda}-u)}{\sin(2\overline{\lambda})\sin(3\overline{\lambda})} \\
\rho_2(u) &= \rho_3(u) = \frac{\sin(3\overline{\lambda}-u)}{\sin(3\overline{\lambda})} \\
\rho_4(u) &= \rho_5(u) = \frac{\sin u}{\sin(3\overline{\lambda})} \\
\rho_6(u) &= \rho_7(u) = \frac{\sin u \sin(3\overline{\lambda}-u)}{\sin(2\overline{\lambda})\sin(3\overline{\lambda})} \\
\rho_8(u) &= \frac{\sin(2\overline{\lambda}-u)\sin(3\overline{\lambda}-u)}{\sin(2\overline{\lambda})\sin(3\overline{\lambda})} \\
\rho_9(u) &= -\frac{\sin u \sin(\overline{\lambda}-u)}{\sin(2\overline{\lambda})\sin(3\overline{\lambda})}.
\end{aligned} \tag{4.9}$$

The effective adjacency graph is given by adding a loop to each node of the $A$–$D$–$E$ graphs, that is, the spin states at adjacent sites of the lattice are either the same or adjacent on the $A$–$D$–$E$ graph.

The dilute $A$–$D$–$E$ models admit two distinct branches given by

$$\overline{\lambda} = \begin{cases} \dfrac{(h-1)\pi}{4h}, & \text{branch 1} \\ \dfrac{(h+1)\pi}{4h}, & \text{branch 2.} \end{cases} \tag{4.10}$$

The central charges of these models are given by[17]

$$c = \begin{cases} 1 - \dfrac{6}{h(h+1)}, & \text{branch 1} \\ 1 - \dfrac{6}{h(h-1)}, & \text{branch 2.} \end{cases} \tag{4.11}$$

This suggests identifying the universality classes of the first few dilute $A$–$D$–$E$ models as follows:

$$\begin{array}{lll}
\text{branch 2:} & A_3 = \text{critical Ising} & c = 1/2 \\
\text{branch 1:} & A_3 = \text{tricritical Ising} & c = 7/10 \\
\text{branch 2:} & D_4 = \text{critical 3-state Potts} & c = 4/5 \\
\text{branch 1:} & D_4 = \text{tricritical 3-state Potts} & c = 6/7
\end{array} \tag{4.12}$$

Notice that the dilute $A_3$ and $D_4$ are not the usual Ising and 3-state Potts models, they just have the same $\mathbb{Z}_2$ and $\mathbb{Z}_3$ symmetries and therefore lie in the same universality classes.

The dilute $A$–$D$–$E$ lattice models in branch 2 in fact give a second realization of the $(A, G')$ series of Capelli, Itzykson and Zuber. More importantly, it has been shown very recently[18] that the modular invariant partition functions of the dilute $A$–$D$–$E$ lattice models in branch 1 precisely reproduce the missing $(G, A')$ series. The dilute $A$–$D$–$E$ models thus provide lattice realizations of all the unitary minimal conformal field theories!

## 4.3. *Off-Critical Order Parameters*

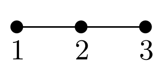
The ABF and dilute $A$ models can also be solved off-criticality since the solution to the Yang-Baxter equations admits an elliptic extension. In particular, this allows the local state probabilities and order parameters $R^{(k)}$ to be calculated by corner transfer matrix methods[9]. For the ABF models, the elliptic nome plays the role of a temperature-like variable $t$ which measures the departure from criticality. In contrast, for the dilute $A$ models, the elliptic nome plays the role of a magnetic field $h$. Remarkably, the Virasoro characters appear in the expressions for the local state probabilities in both cases. Since the off-critical models are not conformally invariant, the appearance of these characters is somewhat mysterious. We will not give the complete results for the local state probabilities here but merely summarize the resulting critical behaviour.

### 4.3.1. *ABF Models:*

The results of Andrews, Baxter and Forrester[13,14] lead to the following critical behaviour for the order parameters of the $A_L$ models

$$R^{(k)} \sim t^{\beta_k}, \qquad \beta_k = \frac{(k+1)^2 - 1}{8L}, \qquad k = 1, 2, \ldots, L-2. \tag{4.13}$$

In particular, specializing to the case of $A_3$ with $k = 1$,




$=$ Ising model (4.14)

we find the well known result

$$m \sim t^{\beta}, \qquad \beta = 1/8 \tag{4.15}$$

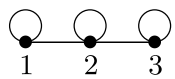
in agreement with the result of Yang[10].

### 4.3.2. *Dilute A Models:*

Analogous results have recently been obtained[19] for the dilute $A_L$ models at least when $L$ is odd. The critical behaviour of the relevant order parameters for these models is given by

$$\overline{R}^{(k)} \sim h^{1/\delta_k}, \qquad \delta_k = \frac{3L(L+2)}{(k+1)^2 - 1}, \qquad k = 1, 2, \ldots, L-2. \tag{4.16}$$

The dilute $A_3$ model is of particular interest because it lies in the universality class of the Ising model in a magnetic field



$$= \left\{ \begin{array}{l} \text{Universality class of Ising} \\ \text{model in a magnetic field.} \end{array} \right. \tag{4.17}$$

Specializing the result to $L = 3$ and $k = 1$ yields

$$m \sim h^{1/\delta}, \qquad \delta = 15. \tag{4.18}$$

This provides the first direct calculation of the exponent $\delta = 15$ for the Ising universality class without invoking scaling relations. The dilute $A$–$D$–$E$ models have thus yielded a new result for the Ising model from whence we started.

## 5. Summary

An overview has been given of the progress in studying the critical behaviour of two-dimensional lattice models, starting with the Ising model and ending with the dilute $A$–$D$–$E$ lattice models. The latter models are particularly significant because, at criticality, they realize all possible unitary minimal conformal field theories and thus they provide integrable representatives of all possible universality classes of critical behaviour with central charges $c < 1$.

There are many other topics of interest in modern mathematical physics that are connected to the study of solvable lattice models and their connections with conformal field theory. These include quantum groups, Yang-Baxter algebras, affine Lie algebras, braid-monoid algebras and knot theory just to name a few. This review is too short to delve into these fascinating areas but I am sure if Bert Green and Angas Hurst were starting out again today these topics would rank high among their priorities.

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## References

1. W. Lenz, Z. Phys. **21** (1920) 613.
2. E. Ising, Z. Phys. **31** (1925) 253.
3. L. Onsager, Phys. Rev. **65** (1944) 117.
4. H. S. Green and C. A. Hurst, “Order-Disorder Phenomena”, Interscience, New York, 1964.
5. A. Capelli, C. Itzykson and J.-B. Zuber, Nucl. Phys. **B280** (1987) 445; Comm. Math. Phys. **113** (1987) 1
6. V. Pasquier, Nucl. Phys. **B28** (1987) 162; J. Phys. A **20** (1987) L1229, 5707.
7. S. O. Warnaar, B. Nienhuis and K. A. Seaton, Phys. Rev. Lett. **69** (1992) 710.
8. Ph. Roche, Phys. Lett. **B4** (1992) 929.
9. R. J. Baxter, “Exactly Solved Models in Statistical Mechanics”, Academic Press, London, 1982.

10. C. N. Yang, Phys. Rev. **85** (1952) 809.
11. C. A. Hurst and H. S. Green, J. Chem. Phys. **33** (1960) 1059.
12. A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. B **241** (1984) 333.
13. G. E. Andrews, R. J. Baxter and P. J. Forrester, J. Stat. Phys. **35** (1984) 193.
14. D. A. Huse, Phys. Rev. B **30** (1984) 3908.
15. D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. **52** (1984) 1575; in "Vertex Operators in Mathematics and Physics", eds. J. Lepowsky, S. Mandelstam and I.M. Singer, Springer, 1984.
16. J. L. Cardy, Nucl. Phys. **B270** (1986) 186; Nucl. Phys. **B275** (1986) 200.
17. S. O. Warnaar, M. T. Batchelor and B. Nienhuis, J. Phys. A **25** (1992) 3077.
18. O. E. Foda, D. L. O'Brien and P. A. Pearce, "Lattice Realizations of $A$–$D$–$E$ Unitary Minimal Modular Invariant Partition Functions", to be published (1994).
19. S. O. Warnaar, P. A. Pearce, K. A. Seaton and B. Nienhuis, J. Stat. Phys. **74** (1994) 469.