Fusion of Dilute A_L Lattice Models

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Abstract

The fusion procedure is implemented for the dilute A_L lattice models. A fusion hierarchy of functional equations is derived for the fused transfer matrices. This hierarchy has an $su(3)$ structure. Then we present the Bethe ansatz to diagonalize the transfer matrices of the fused models. The eigenvalues and Bethe ansatz equations of the transfer matrices have been given.

1 Introduction

The fusion procedure as an useful method to find new solution of Yang-Baxter equation [1, 2] based on a fundamental solution or solvable lattice models was found first in [3]. It has been successfully applied to many solvable models in two-dimensional statistical mechanics. Specially, the fusion for the Baxter's eight-vertex model and ABF's SOS models [1, 4] in [5], for the critical D and E face models [12] in [21] and for Belavin's $\mathbb{Z}_n$ symmetric vertex models [7] and JMO's IRF models [6] in [9, 8]. The Fusion has been constructed for the correspondence relations between the the Baxter's eight-vertex models and the ABF's SOS models in [5] and between the Belavin's vertex models and JMO's faces models in [10] and for the intertwining relations [24, 15] between two A–D–E models in [13].

Recently the fusion procedure has been paid more attention to find the fusion hierarchies of solvable lattice models. These fusion hierarchies satisfy a group of functional

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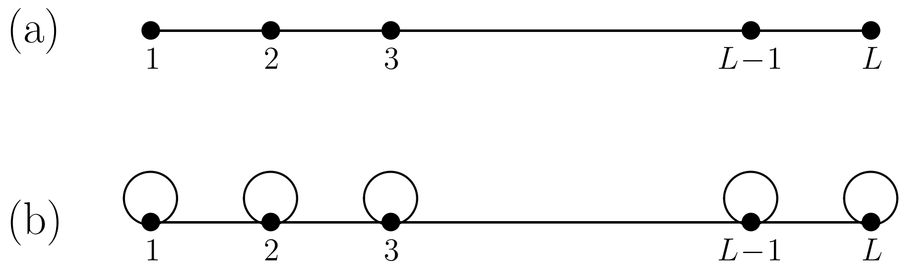


Figure 1. The A_L Dynkin diagram (a) and the effective adjacency diagram of the dilute A_L models (b).

relations which includes the information of eigenvalue spectra of the fused models. The finite-size corrections of eigenvalue spectra of transfer matrices of the fused models can be given by solving these functional relations. Thus we can extract the value of central charge and conformal weights of underlying conformal field theories from the finite-size corrections. For example, the central charges and conformal weights of the ABF's restricted SOS models and the JMO's restricted IRF models have been given by finding the finite-size corrections of eigenvalue spectra in [18, 17, 16, 27].

The dilute lattice models as a generalization of the ABF's restricted SOS models have been found in [14]. Recent studies have shown that the dilute A_L models present many interesting aspects. In an appropriate regime the dilute A_3 model lies in the universality class of the Ising model in a magnetic field and gives the magnetic exponent $\delta = 15$ [25]. Also the A_3 model shows the E_8 scattering theory for massive excitations over the ground state [23, 26]. In present paper we construct the fusion procedure of the dilute models.

In the next subsection we describe the dilute lattice models. Then the $su(3)$-structure of the fusion hierarchies has been shown based on the Bethe ansatz solution [23]. In section 2 we show that the face weights satisfy a group of special properties which ensure that they can be taken as the elementary blocks for fusion. Also it is shown that there are two projectors and thus two different fusions of $su(2)$-type and $su(3)$-type exist on the elementary models. In section 3 we give in detail the procedure for constructing the level 2 fully symmetric and level $2, 3$ fully antisymmetric fused face weights of the $su(3)$-type. Then we describe the general procedure for finding the the $su(3)$ fused face weights of level (n, m). This is accomplished by introducing parities for the fusion projectors. The fusion procedure of the $su(2)$-type have been given in [22]. In section 3 we describe the $su(3)$ fusion hierarchies satisfied by the fused dilute A_L row transfer matrices. Then as an application of these fusion hierarchy we find that the Bethe ansatz equations and eigenvalues of these fused transfer matrices. Thus we generalize the discussion given in subsection 1.3 to the general cases. In the final section a brief discussion is presented.

1.1 Dilute $\boldsymbol{A}_L$ Lattice Models

The dilute A_L lattice models [14] are restricted solid–on–solid (RSOS) models with L heights built on the A_L Dynkin diagram as shown in Figure 1(a). The elements $A_{a,b}$ of the adjacency matrix for this diagram are given by

$$A_{a,b} \;=\; A_{b,a} \;=\; \begin{cases} 1, & |a-b|=1 \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

The face weights of the dilute A_L models are

$$\begin{aligned}
W\!\left(\begin{matrix} a & a \\ a & a \end{matrix}\,\middle|\, u\right) &= \frac{\vartheta_1(6\lambda-u)\vartheta_1(3\lambda+u)}{\vartheta_1(6\lambda)\vartheta_1(3\lambda)} \\
&\quad -\left(\frac{S_{a+1}}{S_a}\frac{\vartheta_4(2a\lambda-5\lambda)}{\vartheta_4(2a\lambda+\lambda)}+\frac{S_{a-1}}{S_a}\frac{\vartheta_4(2a\lambda+5\lambda)}{\vartheta_4(2a\lambda-\lambda)}\right)\frac{\vartheta_1(u)\vartheta_1(3\lambda-u)}{\vartheta_1(6\lambda)\vartheta_1(3\lambda)} \\
W\!\left(\begin{matrix} a\pm1 & a \\ a & a \end{matrix}\,\middle|\, u\right) &= W\!\left(\begin{matrix} a & a \\ a & a\pm1 \end{matrix}\,\middle|\, u\right) = \frac{\vartheta_1(3\lambda-u)\vartheta_4(\pm2a\lambda+\lambda-u)}{\vartheta_1(3\lambda)\vartheta_4(\pm2a\lambda+\lambda)} \\
W\!\left(\begin{matrix} a & a \\ a\pm1 & a \end{matrix}\,\middle|\, u\right) &= W\!\left(\begin{matrix} a & a\pm1 \\ a & a \end{matrix}\,\middle|\, u\right) = \left(\frac{S_{a\pm1}}{S_a}\right)^{1/2}\frac{\vartheta_1(u)\vartheta_4(\pm2a\lambda-2\lambda+u)}{\vartheta_1(3\lambda)\vartheta_4(\pm2a\lambda+\lambda)} \\
W\!\left(\begin{matrix} a & a\pm1 \\ a & a\pm1 \end{matrix}\,\middle|\, u\right) &= W\!\left(\begin{matrix} a\pm1 & a\pm1 \\ a & a \end{matrix}\,\middle|\, u\right) \\
&= \left(\frac{\vartheta_4(\pm2a\lambda+3\lambda)\vartheta_4(\pm2a\lambda-\lambda)}{\vartheta_4^2(\pm2a\lambda+\lambda)}\right)^{1/2}\frac{\vartheta_1(u)\vartheta_1(3\lambda-u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \qquad (1.2)\\
W\!\left(\begin{matrix} a\pm1 & a \\ a & a\mp1 \end{matrix}\,\middle|\, u\right) &= \frac{\vartheta_1(2\lambda-u)\vartheta_1(3\lambda-u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \\
W\!\left(\begin{matrix} a & a\mp1 \\ a\pm1 & a \end{matrix}\,\middle|\, u\right) &= -\left(\frac{S_{a-1}S_{a+1}}{S_a^2}\right)^{1/2}\frac{\vartheta_1(u)\vartheta_1(\lambda-u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \\
W\!\left(\begin{matrix} a & a\pm1 \\ a\pm1 & a \end{matrix}\,\middle|\, u\right) &= \frac{\vartheta_1(3\lambda-u)\vartheta_1(\pm4a\lambda+2\lambda+u)}{\vartheta_1(3\lambda)\vartheta_1(\pm4a\lambda+2\lambda)}+\frac{S_{a\pm1}}{S_a}\frac{\vartheta_1(u)\vartheta_1(\pm4a\lambda-\lambda+u)}{\vartheta_1(3\lambda)\vartheta_1(\pm4a\lambda+2\lambda)} \\
&= \frac{\vartheta_1(3\lambda+u)\vartheta_1(\pm4a\lambda-4\lambda+u)}{\vartheta_1(3\lambda)\vartheta_1(\pm4a\lambda-4\lambda)} \\
&\quad +\left(\frac{S_{a\mp1}}{S_a}\frac{\vartheta_1(4\lambda)}{\vartheta_1(2\lambda)}-\frac{\vartheta_4(\pm2a\lambda-5\lambda)}{\vartheta_4(\pm2a\lambda+\lambda)}\right)\frac{\vartheta_1(u)\vartheta_1(\pm4a\lambda-\lambda+u)}{\vartheta_1(3\lambda)\vartheta_1(\pm4a\lambda-4\lambda)}
\end{aligned}$$

Here the crossing factors are

$$S_a \;=\; (-1)^a \,\frac{\vartheta_1(4a\lambda)}{\vartheta_4(2a\lambda)} \tag{1.3}$$

and $\vartheta_1(u)$, $\vartheta_4(u)$ are standard theta functions of nome p with $|p| < 1$. Note that the effective adjacency matrix in the face weights is $I + A$, see Figure 1(b). We denote by $\mathrm{val}(a)$ the number of allowed neighbours of height a,

$$\mathrm{val}(a) \;=\; \sum_b \,(I+A)_{a,b} \tag{1.4}$$

which is nothing but the sum of the elements of the corresponding row (respectively column) of the effective adjacency matrix. Obviously, we have $\mathrm{val}(a) \le 3$ for the dilute A_L models. The weights (1.2) are normalized such that

$$W\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, 0\right) \;=\; (I+A)_{a,b}\,(I+A)_{a,d}\,\delta_{a,c}\;, \tag{1.5}$$

are invariant under reflections along the diagonals

$$W\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) \;=\; W\!\left(\begin{matrix} d & a \\ c & b \end{matrix}\,\middle|\, u\right) \;=\; W\!\left(\begin{matrix} b & c \\ a & d \end{matrix}\,\middle|\, u\right) \;=\; W\!\left(\begin{matrix} b & a \\ c & d \end{matrix}\,\middle|\, u\right) \tag{1.6}$$

and satisfy the crossing symmetry

$$W\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, 3\lambda-u\right) \;=\; \left(\frac{S_aS_c}{S_bS_d}\right)^{1/2} W\!\left(\begin{matrix} c & b \\ d & a \end{matrix}\,\middle|\, u\right)\;. \tag{1.7}$$

The dilute models admit four different physical branches. The spectral parameter u and the crossing parameter λ in the four branches take values

$$\begin{array}{lcl}
\text{branch 1:} & 0<u<3\lambda & \lambda = \dfrac{\pi}{4}\dfrac{L}{L+1} \\[2ex]
\text{branch 2:} & 0<u<3\lambda & \lambda = \dfrac{\pi}{4}\dfrac{L+2}{L+1} \\[2ex]
\text{branch 3:} & -\pi+3\lambda<u<0 & \lambda = \dfrac{\pi}{4}\dfrac{L+2}{L+1} \\[2ex]
\text{branch 4:} & -\pi+3\lambda<u<0 & \lambda = \dfrac{\pi}{4}\dfrac{L}{L+1}
\end{array} \tag{1.8}$$

At criticality, the face weights simplify to [14, 15]

$$
\begin{aligned}
W\left(\begin{matrix} d & c \\ a & b \end{matrix} \,\middle|\, u\right) = \begin{array}{ccc} d & & c \\ & \boxed{u} & \\ a & & b \end{array} &= \rho_1(u)\delta_{a,b,c,d} + \rho_2(u)\delta_{a,b,c}A_{a,d} + \rho_3(u)\delta_{a,c,d}A_{a,b} \\
&+ \sqrt{\frac{S_a}{S_b}}\rho_4(u)\delta_{b,c,d}A_{a,b} + \sqrt{\frac{S_c}{S_a}}\rho_5(u)\delta_{a,b,d}A_{a,c} + \rho_6(u)\delta_{a,b}\delta_{c,d}A_{a,c} \\
&+ \rho_7(u)\delta_{a,d}\delta_{c,b}A_{a,b} + \rho_8(u)\delta_{a,c}A_{a,b}A_{a,d} + \sqrt{\frac{S_aS_c}{S_bS_d}}\rho_9(u)\delta_{b,d}A_{a,b}A_{b,c}
\end{aligned} \tag{1.9}
$$

with

$$
\begin{aligned}
\rho_1(u) &= \frac{\sin 2\lambda \sin 3\lambda + \sin u \sin(3\lambda - u)}{\sin 2\lambda \sin 3\lambda} \\
\rho_2(u) &= \rho_3(u) = \frac{\sin(3\lambda - u)}{\sin 3\lambda} \\
\rho_4(u) &= \rho_5(u) = \frac{\sin u}{\sin 3\lambda} \\
\rho_6(u) &= \rho_7(u) = \frac{\sin u \sin(3\lambda - u)}{\sin 2\lambda \sin 3\lambda} \\
\rho_8(u) &= \frac{\sin(2\lambda - u)\sin(3\lambda - u)}{\sin 2\lambda \sin 3\lambda} \\
\rho_9(u) &= -\frac{\sin u \sin(\lambda - u)}{\sin 2\lambda \sin 3\lambda}.
\end{aligned} \tag{1.10}
$$

Moreover, the crossing factors reduce to the nonnegative elements of the Perron-Frobenius eigenvectors of the adjacency matrix given by

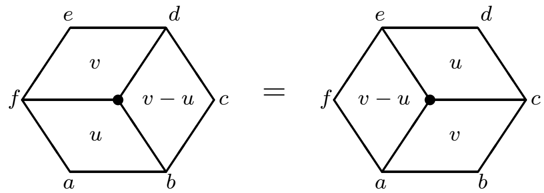
$$
\sum_b A_{a,b}S_b = 2\cos\frac{\pi}{L+1}\, S_a\ . \tag{1.11}
$$

1.2 Commuting Transfer Matrices and Bethe Ansatz Equations

The dilute A models are exactly solvable because their face weights satisfy the Yang-Baxter equations

$$
\begin{aligned}
&\sum_{g=1}^{L} W\left(\begin{matrix} f & g \\ a & b \end{matrix} \,\middle|\, u\right) W\left(\begin{matrix} e & d \\ f & g \end{matrix} \,\middle|\, v\right) W\left(\begin{matrix} d & c \\ g & b \end{matrix} \,\middle|\, v-u\right) \\
&= \sum_{g=1}^{L} W\left(\begin{matrix} e & g \\ f & a \end{matrix} \,\middle|\, v-u\right) W\left(\begin{matrix} g & c \\ a & b \end{matrix} \,\middle|\, v\right) W\left(\begin{matrix} e & d \\ g & c \end{matrix} \,\middle|\, u\right)
\end{aligned} \tag{1.12}
$$

Diagrammatically, this equation is represented as follows



(1.13)

where the solid circle denotes summation over heights. This implies that the row transfer matrices $\boldsymbol{T}(u)$ commute. Here the elements of $\boldsymbol{T}(u)$ are given by

$$\langle\sigma|\boldsymbol{T}(u)|\sigma'\rangle = \prod_{j=1}^{N} W\begin{pmatrix} \sigma'_j & \sigma'_{j+1} \\ \sigma_j & \sigma_{j+1} \end{pmatrix}\!\!\left.\vphantom{\begin{pmatrix}a\\b\end{pmatrix}}\right|\, u\Bigg) \tag{1.14}$$

where the paths $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_N\}$ and $\sigma' = \{\sigma'_1, \sigma'_2, \ldots, \sigma'_N\}$ are allowed configurations of heights along a row with periodic boundary conditions $\sigma_{N+1} = \sigma_1$ and $\sigma'_{N+1} = \sigma'_1$.

The eigenvalues $T(u)$ of the row transfer matrices $\boldsymbol{T}(u)$ can be calculated using a Bethe ansatz. Explicitly, the eigenvalues are given by

$$\begin{aligned} T(u) &= \omega\,\frac{s(u-2\lambda)s(u-3\lambda)}{s(2\lambda)s(3\lambda)}\frac{Q(u+\lambda)}{Q(u-\lambda)} + \frac{s(u)s(3\lambda-u)}{s(2\lambda)s(3\lambda)}\frac{Q(u)Q(u-3\lambda)}{Q(u-\lambda)Q(u-2\lambda)} \\ &\quad +\omega^{-1}\,\frac{s(u)s(u-\lambda)}{s(2\lambda)s(3\lambda)}\frac{Q(u-4\lambda)}{Q(u-2\lambda)} \end{aligned} \tag{1.15}$$

where

$$s(u) = \vartheta_1(u)^N, \qquad Q(u) = \prod_{j=1}^{N}\vartheta_1(u-u_j) \tag{1.16}$$

and the zeros $\{u_j\}$ satisfy the Bethe ansatz equations

$$\omega^{-1}\left[\frac{\vartheta_1(u_j+\lambda)}{\vartheta_1(u_j-\lambda)}\right]^N = -\prod_{k=1}^{N}\frac{\vartheta_1(u_j-u_k+2\lambda)\vartheta_1(u_j-u_k-\lambda)}{\vartheta_1(u_j-u_k-2\lambda)\vartheta_1(u_j-u_k+\lambda)} \tag{1.17}$$

with $j = 1, \ldots, N$ and $\omega = \exp(i\pi\ell/(L+1))$, $\ell = 1, \ldots, L$. At criticality, these equations reduce, apart from the phase factors, to the Bethe ansatz equations of the Izergin-Korepin model [28, 30]. The Bethe ansatz equations ensure that the eigenvalues $T(u)$ are entire functions of u.

1.3 Fusion Hierarchy

Before discussing the fusion hierarchy, we recall some basic facts concerning $su(3)$. Let (n, m), where n and m are nonnegative integers, denote the highest weight irreducible

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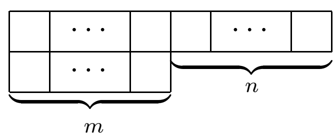
Figure 2. The $su(3)$ weight lattice at level $l = 5$. Each dot corresponds to a member of the fusion hierarchy of the dilute A_3 model labeled by the Young diagram of the respective representation of $su(3)$.

representations of $su(3)$. Then the decomposition of the basic tensor product representations into irreducible representations is given by

$$\begin{aligned} (n,m)\otimes(1,0) &= (n+1,m)\oplus(n-1,m+1)\oplus(n,m-1) \\ (n,m)\otimes(0,1) &= (n,m+1)\oplus(n+1,m-1)\oplus(n-1,m). \end{aligned} \tag{1.18}$$

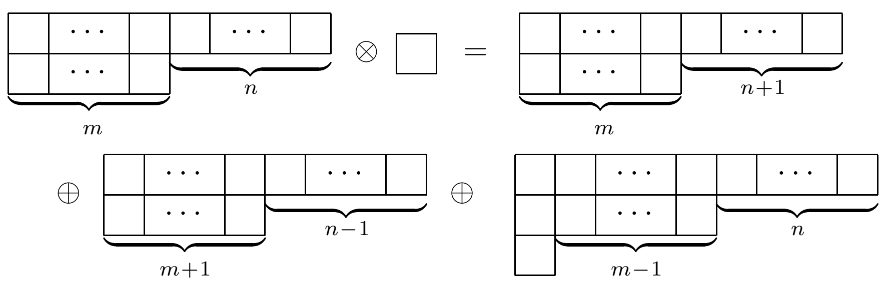
The irreducible representations can be represented by Young tableaux

$(n,m) =$



(1.19)

so that, for example,



(1.20)

where the triple box in a column corresponds to the trivial representation and can be omitted. The relation between these representations is encapsulated in the $su(3)$ weight lattice, which is shown in Figure 2 for the case of level $l = m + n = 5$.

The Bethe ansatz equations (1.15) can be thought of as matrix equations in $\boldsymbol{T}(u)$ and an auxiliary matrix family $\boldsymbol{Q}(u)$ which commutes with $\boldsymbol{T}(u)$. These matrix equations imply the fusion hierarchy

$$\boldsymbol{T}_0^{(n,0)}\boldsymbol{T}_n^{(1,0)} = \boldsymbol{T}_0^{(n-1,1)} + \boldsymbol{T}_0^{(n+1,0)} \tag{1.21}$$

$$\boldsymbol{T}_1^{(0,m)}\boldsymbol{T}_0^{(0,1)} = \boldsymbol{T}_0^{(0,m+1)} + f_0\boldsymbol{T}_1^{(1,m-1)} \tag{1.22}$$

$$\boldsymbol{T}_0^{(n,0)}\boldsymbol{T}_n^{(0,m)} = \boldsymbol{T}_0^{(n,m)} + f_{n-1}\boldsymbol{T}_0^{(n-1,0)}\boldsymbol{T}_{n+1}^{(0,m-1)} \tag{1.23}$$

where $\boldsymbol{T}_k^{(n,m)} = \boldsymbol{T}^{(n,m)}(u+2k\lambda)$ is the row transfer matrix of the fused model of fusion type $(1,0)\times(n,m)$ in the horizontal and vertical directions. Here we have suppressed the horizontal fusion level and in this case $\boldsymbol{T}_k^{(0,0)} = \boldsymbol{I}$, $\boldsymbol{T}_k^{(1,0)} = \boldsymbol{T}(u+2k\lambda)$ and $f_k = f(u+2k\lambda)$

with

$$f(u) = (-1)^N \frac{s(u-3\lambda)s(u-2\lambda)s(u-\lambda)s(u+2\lambda)s(u+3\lambda)s(u+4\lambda)}{s(2\lambda)^3 s(3\lambda)^3}. \tag{1.24}$$

We will later show that the fusion equations (1.21)–(1.23) hold for arbitrary fusion of type $(n', m') \times (n, m)$, again with the horizontal fusion type suppressed. In this general case, the fused face weights involve a rectangular block of $(n'+2m')\times(n+2m)$ elementary faces. For the moment, however, we only consider $(n', m') = (1, 0)$ for simplicity.

To derive the fusion hierarchy, we use semi-standard Young tableaux [19, 18, 20] and set

$$\begin{aligned}
\boxed{1}^{\,k} &= \omega \frac{s(u+2k\lambda-2\lambda)s(u+2k\lambda-3\lambda)}{s(2\lambda)s(3\lambda)} \frac{Q(u+2k\lambda+\lambda)}{Q(u+2k\lambda-\lambda)} \\
\boxed{2}^{\,k} &= \frac{s(u+2k\lambda)s(u+2k\lambda-3\lambda)}{s(2\lambda)s(3\lambda)} \frac{Q(u+2k\lambda)Q(u+2k\lambda-3\lambda)}{Q(u+2k\lambda-\lambda)Q(u+2k\lambda-2\lambda)} \\
\boxed{3}^{\,k} &= \omega^{-1} \frac{s(u+2k\lambda)s(u+2k\lambda-\lambda)}{s(2\lambda)s(3\lambda)} \frac{Q(u+2k\lambda-4\lambda)}{Q(u+2k\lambda-2\lambda)}
\end{aligned} \tag{1.25}$$

so that

$$T_0^{(1,0)} = \boxed{1}^{\,0} + \boxed{2}^{\,0} + \boxed{3}^{\,0} = \sum \boxed{}^{\,0} \tag{1.26}$$

where such summations are performed over all allowed numberings of the boxes using the numbers 1, 2, and 3. For a general Young tableau, the numbers must not decrease moving to the right along a row and must strictly increase moving down a column:

$$\begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 2 & 2 & 3 \\
\hline
2 & 3 & 3 \\
\cline{1-3}
\end{array} \tag{1.27}$$

Such a Young tableau denotes the product of the eight labeled boxes as given by (1.25) where it is understood that the relative shifts in the arguments are given by

$$\begin{array}{|c|c|c|c|c|}
\hline
u+8\lambda & u+6\lambda & u+4\lambda & u+2\lambda & u \\
\hline
u+10\lambda & u+8\lambda & u+6\lambda \\
\cline{1-3}
\end{array}^{\,0} \tag{1.28}$$

and the zero superscript gives the shift in the top right box.

Using this notation, the eigenvalues of the fused row transfer matrix at level (n, m) can be written as

$$T_0^{(n,m)} = \sum \underbrace{\begin{array}{|c|c|c|}
\hline
 & \cdots & \\
\hline
 & \cdots & \\
\hline
\end{array}}_{m}\!\!\begin{array}{c}
\begin{array}{|c|c|c|}
\hline
 & \cdots & \\
\hline
\end{array}^{\,0} \\
\underbrace{\hphantom{xxxxxxx}}_{n}
\end{array} \tag{1.29}$$

where the number of terms in the sum is given by the dimension of the irreducible representations of $su(3)$

$$\dim(n,m) = (n+1)(m+1)(n+m+2)/2. \tag{1.30}$$

It is straightforward to show that these satisfy the fusion equations (1.21)–(1.23) with

$$f_0 = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}^{\,0}\ ,\quad T_0^{(0,1)} = \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}^{\,0} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}^{\,0} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}^{\,0} = (-1)^N \frac{s(u+2\lambda)s(u-3\lambda)}{s(2\lambda)s(3\lambda)} T_{1/2}^{(1,0)} \tag{1.31}$$

The product of factors on the left side of the fusion equations can be precisely partitioned into the two terms on the right side. Although it is not evident from these equations, the $T_0^{(n,m)}$ inductively defined are in fact entire functions of u. Furthermore, these equations close at level $n+m = 2L$ with

$$T_0^{(n,m)} = 0 \qquad \text{if } n+m \geq 2L. \tag{1.32}$$

These facts will be established for the dilute A_L lattice models by carrying out the fusion procedure directly at the level of the face weights in the sequel.

The fusion equations completely determine the fused transfer matrices. Indeed, the solution of (1.21)–(1.23) for the fused transfer matrices $T_0^{(n,m)}$ can be written in the determinantal form

$$T_0^{(n,m)} = \left|\begin{array}{ccccc|ccccc}
T^{(0,1)}_{n+m-1} & f_{n+m-2} & & & & & & & & \\
T^{(1,0)}_{n+m-1} & \ddots & \ddots & & & & & & & \\
1 & \ddots & \ddots & \ddots & & & & & & \\
 & \ddots & \ddots & \ddots & f_n & & & & & \\
 & & 1 & T^{(1,0)}_{n+1} & T^{(0,1)}_{n} & f_{n-1} & & & & \\
\hline
 & & & & 1 & T^{(1,0)}_{n-1} & T^{(0,1)}_{n-2} & f_{n-3} & & \\
 & & & & & 1 & \ddots & \ddots & \ddots & \\
 & & & & & & \ddots & \ddots & \ddots & f_0 \\
 & & & & & & & 1 & T^{(1,0)}_{1} & T^{(0,1)}_{0} \\
 & & & & & & & & 1 & T^{(1,0)}_{0}
\end{array}\right| \tag{1.33}$$

This can be directly verified using standard properties of determinants. In particular, it immediately follows from (1.15) and (1.31) that $T_0^{(n,m)}$ are entire functions of u.

2 Elementary Fusion

Fusion is a process [3] to build up new solutions of the Yang-Baxter equation. The essential idea is to form Z-invariant [44, 1] $p \times q$ blocks of elementary face weights. Then new solutions of the Yang-Baxter equation with distinct critical behavior are obtained by applying suitable projectors. To fuse the dilute A_L models, we will follow the detailed methods of Zhou and Pearce [21]. However, we still feel that at least the basic example of 1×2 fusion should be presented in some detail, allowing us to introduce our notation properly and to keep the paper self-contained.

This is the purpose of this section, where we consider the elementary fusion of a row of two and three faces corresponding to fusion levels $(2,0)$, $(0,1)$ and $(0,0,1)$, the latter corresponding to a Young diagram with three vertically arranged boxes, which in the $su(3)$ case reduces to the trivial representation. Thereafter, in the following section, we treat the more general case of level $(n,0)$ and $(0,n)$ fusion, in both horizontal and vertical directions. Subsequently, in section 4, we finally arrive at the general case of fusion level (n,m).

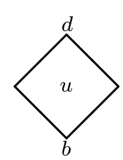
2.1 Projectors

Let us define local face transfer operators $X_j(u)$ with elements

$$\langle\sigma|X_j(u)|\sigma'\rangle = W\left(\begin{matrix}\sigma_{j-1} & \sigma'_j \\ \sigma_j & \sigma_{j+1}\end{matrix}\,\middle|\, u\right) \prod_{k\neq j} \delta_{\sigma_j,\sigma'_j} \tag{2.1}$$

where σ and σ' are allowed paths. The matrix $X_j(u)$ is block diagonal and we denote the blocks for fixed j by

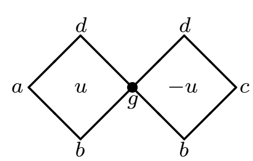
$$X^{(b,d)}(u) \quad = \tag{2.2}$$



The dimension of this block is given by the number of allowed two-step paths from d to b which is $[(I+A)^2]_{b,d} \leq 3$.

In addition to the Yang-Baxter equation the face weights of the dilute A_L models also satisfy the local inversion relation

$$\sum_{g=1}^{L} W\left(\begin{matrix}d & g \\ a & b\end{matrix}\,\middle|\, u\right) W\left(\begin{matrix}d & c \\ g & b\end{matrix}\,\middle|\, -u\right) \quad = \quad [\text{diagram}] \quad = \quad \rho(u)\rho(-u)\delta_{a,c} \tag{2.3}$$



where

$$\rho(u)=\frac{\vartheta_1(2\lambda-u)\vartheta_1(3\lambda-u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)}. \tag{2.4}$$

In terms of matrix multiplication of the face transfer operators, this relation takes the form

$$X^{(b,d)}(u)X^{(b,d)}(-u)=\rho(u)\rho(-u)I. \tag{2.5}$$

It follows that the two pairs of block matrices

(2.6)

are singular. In fact, after appropriate normalization, $X^{(b,d)}(3\lambda)$ and $X^{(b,d)}(2\lambda)$ are projection operators. Although the other two are not strictly projectors, we will subsequently refer to these four singular operators as projectors. Within each pair, these operators are orthogonal as a consequence of the inversion relation.

Either of the above pairs of projectors can be used to construct new solvable models by the fusion procedure [22]. These two fusions are very different in nature. The first pair of projectors leads to $su(2)$ type fusion with adjacency matrices $A^{(n)}$ of the fused lattice models at level n given by the $su(2)$ fusion rules

$$\begin{gathered}A^{(0)}=I,\qquad A^{(1)}=I+A\\ A^{(n)}A^{(1)}=A^{(n-1)}+A^{(n+1)}\quad n=1,2,3,\ldots\end{gathered} \tag{2.7}$$

without any closure. The second pair of projectors leads to $su(3)$ type fusion with adjacency matrices $A^{(n,m)}$ of the fused lattice models at level (n,m) given by $su(3)$ fusion rules

$$\begin{gathered}A^{(n,m)}=0\quad\text{if } n<0 \text{ or } m<0,\qquad A^{(n,m)}=A^{(m,n)}\\ A^{(0,0)}=I,\qquad A^{(1,0)}=I+A\\ A^{(n,m)}A^{(1,0)}=A^{(n+1,m)}+A^{(n-1,m+1)}+A^{(n,m-1)}\quad n,m=0,1,2,\ldots\end{gathered} \tag{2.8}$$

These equations close with

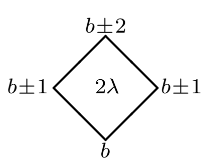
$$A^{(n,m)}=0\quad\text{for } n+m\geq 2L. \tag{2.9}$$

Note that the fusion hierarchy equations (1.21)–(1.23) yield valid adjacency equations if the fused transfer matrices are replaced by the fused adjacency matrices, the shifts are discarded and the functions f_k are set to one. The elements of the fused adjacency matrices can in general be nonnegative integers greater than one. In this case we distinguish the edges of the adjacency diagram joining two given sites by bond variables $\alpha = 1, 2, \ldots$ If there is just one edge then the corresponding bond variable is $\alpha = 1$. More generally, the entries $A^{(n,m)}_{a,b}$ of the fused adjacency matrices give the number of admissible bonds joining states a and b in the fused models at level (n,m). Explicitly, the fused adjacency matrices $A^{(n,m)} = A^{(m,n)}$ for A_3 are given by

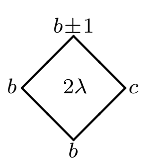
$$
\begin{aligned}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= A^{(0,0)} = A^{(5,0)} = A^{(0,5)} \\
\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} &= A^{(1,0)} = A^{(4,1)} = A^{(0,4)} = A^{(4,0)} = A^{(1,4)} = A^{(0,1)} \\
\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} &= A^{(2,0)} = A^{(3,2)} = A^{(0,3)} = A^{(3,0)} = A^{(2,3)} = A^{(0,2)} \\
\begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix} &= A^{(1,1)} = A^{(3,1)} = A^{(1,3)} \\
\begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 1 \end{pmatrix} &= A^{(1,2)} = A^{(2,1)} = A^{(2,2)}
\end{aligned}
\tag{2.10}
$$

where the $\mathbb{Z}_3$ symmetry of the weight lattice about the fusion level (m,n) is apparent, see Figure 2.

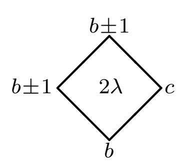
Let us list some properties of the projectors useful for the implementation of $su(3)$ fusion. These either follow from the explicit form of the face weights or from the inversion relation (2.3) with $u = -2\lambda$. From this, the first group is obtained by inserting the face weights at $u = -2\lambda$ for an arbitrary value of a which gives



$= 0$



$$= \sqrt{\frac{\vartheta_4(2b\lambda \mp \lambda)}{\vartheta_4(2b\lambda \pm 3\lambda)}} \tag{2.11}$$



$$\begin{array}{ccc} & b & \\ b\pm1 & 2\lambda & c \\ & b & \end{array} \quad = \quad A_{b\pm1,b}\,\frac{\vartheta_1(\lambda)}{\vartheta_1(2\lambda)}\,\frac{S(b\pm1,b)}{S(b,b)}\;\begin{array}{ccc} & b & \\ b & 2\lambda & c \\ & b & \end{array}$$

for any value of c. Analogously, we obtain a second group of relations by inserting the face weights at $u = 2\lambda$ choosing a value of c

$$\begin{array}{ccc} & b\pm2 & \\ b\pm1 & -2\lambda & b\pm1 \\ & b & \end{array} \quad = \quad \frac{\vartheta_1(4\lambda)\vartheta_1(5\lambda)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)}$$

$$\begin{array}{ccc} & b\pm1 & \\ a & -2\lambda & b \\ & b & \end{array} \quad = \quad -\sqrt{\frac{\vartheta_4(2b\lambda\pm3\lambda)}{\vartheta_4(2b\lambda\mp\lambda)}}\;\begin{array}{ccc} & b\pm1 & \\ a & -2\lambda & b\pm1 \\ & b & \end{array} \tag{2.12}$$

$$A_{b,b-1}\,\frac{S(b-1,b)}{S(b+1,b)}\;\begin{array}{ccc} & b & \\ a & -2\lambda & b-1 \\ & b & \end{array} + \frac{\vartheta_1(2\lambda)}{\vartheta_1(\lambda)}\,\frac{S(b,b)}{S(b+1,b)}\;\begin{array}{ccc} & b & \\ a & -2\lambda & b \\ & b & \end{array} + A_{b,b+1}\;\begin{array}{ccc} & b & \\ a & -2\lambda & b+1 \\ & b & \end{array} = 0$$

where now a is arbitrary. Here we introduced the compact notation

$$S(a,b) \quad = \quad \sqrt{S_a}\,\vartheta_4(2b\lambda + (b-a)\lambda). \tag{2.13}$$

2.2 Level (2,0) Fusion

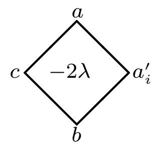
The projector of symmetric 1×2 fusion is the local face transfer operator $X_j(-2\lambda)$, or more explicitly, the blocks $X^{(b,d)}(-2\lambda)$ in (2.2). The action of these projectors is to project out certain two-step paths from d to b. The number of remaining paths are then given by the entries of the fused adjacency matrix $A^{(2,0)}$.

To be more precise, let us denote the set of two-step paths (a,a',b) from a to b by $\text{path}(a,b;2)$. We refer to a path (a,a',b) as *dependent* on a set of paths $\{(a,a'_i,b)\}$ (with respect to $X^{(b,a)}(-2\lambda)$) if there exist $\phi(a,a'_i,b)$ such that

$$\begin{array}{ccc} & a & \\ c & -2\lambda & a' \\ & b & \end{array} \quad = \quad \sum_i \phi(a,a'_i,b)\;\begin{array}{ccc} & a & \\ c & -2\lambda & a'_i \\ & b & \end{array} \tag{2.14}$$

holds for any c. In other words, a set of paths $\{(a, a'_i, b)\}$ is called *independent* if

$$\sum_i \phi(a, a'_i, b)$$



$$= \quad 0 \qquad (2.15)$$

implies that all coefficients vanish, i.e., $\phi(a, a'_i, b) = 0$. Note that this definition of dependent and independent paths obviously depends on the projector under consideration.

Given a dependent set of paths, the choice of a maximal independent subset is by no means unique. However, this does not matter for our purpose as different choices result in equivalent fused models related to each other by a local gauge. This allows us to use arbitrary sets of independent paths in the construction of the fused face weights. We denote such a set of independent two-step paths from a to b (w.r.t. $X^{(b,a)}(-2\lambda)$) by $\mathrm{indpath}_{(2,0)}[a, b]$. The actual number of these paths is of course determined by the number of non-zero eigenvalues[4] of the block $X^{(b,a)}(-2\lambda)$. The eigenvector for the zero eigenvalue of $X^{(b,a)}(-2\lambda)$ are given in the second and third equation in (2.12). As shown before, these immediately follow from the inversion relation. Note, however, that we do *not* have to use the explicit form of the eigenvectors for non-zero eigenvalues in what follows.

For our present discussion, this means that there is only one case where we have more than one independent path to consider, namely if $a = b$ and $\mathrm{val}(a) = 3$ (1.4), i.e., if $2 \le a \le L-1$. In this case there are three allowed paths $(a, a-1, a)$, (a, a, a) and $(a, a+1, a)$ which are dependent by eq. (2.12), so one is left with two independent paths in this case. For the fused face weights, this implies that one has to consider two different kinds of bonds labeled by a *bond variable* α which takes two values $\alpha = 1, 2$ corresponding to the two independent paths.

The allowed two-step paths on the adjacency diagram of the dilute A_L models shown in Figure 1(b) are

$$\mathrm{path}(a, b; 2) \quad = \quad \begin{cases} \{(a, a{\pm}1, a{\pm}2)\} & \text{if } b = a \pm 2 \\ \{(a, a, a{\pm}1), (a, a{\pm}1, a{\pm}1)\} & \text{if } b = a \pm 1 \\ \{(1, 2, 1), (1, 1, 1)\} & \text{if } a = b = 1 \\ \{(L, L, L), (L, L{-}1, L)\} & \text{if } a = b = L \\ \{(a, a{+}1, a), (a, a, a), (a, a{-}1, a)\} & \text{if } a = b,\ \mathrm{val}(a) = 3 \end{cases} \qquad (2.16)$$

[4]Remember that $X^{(b,a)}(-2\lambda)$ is not strictly a projector, hence its eigenvalues do not have to be 0 or 1.

¿From those, let us choose the independent paths for the construction of the $(2,0)$ fused face weights as follows

$$\mathrm{indpath}_{(2,0)}[a,b] \;=\; \begin{cases} \{(a,a{\pm}1,a{\pm}2)\} & \text{if } b=a\pm 2 \\ \{(a,a{+}1,a{+}1)\} & \text{if } b=a+1 \\ \{(a,a,a{-}1)\} & \text{if } b=a-1 \\ \{(1,2,1)\} & \text{if } a=b=1 \\ \{(L,L{-}1,L)\} & \text{if } a=b=L \\ \{(a,a{+}1,a),(a,a{-}1,a)\} & \text{if } a=b,\ \mathrm{val}(a)=3 \end{cases} \tag{2.17}$$

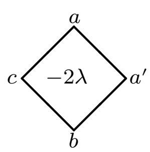
or, in other words,

$$\mathrm{indpath}_{(2,0)}[a,b] \;=\; \{(a,a',b)\in \mathrm{path}(a,b;2)\,|\,a'_\alpha \neq \min(a,b)\}\ . \tag{2.18}$$

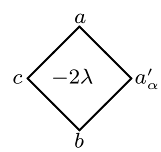
Obviously, the number of independent paths is

$$\Big|\mathrm{indpath}_{(2,0)}[a,b]\Big| \;=\; A^{(2,0)}_{a,b} \tag{2.19}$$

where $A^{(2,0)}$ is the fused adjacency matrix of eq. (2.8). We define $\phi_{(2,0)}(a,a',b|\alpha)$ as the coefficients of the path (a,a',b) in terms of the "basis" of independent paths $\mathrm{indpath}_{(2,0)}[a,b]$, where $1\le\alpha\le A^{(2,0)}_{a,b}$ denotes the first and, if existing, second element of $\mathrm{indpath}_{(2,0)}[a,b]$, respectively. This means



$$= \sum_{\alpha=1}^{A^{(2,0)}_{a,b}} \phi_{(2,0)}(a,a',b|\alpha) \tag{2.20}$$



where (a,a'_α,b) is the corresponding independent path. The coefficients can be read off from eqs. (2.12), explicitly they are given by

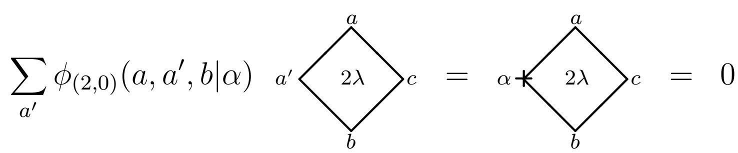
$$\phi_{(2,0)}(a,a',b|\alpha) = \begin{cases} \delta_{\alpha,1} & \text{if } |a-b|=2 \\ \left(-\sqrt{\dfrac{\vartheta_4(2a'_\alpha\lambda+\lambda)}{\vartheta_4(2a'_\alpha\lambda-3\lambda)}}\right)^{a'_\alpha-a'}\delta_{\alpha,1} & \text{if } |a-b|=1 \\ \left(-\dfrac{\vartheta_1(\lambda)}{\vartheta_1(2\lambda)}\,\dfrac{S(a'_\alpha,a)}{S(a,a)}\right)^{\delta_{a,a'}}\delta_{\alpha,1} & \text{if } a{=}b,\ \mathrm{val}(a){=}2 \\ \left(-\dfrac{\vartheta_1(\lambda)}{\vartheta_1(2\lambda)}\,\dfrac{S(a'_\alpha,a)}{S(a,a)}\right)^{\delta_{a,a'}} & \text{if } a{=}b,\ \mathrm{val}(a){=}3,\ |a'_\alpha{-}a'|{<}2 \\ 0 & \text{otherwise} \end{cases} \tag{2.21}$$

where it is understood that $\phi_{(2,0)}(a,a',b|\alpha)=0$ if $(a,a',b)\notin\mathrm{path}(a,b;2)$ and a'_α is defined as above.

To phrase it differently, and maybe more clearly, the blocks $X^{(b,a)}(-2\lambda)$ are square matrices in the basis given by path$(a, b; 2)$. From the inversion relation, one obtains eigenvectors of $X^{(b,a)}(-2\lambda)$ with eigenvalue 0 (if $|b-a| < 2$), eqs. (2.12) give their components in the basis of the paths up to an arbitrary normalization. The $\phi_{(2,0)}(a, a', b|\alpha)$ defined above are nothing else than components of α linearly independent vectors which span the orthogonal complement of the zero eigenvalue eigenspace of $X^{(b,a)}(-2\lambda)$. Note that these are in general not eigenvectors of $X^{(b,a)}(-2\lambda)$ and therefore the vectors for different α are not necessarily mutually orthogonal; for instance, for $a = b$, $\mathrm{val}(a) = 3$, the three vectors $|\alpha\rangle$ ($\alpha = 0, 1, 2$, $\alpha = 0$ denoting the eigenvector with zero eigenvalue) have the following components in the basis path$(a, a; 2)$ (2.16)

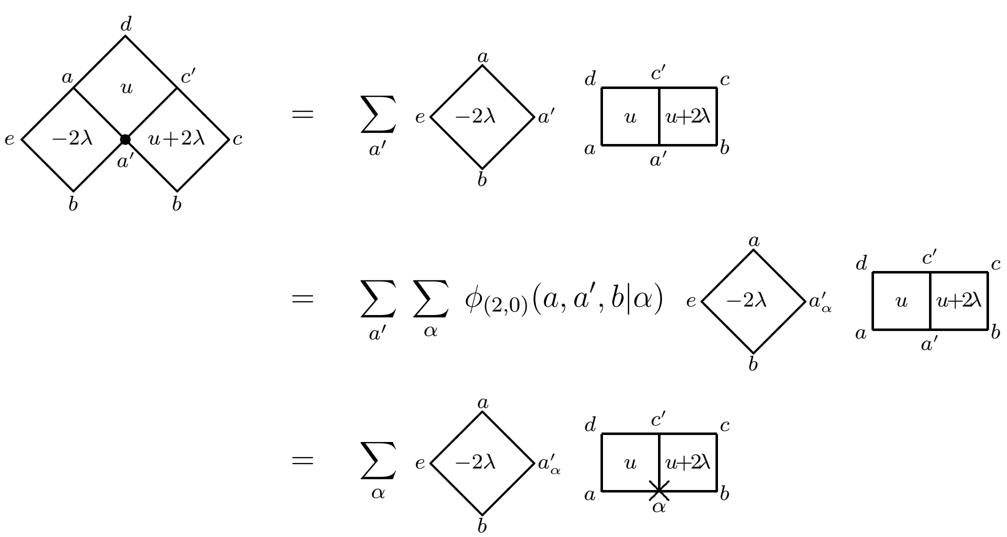
$$\begin{aligned} |0\rangle &= \left(S(a+1,a), \frac{\vartheta_1(2\lambda)}{\vartheta_1(\lambda)} S(a,a), S(a-1,a) \right) \\ |1\rangle &= \left(1, -\frac{\vartheta_1(\lambda)}{\vartheta_1(2\lambda)} \frac{S(a+1,a)}{S(a,a)}, 0 \right) \\ |2\rangle &= \left(0, -\frac{\vartheta_1(\lambda)}{\vartheta_1(2\lambda)} \frac{S(a-1,a)}{S(a,a)}, 1 \right) \end{aligned} \tag{2.22}$$

where $|1\rangle$ and $|2\rangle$ are both orthogonal to $|0\rangle$ but not orthogonal to each other. We can split any summation over paths (a, a', b) into a summation over the zero eigenvalue eigenvectors of $X^{(b,a)}(-2\lambda)$ and its orthogonal complement which yields a sum over α with coefficients $\phi_{(2,0)}(a, a', b|\alpha)$ by choosing a suitable basis. Clearly, eqs. (2.11) are just



(2.23)

for $\alpha = 1, 2$ and any value of c. We refer to the sum with coefficients $\phi_{(2,0)}(a, a', b|\alpha)$ as the *symmetric sum* and denote it by a cross with label α. In particular, we find the following decomposition ("split property")



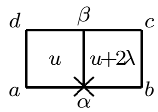
(2.24)

where in the last step we performed the summation over a'. Here, the sum on the RHS includes only the independent paths (a, a'_α, b), $\alpha \in \{1, 2\}$.

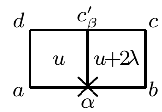
After these preliminary remarks, we are finally in the position to define the fused face weights for the elementary symmetric fusion. These are basically the symmetric sums on the RHS of eq. (2.24), but we still have to make a choice how to relate bond variable β to the value of c'. In the case $c = d$ and $\mathrm{val}(c) = 3$ this choice is not completely free, due to our selection of independent paths we have to exclude $c' = c$, because the path (c, c, c) has non-zero coefficients in terms of both independent paths. For definiteness and simplicity, we choose c' such that (d, c', c) is path β in the set $\mathrm{indpath}_{(2,0)}[d, c]$.

Lemma 2.1 (Elementary Symmetric Fusion) *If (a, b) and (d, c) are admissible edges at fusion level $(2, 0)$, we define the 1×2 fused weights by*

$$W_{(2,0)}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix} \,\middle|\, u\right) \quad = \quad$$



$$=$$



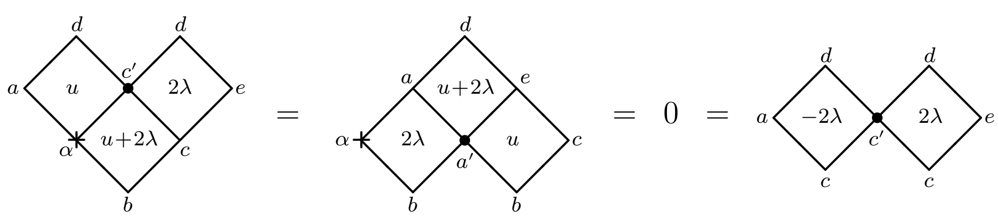
$$= \sum_{a'} \phi_{(2,0)}(a, a', b|\alpha)\; W\left(\begin{matrix} d & c'_\beta \\ a & a' \end{matrix} \,\middle|\, u\right) W\left(\begin{matrix} c'_\beta & c \\ a' & b \end{matrix} \,\middle|\, u{+}2\lambda\right) \qquad (2.25)$$

where the sum is over all allowed spins a', the bond variables take values $1 \le \alpha \le A^{(2,0)}_{a,b}$, $1 \le \beta \le A^{(2,0)}_{c,d}$, and where the coefficients $\phi_{(2,0)}(a, a', b|\alpha)$ are those of eq. (2.21). Furthermore, the value of c' on the RHS is chosen such that $(d, c', c) \in \mathrm{indpath}_{(2,0)}[d, c]$, β labeling the elements of $\mathrm{indpath}_{(2,0)}[d, c]$. The such defined weights satisfy the Yang-Baxter equation. In particular, we note that

$$W_{(2,0)}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix} \,\middle|\, 0\right) = W_{(2,0)}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix} \,\middle|\, \lambda\right) = 0 \qquad (2.26)$$

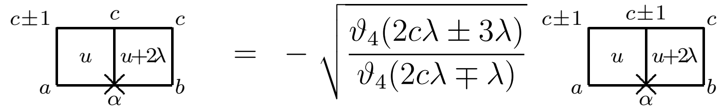
for all $a, b, c, d, \alpha, \beta$.

To show that the Yang-Baxter equation is indeed satisfied, we proceed as follows. From the Yang-Baxter equation of the elementary weights and eq. (2.23), one obtains

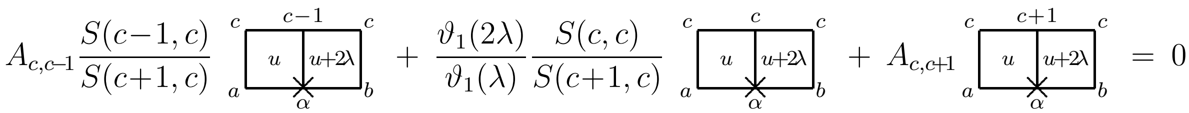


(2.27)

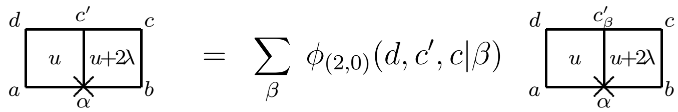
This establishes the "push-through" property, i.e., the dependence of the symmetric sum in (2.27) on the path (d, c', c) is obviously the same as that of the projector $X^{(c,d)}(-2\lambda)$, see eq. (2.12). Explicitly, this yields



(2.28)

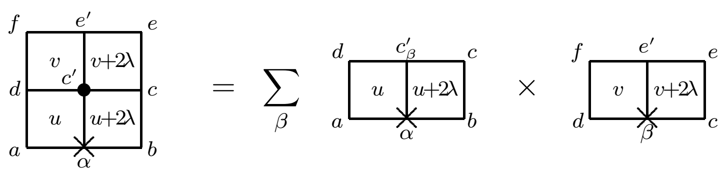


or, for short,



(2.29)

In complete analogy to the decomposition in eq. (2.24), one finds



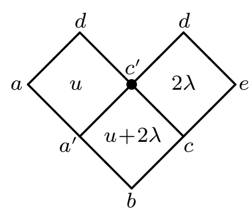
(2.30)

and therefore the Yang-Baxter equation for the fused weights follows immediately from that for the elementary faces (1.13). This also shows why it is most convenient to choose the c' in the definition of the fused weights (2.25) to correspond to an independent path and that one has to be careful if there is more than one independent path.

From the initial condition (1.5), the fused weights (2.25) at $u = 0$ reduce to the symmetric sum of the elementary face weight at $u = 2\lambda$ which vanishes due to eq. (2.23). Similarly, one obtains the second part of eq. (2.26) using the crossing symmetry (1.7). This implies that the fused weights contain an overall factor of $\vartheta_1(u)\vartheta_1(u-\lambda)$.

2.3 *Level (0,1) Fusion*

The antisymmetric 1×2 fusion is constructed by using the blocks $X^{(c,d)}(2\lambda)$ as projectors. This means that the fused weights are basically given by the products



(2.31)

Here, things are somewhat simpler then for the $(2,0)$ fusion, since the blocks $X^{(c,d)}(2\lambda)$, up to normalization, really are projectors onto at most one-dimensional spaces. In particular, this implies that we do not need to introduce a bond variable.

Note that we define the $(0,1)$ fusion by summing over the variable c' instead a'. This convention allows us to use the same product of elementary face weights in both cases (otherwise we would have to interchange u and $u+2\lambda$). This also means that the definition of independent paths $\mathrm{indpath}_{(0,1)}[d,c]$ apparently involves "the other side" of the projector which however makes no difference since the weights (1.2) are symmetric under reflection, see eq. (1.6). Let us choose the following sets of independent paths

$$\mathrm{indpath}_{(0,1)}[d,c] \quad = \quad \{(d,c',c) \in \mathrm{path}(d,c;2) | c' = \min(c,d)\} \tag{2.32}$$

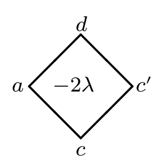
which is just the complement of the set of independent paths for level $(2,0)$ fusion, see eqs. (4.3)–(2.18). Clearly, the corresponding adjacency matrix $A^{(0,1)}$ for the fused weights coincides with that of the elementary face weights, i.e., $A^{(0,1)} = A^{(1,0)}$ (2.8). From eq. (2.11), we obtain

$$\phi_{(0,1)}(d,c',c) = \begin{cases} \left(\dfrac{\vartheta_4(2c_1'\lambda+3\lambda)}{\vartheta_4(2c_1'\lambda-\lambda)}\right)^{(c'-c_1')/2} & \text{if } |c-d|=1 \\ \left(\dfrac{\vartheta_1(\lambda)}{\vartheta_1(2\lambda)}\dfrac{S(c',d)}{S(d,d)}\right)^{|c'-d|} & \text{if } c=d \\ 0 & \text{otherwise} \end{cases} \tag{2.33}$$

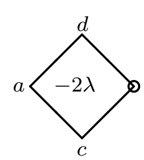
for all $(d,c',c) \in \mathrm{path}(d,c;2)$ (2.16) and zero otherwise, where $c_1' = \min(c,d)$.

We refer to the sum over c' with coefficients $\phi_{(0,1)}(d,c',c)$ as *antisymmetric sum* and denote it by a circle (without any further label) in our diagrammatical notation. Obviously, eq. (2.12) becomes

$$\sum_c \phi_{(0,1)}(d,c',c)$$

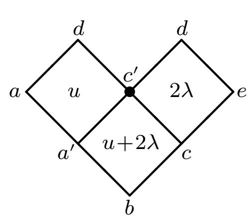


$=$

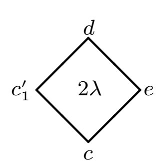


$$= \quad 0 \tag{2.34}$$

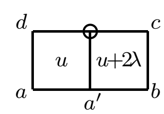
and the split relation analogous to eq. (2.24) is simply



$=$

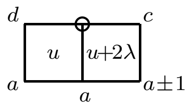


$\times$

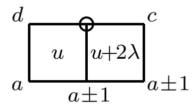


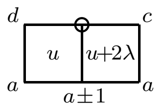
(2.35)

where (d, c_1', c) is the corresponding independent path, i.e., $c_1' = \min(c, d)$. As before, we define the fused weights to be the object on the RHS of the above equation, up to a choice on the values of a'. By the same arguments as above, the dependence on a' can be read off directly from eq. (2.11), yielding

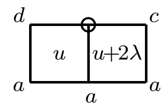


$$= \sqrt{\frac{\vartheta_4(2a\lambda \mp \lambda)}{\vartheta_4(2a\lambda \pm 3\lambda)}}$$

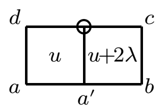




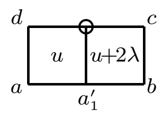
$$= A_{a\pm1,a} \frac{\vartheta_1(\lambda)}{\vartheta_1(2\lambda)} \frac{S(a\pm1, a)}{S(a, a)} \tag{2.36}$$



which is just



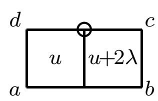
$$= \phi_{(0,1)}(a, a', b) \tag{2.37}$$



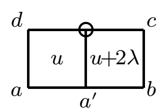
with $a_1' = \min(a, b)$. Again, we choose a' (2.35)such that $(a, a', b) \in \mathrm{indpath}_{(0,1)}[a, b]$ to define the fused weights.

Lemma 2.2 (Elementary Antisymmetric Fusion) *Define*

$$W_{(0,1)}\left(\begin{matrix} d & c \\ a & b \end{matrix} \,\middle|\, u\right) =$$



$$=$$



$$= \sum_{c'} \phi_{(0,1)}(d, c', c)\, W\left(\begin{matrix} d & c' \\ a & a' \end{matrix} \,\middle|\, u\right) W\left(\begin{matrix} c' & c \\ a' & b \end{matrix} \,\middle|\, u+2\lambda\right) \tag{2.38}$$

where we sum over all allowed values of c' and where $a' = \min(a, b)$. Then the fused weights satisfy

$$\begin{aligned} W_{(0,1)}\left(\begin{matrix} d & c \\ a & b \end{matrix} \,\middle|\, u\right) &= -\, s_1^1\, s_{-3/2}^1 \frac{g(d, c)}{g(a, b)}\, W\left(\begin{matrix} d & c \\ a & b \end{matrix} \,\middle|\, u+\lambda\right) \\ &= -\, r_1^1 \frac{g(d, c)}{g(a, b)}\, W\left(\begin{matrix} d & c \\ a & b \end{matrix} \,\middle|\, u+\lambda\right) \end{aligned} \tag{2.39}$$

where the gauge factors $g(a, b)$ are given by

$$g(a, b) = \left(\frac{\vartheta_1(2\lambda)}{\vartheta_1(\lambda)}\right)^{|a-b|} \times \begin{cases} \sqrt{\dfrac{\vartheta_4(2b\lambda + \lambda)}{\vartheta_4(2b\lambda - \lambda)}} & \text{for } a > b \\ \sqrt{\dfrac{\vartheta_4(2b\lambda - \lambda)}{\vartheta_4(2b\lambda - 3\lambda)}} & \text{for } a < b \\ \sqrt{\dfrac{\vartheta_4(2b\lambda + \lambda)\vartheta_4(2b\lambda - \lambda)}{\vartheta_4^2(2b\lambda)}} & \text{for } a = b \end{cases} \tag{2.40}$$

and s_n^m and r_n^m are functions of u defined as

$$
\begin{aligned}
s_n^m &= \prod_{j=0}^{m-1} \frac{\vartheta_1(u+2(n-j)\lambda)}{\sqrt{\vartheta_1(2\lambda)\vartheta_1(3\lambda)}} && (2.41)\\
r_n^m &= s_n^m\, s_{n-5/2}^m = \prod_{j=0}^{m-1} \frac{\vartheta_1(u+2(n-j)\lambda)\vartheta_1(u+(2(n-j)-5)\lambda)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)}\,. && (2.42)
\end{aligned}
$$

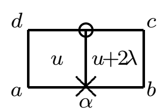
In particular, this implies that

$$
W_{(0,1)}\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, -2\lambda\right) = W_{(0,1)}\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, 3\lambda\right) = 0 \qquad (2.43)
$$

for all a, b, c, d.

In other words, up to a gauge and some overall factors the $(0,1)$ fused weights are nothing but the elementary face weights (1.2) we started with. This of course already implies that the Yang-Baxter equation is fulfilled which alternatively follows from the push-through properties (2.36) as in the symmetric case. We omit the proof of the lemma since it reduces to the explicit computation of the fused weights. However, the appearance of the overall factor $r_1^1 = s_1^1 s_{-3/2}^1$ in (2.39) can be understood directly from eq. (2.43) which follows from eq. (2.34) using the initial condition (1.5) and crossing symmetry (1.7).

The symmetric and antisymmetric fusion that we introduced so far are "orthogonal" in the sense that



$$= \quad 0 \qquad (2.44)$$

for all a, b, c, d and α. This follows for example from eq. (2.27), which basically is just the inversion relation (2.5) for $u = \pm 2\lambda$. But this is not all, the two projectors act also complementary in the sense that there is no non-trivial subspace that is annihilated by both the blocks $X^{(b,d)}(2\lambda)$ and $X^{(b,d)}(-2\lambda)$, provided that $|b-d| \leq 2$. This follows from the inversion relation (2.5) since the RHS has only simple zeros in the spectral parameter u. If both blocks really were projectors, this would mean that they add up to the identity matrix, but of course that is not the case here. However, it implies that we can choose the independent paths such that the two sets of independent paths $\mathrm{indpath}_{(2,0)}(a,b)$ and $\mathrm{indpath}_{(0,1)}(a,b)$ are disjoint and that their union consists of all allowed paths, i.e.,

$$
\begin{aligned}
\mathrm{indpath}_{(2,0)}(a,b) \,\cap\, \mathrm{indpath}_{(0,1)}(a,b) &= \emptyset \\
\mathrm{indpath}_{(2,0)}(a,b) \,\cup\, \mathrm{indpath}_{(0,1)}(a,b) &= \mathrm{path}(a,b;2) && (2.45)
\end{aligned}
$$

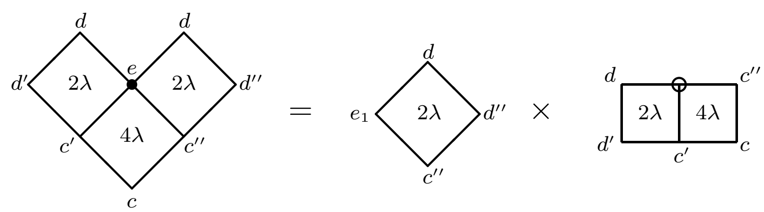
which is exactly what we did, see eqs. (4.3)–(2.18) and (2.32). In particular, this implies the relation

$$\left(A^{(1,0)}\right)^2 \;=\; A^{(2,0)} \;+\; A^{(0,1)} \tag{2.46}$$

for the fused adjacency matrices.

2.4 Antisymmetric 1×3 Fusion: Level (0,0,1)

Before we move on to the higher level symmetric fusion, we first have a look at the completely antisymmetric 1×3 fused weights. The corresponding projector is the following product of elementary face weights



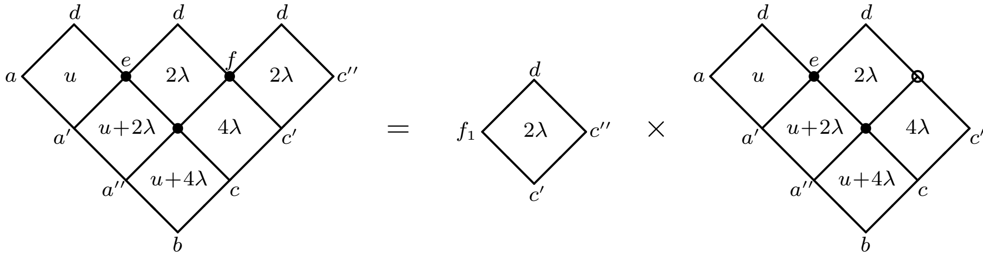
(2.47)

where the above equality is just eq. (2.35) with $u = 2\lambda$ and e_1 is defined accordingly.

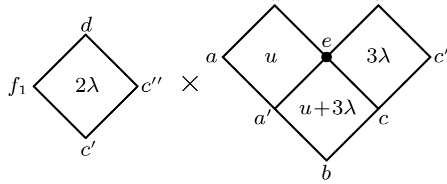
Of course, eq. (2.47) means that we can equally well regard the antisymmetric sum on the RHS as the projector defining the fused weights. By Lemma (2.2), it is proportional to the elementary face weights at $u = 3\lambda$ which in turn is crossing-related to $u = 0$ and hence proportional to $\delta_{c,d}$. Furthermore, the local face operators (2.1) at $u = 3\lambda$ represent the Temperley–Lieb algebra [45] and its blocks are (up to normalization) projectors onto at most one dimension. This implies that the same is true for our projector (2.47), viewed as a matrix acting from path (d, d', c', c) to (d, d'', c'', c), hence we have to deal with only one independent path here. Let us choose this path to be (c, c, c, c). Then the only non-vanishing coefficients $\phi_{(0,0,1)}(d, d', c', c)$ are given by $\phi_{(0,0,1)}(c, c, c, c) = 1$ and

$$\begin{aligned}\phi_{(0,0,1)}(c, c{\pm}1, c, c) \;&=\; \phi_{(0,0,1)}(c, c, c{\pm}1, c) \\ &=\; \sqrt{\frac{\vartheta_4(2c\lambda \mp \lambda)}{\vartheta_4(2c\lambda \pm 3\lambda)}}\;\phi_{(0,0,1)}(c, c{\pm}1, c{\pm}1, c) \;=\; \frac{\vartheta_1(\lambda)}{\vartheta_1(2\lambda)}\frac{S(c{\pm}1, c)}{S(c, c)}\end{aligned} \tag{2.48}$$

The fused weights are deduced from the product of elementary faces with the projector of eq. (2.47). From the above remarks, we find



$$= \; h(u)\, \phi_{(0,1)}(a',a'',b)\, \frac{g(d,c')}{g(a',b)} \tag{2.49}$$



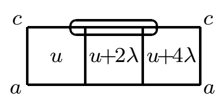
where

$$h(u) \;\; = \;\; -\,\frac{\vartheta_1(\lambda)\vartheta_1(4\lambda)\vartheta_1(u-\lambda)\vartheta_1(u+4\lambda)}{\vartheta_1^2(2\lambda)\,\vartheta_1^2(3\lambda)} \tag{2.50}$$

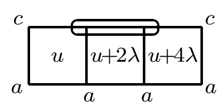
Hence the fused weights are basically the same as those of antisymmetric $su(2)$ type fusion with the projector $X^{(c,d)}(3\lambda)$. This shows that the fused weights are essentially trivial, in particular they vanish unless $c = d$ and also $a = b$, which follows using the Yang-Baxter equation (1.13). We obtain the following result for the fused weights.

Lemma 2.3 (Antisymmetric 1×3 Fusion) *The non-zero fused weights defined by*

$$W_{(0,0,1)}\left(\begin{matrix} c & c \\ a & a \end{matrix}\,\middle|\, u\right) \;\; = $$



$$=$$



$$= \sum_{d',c'} \phi_{(0,0,1)}(c,d',c',c)\; W\left(\begin{matrix} c & d' \\ a & a \end{matrix}\,\middle|\, u\right) W\left(\begin{matrix} d' & c' \\ a & a \end{matrix}\,\middle|\, u+2\lambda\right) W\left(\begin{matrix} c' & c \\ a & a \end{matrix}\,\middle|\, u+4\lambda\right) \tag{2.51}$$

have the following simple form

$$W_{(0,0,1)}\left(\begin{matrix} c & c \\ a & a \end{matrix}\,\middle|\, u\right) \;\; = \;\; f_0^1\, \frac{g(c)}{g(a)} \tag{2.52}$$

where we define functions f_n^m of u by

$$f_k^m \;\; = \;\; (-1)^m\, r_{k+1}^m\, r_{k+3/2}^m\, r_{k+2}^m \;\; = \;\; (-1)^m\, s_{k-3/2}^m\, s_{k-1}^m\, s_{k-1/2}^m\, s_{k+1}^m\, s_{k+3/2}^m\, s_{k+2}^m \tag{2.53}$$

and $g(a)$ is given by

$$g(a) = \sqrt{\frac{\vartheta_4(2a\lambda+\lambda)\vartheta_4(2a\lambda-\lambda)}{\vartheta_4^2(2a\lambda)}}\;. \tag{2.54}$$

Clearly, one has again the split and push-through properties for the fused weights as in the two previous cases. However, we do not list them here as there is nothing to prove apart from an explicit calculation of the weights.

Note that the $g(a)$ act as a simple gauge. This means that the fused weights are basically trivial, in the sense that the transfer matrix constructed from these weights is a multiple of the identity matrix. This is in perfect agreement with the expected $su(3)$ structure of the fusion hierarchy of the dilute A_L models, compare eq. (1.31).

Furthermore, Lemma 2.2 eventually leads to a symmetry between level (n, m) and level (m, n) fusion, the fused weights being related by a shift in the spectral parameter and a gauge transformation. So in fact we would only need one half of the weight lattice of $su(3)$. The reason for this is that we should rather have used the level $l = n+m$ weight lattice of the twisted affine Lie algebra $A_2^{(2)}$, because the dilute A_L models are related to the Izergin-Korepin R matrix [28] which belongs to the quantum algebra $U_q(A_2^{(2)})$ [32]. However, we think it is conceptually advantageous to introduce fused models for all (n, m) and regard the symmetry as an additional property.

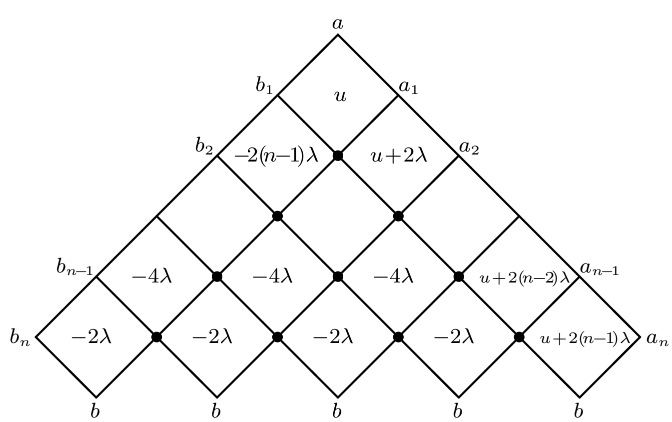
3 Fusion at Levels $(n,0)$ and $(0,n)$

We now want to apply the fusion procedure to higher level, starting with the discussion of symmetric fusion. To do so, we commence by generalizing our notation of paths and independent paths.

3.1 Projectors and Paths

Let us define graphically

$$P_{(n,0)}(u)^{(a,a_1,a_2,\cdots,a_n,b)}_{(a,b_1,b_2,\cdots,b_n,b)} \quad = \tag{3.1}$$



where we regard $P_{(n,0)}(u)$ as an operator acting on the (n+1)-step path $(a, b_1, b_2, \cdots, b_n, b)$ to $(a, a_1, a_2, \cdots, a_n, b)$. As we will see in what follows, $P_{(n,0)}(u)$ will basically define the weights of level $(n, 0)$ fusion, and $P_{(n,0)}(-2n\lambda)$ corresponds to the "projector" of symmetric level $(n+1, 0)$ fusion.

Clearly, $P_{(1,0)}(u)$ is just an elementary block, and $P_{(2,0)}(u)$ is related to the 1×2 symmetric fusion presented in section 2.2. In particular, eq. (2.24) becomes

$$P_{(2,0)}(u)^{(d,c',c,b)}_{(d,a,e,b)} \;=\; \sum_{\alpha} P_{(1,0)}(-2\lambda)^{(a,a'_\alpha,b)}_{(a,e,b)}\; W_{(2,0)}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) \tag{3.2}$$

where the sum labeled by the bond variable $\alpha \in \{1, 2\}$ is over all independent paths $(a, a'_\alpha, b) \in \mathrm{indpath}_{(2,0)}[a, b]$.

Now consider $P_{(n,0)}(u)$. By re-arranging elementary faces using the Yang-Baxter equation (1.13), it is easy to see that any two adjacent faces with the spectral parameters $u + 2j\lambda$ and $u + 2(j+1)\lambda$ in (3.1) can be considered as an instance of level $(2, 0)$ fusion. Therefore the properties (2.24) and (2.28) imply the relations

$$\begin{aligned} &P_{(n,0)}(u)^{(a,a_1,\cdots,a_j,a_{j+2},a_{j+2},\cdots,a_n,b)}_{(a,b_1,\cdots,b_j,b_{j+1},b_{j+2},\cdots,b_n,b)} \\ &\qquad = \; -\sqrt{\frac{\vartheta_4(2a_{j+2}\lambda \pm 3\lambda)}{\vartheta_4(2a_{j+2}\lambda \mp \lambda)}}\; P_{(n,0)}(u)^{(a,a_1,\cdots,a_j,a_j,a_{j+2},\cdots,a_n,b)}_{(a,b_1,\cdots,b_j,b_{j+1},b_{j+2},\cdots,b_n,b)} \end{aligned} \tag{3.3}$$

for $a_j - a_{j+2} = \pm 1$ and

$$\begin{aligned} &A_{a_j,a_j-1}\,\frac{S(a_j-1, a_j)}{S(a_j+1, a_j)}\; P_{(n,0)}(u)^{(a,a_1,\cdots,a_j,a_j-1,a_{j+2},\cdots,a_n,b)}_{(a,b_1,\cdots,b_j,b_{j+1},b_{j+2},\cdots,b_n,b)} \\ &\qquad + \frac{\vartheta_1(2\lambda)}{\vartheta_1(\lambda)}\,\frac{S(a_j, a_j)}{S(a_j+1, a_j)}\; P_{(n,0)}(u)^{(a,a_1,\cdots,a_j,a_j,a_{j+2},\cdots,a_n,b)}_{(a,b_1,\cdots,b_j,b_{j+1},b_{j+2},\cdots,b_n,b)} \\ &\qquad + A_{a_j,a_j+1}\; P_{(n,0)}(u)^{(a,a_1,\cdots,a_j,a_j+1,a_{j+2},\cdots,a_n,b)}_{(a,b_1,\cdots,b_j,b_{j+1},b_{j+2},\cdots,b_n,b)} \;=\; 0 \end{aligned} \tag{3.4}$$

for $a_j = a_{j+2}$. These two equations take over both the role of eq. (2.12) in level $(n+1, 0)$ fusion (with $u = -2n\lambda$) and, via a generalized split property (2.24), of eq. (2.28) in level $(n, 0)$ fusion.

Of course, eqs. (3.3) and (3.4) only express that any antisymmetric sum of $P_{(n,0)}(u)^{(a,a_1,\cdots,a_n,b)}_{(a,b_1,\cdots,b_n,b)}$ over any of the variables a_j vanishes. Therefore we can summarize them by

$$\sum_{a_j} \phi_{(0,1)}(a_{j-1}, a_j, a_{j+1})\; P_{(n,0)}(u)^{(a,a_1,\cdots,a_n,b)}_{(a,b_1,\cdots,b_n,b)} \;=\; 0 \tag{3.5}$$

for $1 \le j \le n$, where we set $a_0 = a$ and $a_{n+1} = b$.

As before, we denote the set of n-step paths from a to b on the effective adjacency diagram of Figure 1(b) by $\mathrm{path}(a,b;n)$, or $(a,b;n)$ for short. The number of such paths is given by

$$|\mathrm{path}(a,b;n)| \quad = \quad |(a,b;n)| \quad = \quad [(I+A)^n]_{a,b} \tag{3.6}$$

i.e., by the corresponding element of the n-th power of the effective adjacency matrix $A^{(1,0)} = I + A$. In this basis, $P_{(n,0)}(u)$ becomes a square matrix which we can choose block-diagonal by an appropriate ordering of paths. Let us introduce $(a,b;n|j)$ as a short-hand notation for the j-th element in $\mathrm{path}(a,b;n)$.

The notion of independent paths also generalizes immediately from the discussion of section 2. Here, a set of $(n+1)$-step paths $\{(a,a_1^i,\cdots,a_n^i,b)\}$ is *independent* (w.r.t. $P_{(n,0)}(-2n\lambda)$) if, for any path $(a,b_1,\cdots,b_n,b)$,

$$\sum_i \phi(a,a_1^i,\cdots,a_n^i,b)\; P_{(n,0)}(-2n\lambda)^{(a,a_1^i,\cdots,a_n^i,b)}_{(a,b_1,\cdots,b_n,b)} \quad = \quad 0 \tag{3.7}$$

implies that all coefficients $\phi(a,a_1^i,\cdots,a_n^i,b)$ vanish. We denote by $\mathrm{indpath}_{(n+1,0)}[a,b] = [a,b;(n+1,0)]$ a maximal set of independent paths from a to b, by $|[a,b;(n+1,0)]|$ the number of its elements, and abbreviate its α-th element by $[a,b;(n+1,0)|\alpha]$ (in order to avoid confusion, we will always use greek letters for indices referring to independent paths).

As in section 2, we define coefficients $\phi_{(n+1,0)}$ by

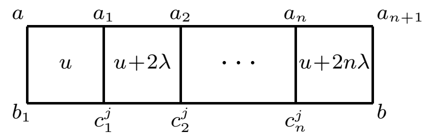
$$P_{(n,0)}(-2n\lambda)^{(a,b;n+1|j)}_{(a,b;n+1|k)} \quad = \quad \sum_{\alpha=1}^{|[a,b;(n+1,0)]|} \phi_{(n+1,0)}[a,b](j|\alpha)\; P_{(n,0)}(-2n\lambda)^{[a,b;(n+1,0)|\alpha]}_{(a,b;n+1|k)} \tag{3.8}$$

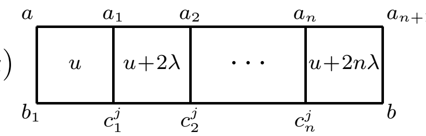
so the $\phi_{(n+1,0)}[a,b](j|\alpha)$ are the "coordinates" of the path $(a,b;n+1|j)$ in the basis of independent paths $[a,b;(n+1,0)|\alpha]$.

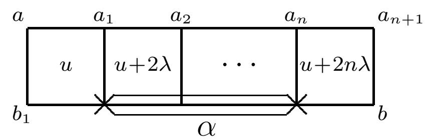
From these definitions, we immediately obtain the generalization of the split property

(2.24) to the $(n,0)$ case. By definition, we have

$$P_{(n+1,0)}(u)^{(a,a_1,\cdots,a_n,a_{n+1},b)}_{(a,b_1,\cdots,b_n,b_{n+1},b)}$$

$$= \sum_{j=1}^{|(b_1,b;n+1)|} P_{(n,0)}(-2n\lambda)^{(b_1,b;n+1|j)}_{(b_1,\cdots,b_{n+1},b)}$$


$$= \sum_{j=1}^{|(b_1,b;n+1)|} \sum_{\alpha=1}^{|[b_1,b;(n+1,0)]|} P_{(n,0)}(-2n\lambda)^{[b_1,b;(n+1,0)|\alpha]}_{(b_1,\cdots,b_{n+1},b)} \, \phi_{(n+1,0)}[b_1,b](j|\alpha)$$


$$= \sum_{\alpha=1}^{|[b_1,b;(n+1,0)]|} P_{(n,0)}(-2n\lambda)^{[b_1,b;(n+1,0)|\alpha]}_{(b_1,\cdots,b_{n+1},b)} \tag{3.9}$$


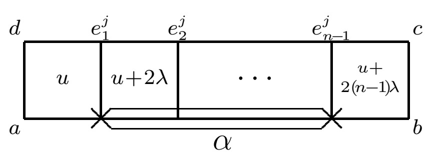
where the c^j_k are given by

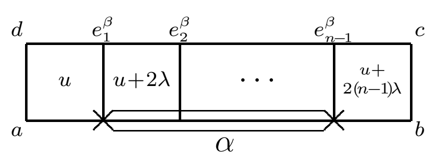
$$(b_1, c^j_1 \cdots, c^j_n, b) \quad = \quad (b_1, b; n{+}1|j)\ . \tag{3.10}$$

In the last line of eq. (3.9), we introduced a symbol for the "symmetric sum" over $c^j_1 \cdots, c^j_n$ labeled by α, which we will use in what follows to define the $(n{+}1,0)$ fused weights.

The other thing we need is the generalization of the push-through property (2.29). Basically, this is clear from our remarks about eq. (3.7), but it is by no means obvious and in fact in general not true that the relations given in eq. (3.7) give indeed all relations between dependent paths.

We make use of the recursive nature of the procedure, in this way proving the push-through property by induction and at the same time obtaining an expression for the number of independent paths. Suppose that the push-through property holds for the symmetric sum of n elementary faces

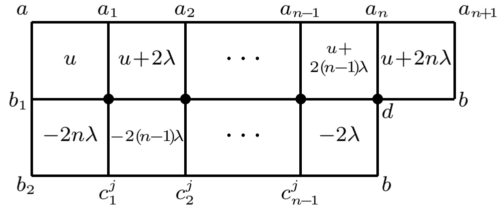


$$= \sum_{\beta=1}^{|[d,c;(n,0)]|} \phi_{(n,0)}[d,c](j|\beta) \tag{3.11}$$


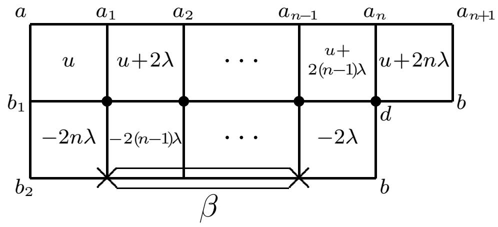
where the paths are labeled by $(d,c;n|j) = (d, e^j_1, \ldots, e^j_{n-1}, c)$ and $[d,c;(n,0)|\beta] = (d, e^\beta_1, \ldots, e^\beta_{n-1}, c)$. As in eq. (3.9), but splitting the product in a different place, we have

$$P_{(n+1,0)}(u)^{(a,a_1,\cdots,a_n,a_{n+1},b)}_{(a,b_1,\cdots,b_n,b_{n+1},b)}$$

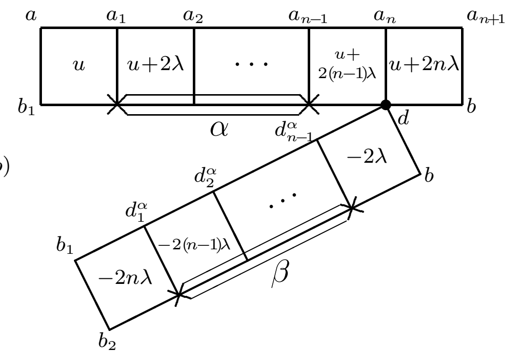
$$= \sum_{j=1}^{|(b_2,b;n)|} P_{(n-1,0)}(-2(n-1)\lambda)^{(b_2,b;n|j)}_{(b_2,\cdots,b_{n+1},b)}$$



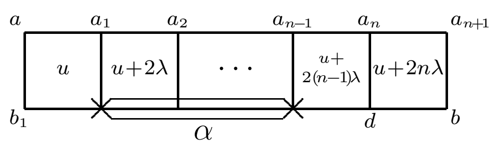
$$= \sum_{\beta=1}^{|[b_2,b;(n,0)]|} P_{(n-1,0)}(-2(n-1)\lambda)^{[b_2,b;(n,0)|\beta]}_{(b_2,\cdots,b_{n+1},b)}$$



$$= \sum_{\beta=1}^{|[b_2,b;(n,0)]|} \sum_{\alpha=1}^{|[b_1,d;(n,0)]|} P_{(n-1,0)}(-2(n-1)\lambda)^{[b_2,b;(n,0)|\beta]}_{(b_2,\cdots,b_{n+1},b)}$$



$$= \sum_{d} \sum_{\alpha=1}^{|[b_1,d;(n,0)]|} P_{(n,0)}(-2n\lambda)^{([b_1,d;(n,0)|\alpha],b)}_{(b_1,\cdots,b_{n+1},b)} \tag{3.12}$$



where we used the push-through property (3.11) and afterwards the split property again. This clearly reflects the recursive nature of the construction of fused weights: the projector of fusion at level $(n+1,0)$ is basically a face weight of level $(n,0)$ fusion. We also see that in order to find the independent paths at level $(n+1,0)$, we only have to consider paths which are independent at level $(n,0)$ and append one step to those.

The number of independent paths is of course given by the difference of the total number of paths and the number of independent equations obtained from eqs. (3.3) and (3.4). This is just the matrix element $A^{(n+1,0)}_{a,b}$ of the fused adjacency matrix defined in eq. (2.8), i.e.,

$$\left|\mathrm{indpath}_{(n+1,0)}[a,b]\right| \;=\; |[a,b;(n+1,0)]| \;=\; A^{(n+1,0)}_{a,b}\,. \tag{3.13}$$

Of course, we in fact have to proof that the recursive formula (2.8) is indeed true and that it defines the adjacency matrices of the fused weights which we construct. This can be done by induction. However, even for the special case we considered so far we need to be able to say anything about fusion levels $(n,1)$, because it enters in eq. (2.8). Although these fusion levels will not be discussed till section 4, we briefly sketch the argument and it will become clear that the definitions of section 4 ensure that it is correct.

Assume that we know the set $\mathrm{indpath}_{(n,0)}[a,a']$, $n \geq 1$, and that is contains $A^{(n,0)}_{a,a'}$ elements. This is clearly fulfilled for $n=1$. Then the number of all allowed $(n{+}1)$-step paths from a to b which are obtained by appending one step to the paths in $\mathrm{indpath}_{(n,0)}[a,a']$, is obviously given by

$$\sum_{a'} A^{(n,0)}_{a,a'}\, A^{(1,0)}_{a',b} \;=\; \left(A^{(n,0)} \cdot A^{(1,0)}\right)_{a,b} . \tag{3.14}$$

However, not all of these are independent. How many relations are there? To see this, we have to look at the last two steps of the paths, and count how many of these are dependent as two-step paths, keeping track of the number of paths in $\mathrm{indpath}_{(n,0)}[a,a']$ which terminate in the corresponding manner. To count these directly appears to be complicated, but remember that for two-step paths one could choose the independent paths for antisymmetric fusion as the complement of those for symmetric fusion. This means that we arrive at a set of independent paths $\mathrm{indpath}_{(n+1,0)}[a,a']$ if we exclude all those paths which are independent w.r.t. fusion containing $(n{-}1)$ symmetric and one antisymmetric sum at the end. We will see later that this is exactly the definition of level $(n{-}1,1)$ fused weights, and hence there are $A^{(n-1,1)}_{a,b}$ of such paths. Therefore

$$\left|\mathrm{indpath}_{(n+1,0)}[a,b]\right| \;=\; \left(A^{(n,0)} \cdot A^{(1,0)} \;-\; A^{(n-1,1)}\right)_{a,b} \;=\; A^{(n+1,0)}_{a,b} \tag{3.15}$$

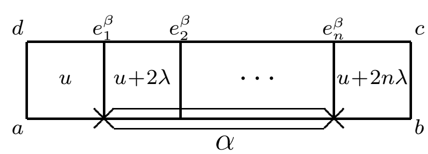
in agreement with eq. (2.8).

3.2 Level ($\boldsymbol{n}$,0) and (0,$\boldsymbol{n}$) Fusion

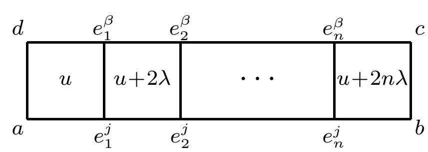
We start by giving the $(n{+}1,0)$ fused weights.

Lemma 3.1 (Symmetric 1×($\boldsymbol{n}$+1) Fusion) *Define the fused weights at level* $(n{+}1,0)$ *by*

$$W_{(n+1,0)}\!\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) \;=$$



$$= \sum_{j=1}^{|(a,b;n+1)|} \phi_{(n+1,0)}[a,b](j|\alpha) \tag{3.16}$$



where

$$(a, e_1^j \cdots, e_n^j, b) \;=\; (a, b; n{+}1|j) \quad , \quad (a, e_1^\beta, \cdots, e_n^\beta, b) \;=\; [a, b; (n{+}1, 0)|\beta] \;,$$

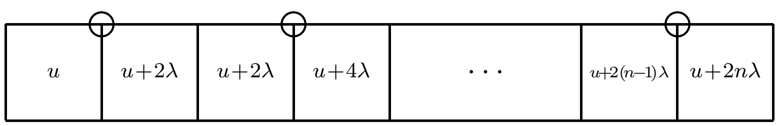
and the bond variables α and β take values

$$1 \,\le\, \alpha \,\le\, |[a, b; (n{+}1, 0)]| \;=\; A_{a,b}^{(n+1,0)} \quad , \quad 1 \,\le\, \beta \,\le\, |[d, c; (n{+}1, 0)]| \;=\; A_{c,d}^{(n+1,0)} \;.$$

The such defined weights satisfy the Yang-Baxter equation.

The lemma follows in the same way as Lemma 2.1 for the case of level $(2, 0)$ fusion by using the push-through (eqs. (3.3)–(3.4)) and split (3.9) properties. For a formal proof one should use induction over n.

The fused weights for level $(0, n)$ are constructed as follows. We consider a row of $2n$ elementary faces with spectral parameters arranged according to the sequence (u, $u+2\lambda$, $u+2\lambda$, $u+4\lambda$, $u+4\lambda$, …, $u+2(n-1)\lambda$, $u+2(n-1)\lambda$, $u+2n\lambda$). We then perform an antisymmetric fusion on each pair of adjacent faces with spectral parameters differing by 2λ, i.e.,



(3.17)

By Lemma 2.2, this yields — apart from a gauge and overall factors — a row of n elementary faces with spectral parameters ($u+\lambda$, $u+3\lambda$, …, $u+(2n-3)\lambda$, $u+(2n-1)\lambda$). These are then fused by the symmetric $(n, 0)$ fusion process described above. Altogether, this means that the $(0, n)$ fused weights differ from the weights at fusion level $(n, 0)$ only by a shift in the spectral parameter, a gauge transformation and some overall factors. Keeping track of the accumulated factors, one obtains

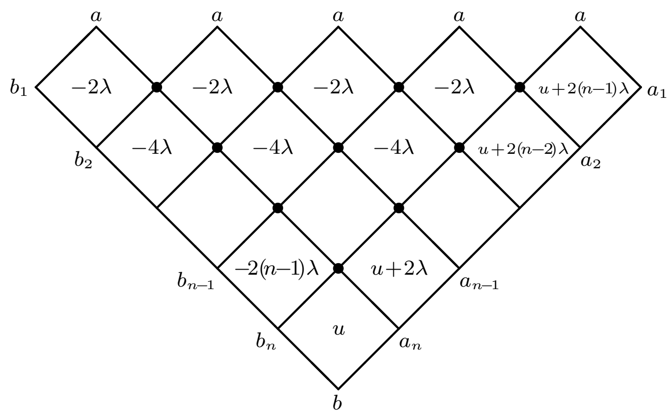
$$W_{(0,n)}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) \;=\; \frac{g(d, c|\beta)}{g(a, b|\alpha)} \left(\prod_{k=1}^{n} (-r_k^1)\right) W_{(n,0)}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix}\,\middle|\, u+\lambda\right) \tag{3.18}$$

where $g(a, b|\alpha)$ denotes the product of the gauge factors of eq. (2.40) along the independent path $[a, b; (n, 0)|\alpha]$.

3.3 $\boldsymbol{n}{\times}$*1 Fusion*

The fusion of a single column of elementary faces is basically the same as that of single rows. Again we can make a choice on which side of the column we perform the symmetric sum, here we choose the left side. The operator in eq. (3.1) is then replaced by

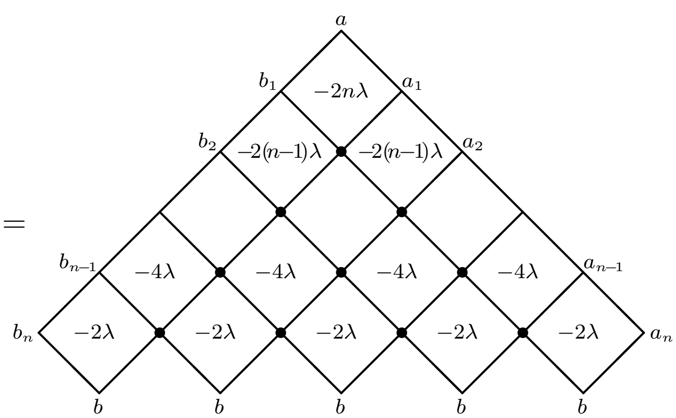
$$P^{(n,0)}(u)^{a,a_1,a_2,\cdots,a_n,b}_{a,b_1,b_2,\cdots,b_n,b} \quad = \tag{3.19}$$



where here and in what follows we use upper indices to denote the fusion level in the vertical direction.

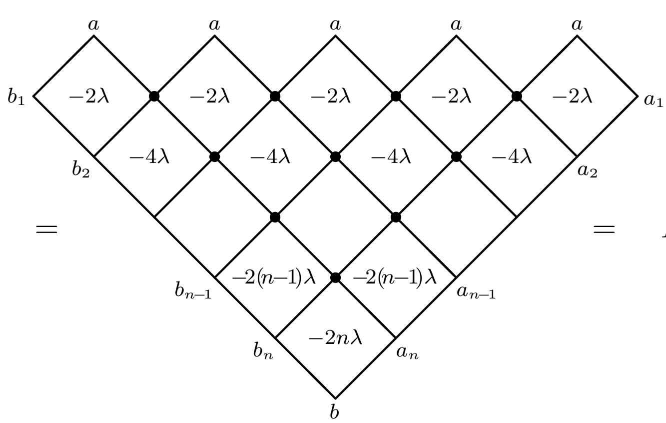
Of course, $P^{(n,0)}(-2n\lambda)$ is now the projector of level $(n+1,0)$ fusion of $n+1$ elementary faces in vertical direction. But, from the Yang-Baxter equation, one finds

$$P_{(n,0)}(-2n\lambda)^{a,a_1,\cdots,a_n,b}_{a,b_1,\cdots,b_n,b} \quad =$$



(3.20)

$$=$$



$$= \quad P^{(n,0)}(-2n\lambda)^{a,a_1,\cdots,a_n,b}_{a,b_1,\cdots,b_n,b}$$

so they are in fact identical.

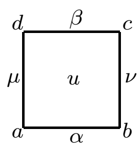
This means we can use the same paths and independent paths to construct the vertically fused weights at level $(n,0)$ as in the horizontal case. Of course, this also holds true for the above discussion about the level $(0,n)$ fused weights. Therefore, we will not go into more detail here and instead move on to discuss directly the more general case of symmetric fusion of rectangular blocks of elementary faces.

3.4 Symmetric $\boldsymbol{m}\times\boldsymbol{n}$ *Fusion*

So far, we considered fusion of a single row respectively a single column of weights, and of those only the fusion types labeled by the irreducible representations $(0,n)$ and $(n,0)$ of $su(3)$. In fact, we want to fuse the dilute models in both the horizontal and vertical direction simultaneously. The corresponding fused weights are labeled by two irreducible representations of $su(3)$ (respectively their Young tableaux) where we use the convention that the lower index of the weights corresponds to the horizontal and the upper index to the vertical fusion level. In general, these fused weights have bond variables not only on the horizontal, but also on the vertical bonds. We now construct the symmetric fusion of a rectangular m by n block of elementary faces.

Let m,n be positive integers and define

$$W^{(m,0)}_{(n,0)}\left(\begin{matrix} d & \beta & c\\ \mu & & \nu\\ a & \alpha & b\end{matrix}\;\middle|\; u\right) =$$



$$= \sum_{j=1}^{|(a,d;m)|}\phi_{(m,0)}[a,d](j|\mu)\sum_{\alpha_2,\cdots,\alpha_h}$$

$$\prod_{k=1}^{m} W_{(n,0)}\left(\begin{matrix}(a,d;m|j)_{k+1} & \alpha_{k+1} & [b,c;(m,0)|\nu]_{k+1}\\ (a,d;m|j)_k & \alpha_k & [b,c;(m,0)|\nu]_k\end{matrix}\;\middle|\; u-2(m-k)\lambda\right), \tag{3.21}$$

where $a=(a,d;m|j)_1$, $b=[b,c;(m,0)|\nu]_1$, $c=[b,c;(m,0)|\nu]_{m+1}$, $d=(a,d;m|j)_{m+1}$, $\alpha=\alpha_1$, $\beta=\alpha_{m+1}$, the summation α_k's are over $1,\cdots,A^{(n,0)}_{(a,d;m|j)_m,[b,c;(m,0)|\nu]_k)}$, and the $(n,0)$ fusion of $1\times n$ faces is

$$W_{(n,0)}\left(\begin{matrix} d & \beta & c\\ a & \alpha & b\end{matrix}\;\middle|\; u\right) =$$

$$\sum_{i=1}^{|(a,d;n)|}\phi_{(n,0)}[a,b](i|\alpha)\prod_{k=1}^{n} W\left(\begin{matrix}[d,c;(n,0)|\beta]_k & [d,c;(n,0)|\beta]_{k+1}\\ (a,b;n|i)_k & (a,b;n|i)_{i,k+1}\end{matrix}\;\middle|\; u+2(k-1)\lambda\right) \tag{3.22}$$

The fused face weights (3.21) can be pictorially described in Fig 3, which depend on both the spin variables a,b,c,d and the bond variables α,β,μ,ν. Moreover, according to subsection 3.2 the fused weights of symmetric $(m,0)\times(0,n)$ fusion and $(0,m)\times(0,n)$ fusion can be simply derived from those of symmetric $(m,0)\times(n,0)$ fusion.

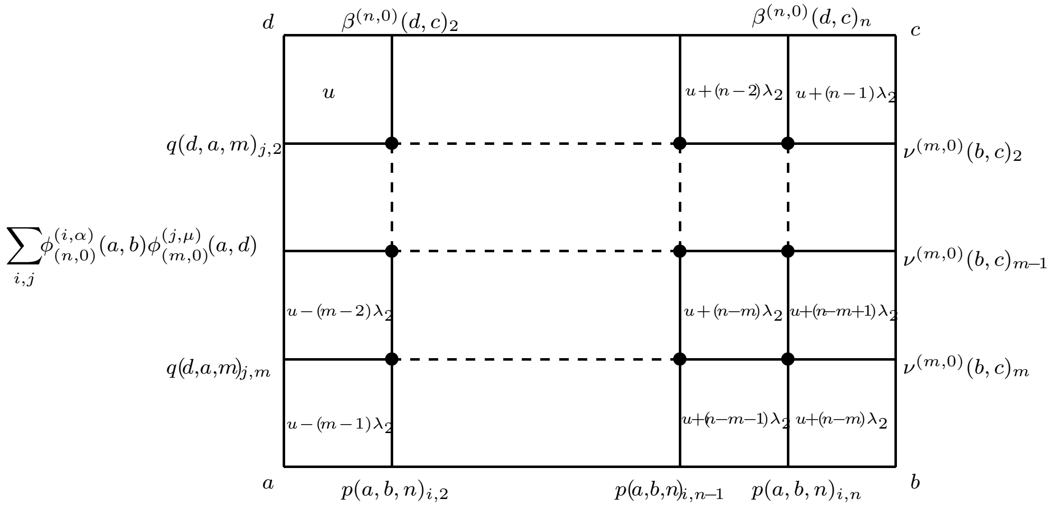


Figure 3. Diagrammatic representation of the face weights of the $n \times m$ fully symmetric fused models. Sites indicated with a solid circle are summed over all possible spin states and $\lambda_2 = 2\lambda$.

$$W^{(m,0)}_{(0,n)}\left(\begin{array}{ccc} d & \beta & c \\ \mu & & \nu \\ a & \alpha & b \end{array}\Big|\, u\right) = W^{(m,0)}_{(n,0)}\left(\begin{array}{ccc} d & \beta & c \\ \mu & & \nu \\ a & \alpha & b \end{array}\Big|\, u+\lambda\right) \prod_{k=1}^{n}\Big((-1)^m r_k^m\Big) \tag{3.23}$$

$$W^{(0,m)}_{(0,n)}\left(\begin{array}{ccc} d & \beta & c \\ \mu & & \nu \\ a & \alpha & b \end{array}\Big|\, u\right) = W^{(m,0)}_{(n,0)}\left(\begin{array}{ccc} d & \beta & c \\ \mu & & \nu \\ a & \alpha & b \end{array}\Big|\, u\right) \times \prod_{k=1}^{n}(-1)^m f_{k-2}^m \tag{3.24}$$

The fused blocks (3.21) is built up from a row of the fused faces. All properties held by one row fused faces carry over to the fused blocks (3.21). From the push-through property (3.11) we have the following lemma.

Lemma 3.2 *For the independent paths $[d,c;(n,0)|\beta]$ and their dependent path $(d,c;n|j)$*

$$W^{(m,0)}_{(n,0)}\left(\begin{array}{ccc} d & j & c \\ \mu & & \nu \\ a & \alpha & b \end{array}\Big|\, u\right) = \sum_{\beta=1}^{A^{(n,0)}_{(d,c)}} \phi_{(n,0)}[d,c](j|\beta)\; W^{(m,0)}_{(n,0)}\left(\begin{array}{ccc} d & \beta & c \\ \mu & & \nu \\ a & \alpha & b \end{array}\Big|\, u\right) \tag{3.25}$$

Similarly, for the independent paths $[b,c;(m,0)|\nu]$ and their dependent path $(b,c;m|j)$

$$W^{(m,0)}_{(n,0)}\left(\begin{array}{ccc} d & \beta & c \\ \mu & & j \\ a & \alpha & b \end{array}\Big|\, u\right) = \sum_{\nu=1}^{A^{(m,0)}_{(b,c)}} \phi_{(m,0)}[b,c](j|\nu)\; W^{(m,0)}_{(n,0)}\left(\begin{array}{ccc} d & \beta & c \\ \mu & & \nu \\ a & \alpha & b \end{array}\Big|\, u\right). \tag{3.26}$$

The above lemma implies the following theorem.

Theorem 3.3 *For a triple of positive inters m, n, l, the fused face weights (3.21) satisfy* *the following Yang-Baxter equation*

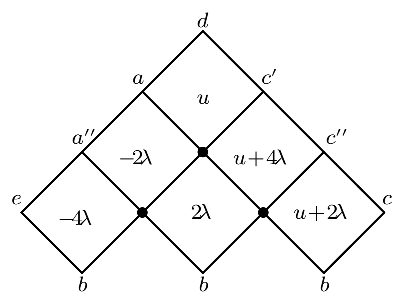
$$\sum_{(\eta_1,\eta_2,\eta_3)}\sum_g W^{(m,0)}_{(n,0)}\left(\begin{matrix} e & \mu & d \\ \nu & & \eta_3 \\ f & \eta_1 & g \end{matrix}\;\middle|\; u\right) W^{(m,0)}_{(l,0)}\left(\begin{matrix} d & \gamma & c \\ \eta_3 & & \beta \\ g & \eta_2 & b \end{matrix}\;\middle|\; u-v\right) W^{(l,0)}_{(n,0)}\left(\begin{matrix} f & \eta_1 & g \\ \rho & & \eta_2 \\ a & \alpha & b \end{matrix}\;\middle|\; v\right)$$

$$= \sum_{(\eta_1,\eta_2,\eta_3)}\sum_g W^{(m,0)}_{(n,0)}\left(\begin{matrix} g & \eta_1 & c \\ \eta_3 & & \beta \\ a & \alpha & b \end{matrix}\;\middle|\; u\right) W^{(m,0)}_{(l,0)}\left(\begin{matrix} e & \eta_2 & g \\ \nu & & \eta_3 \\ f & \rho & a \end{matrix}\;\middle|\; u-v\right) W^{(l,0)}_{(n,0)}\left(\begin{matrix} e & \mu & d \\ \eta_2 & & \gamma \\ g & \eta_1 & c \end{matrix}\;\middle|\; v\right) \quad (3.27)$$

4 Fusion at Level (n, m)

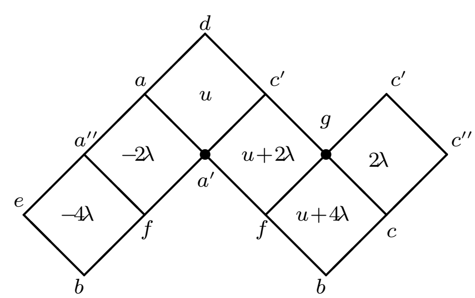
4.1 An Example: Level (1,1) Fusion

We start the discussion of the fusion at the general level (n, m) by considering the simplest non-trivial example $n = m = 1$. Let us consider the following product of elementary face weights



which can be rewritten as

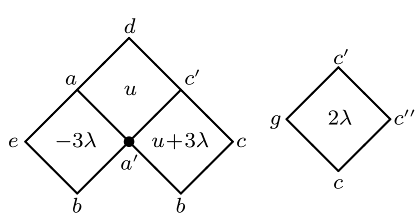
$$\sum_f$$



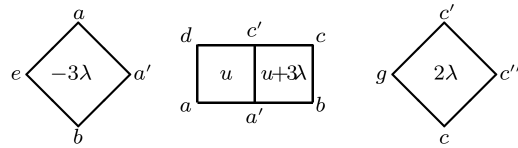
(4.1)

Then by Lemma 2.2 we can simplify the product to be the 1×2 fusion in terms of the projector $X^{(b,d)}(-3\lambda)$

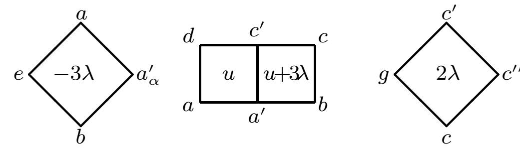
$$h'(u)\frac{g(c',c)}{g(a,e)}$$



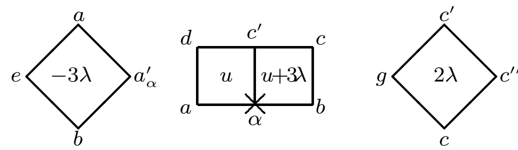
$$= \quad h'(u)\frac{g(c',c)}{g(a,e)}\sum_{a'}$$



$$= \quad h'(u)\frac{g(c',c)}{g(a,e)}\sum_{a'}\sum_{\alpha}\phi_{(1,1)}(a,a',b|\alpha)$$



$$= \quad h'(u)\frac{g(c',c)}{g(a,e)}\sum_{\alpha} \tag{4.2}$$



where $g(a,c)$ is the guage factor (2.40) from the antisymmetric fusion and

$$h'(u) = \frac{\vartheta_1(2\lambda)\vartheta_1(7\lambda)\vartheta_1(u-\lambda)\vartheta_1(u+4\lambda)}{\vartheta_1^2(2\lambda)\,\vartheta_1^2(3\lambda)}\ .$$

In the last step we performed the summation over a'. The sum over α on the RHS includes only the independent paths (a,a'_α,b), $\alpha\in\{1,2\}$. Here, we have taken the the indenpendent paths for the construction of the $(1,1)$ fusion as follows

$$\mathrm{indpath}_{(1,1)}[a,b] \quad = \begin{cases} \{(a,a{\pm}1,a{\pm}2)\} & \text{if } b=a\pm 2\\ \{(1,2,1)\} & \text{if } a=b=1\\ \{(L,L{-}1,L)\} & \text{if } a=b=L\\ \{(a,a,a{+}1),(a,a{+}1,a{+}1)\} & \text{if } b=a+1\\ \{(a,a{-}1,a{-}1),(a,a,a{-}1)\} & \text{if } b=a-1\\ \{(a,a{+}1,a),(a,a{-}1,a)\} & \text{if } a=b,\ \mathrm{val}(a)=3 \end{cases} \tag{4.3}$$

and the coefficients $\phi_{(1,1)}(a,a',b|\alpha)$ as follows

$$\phi_{(1,1)}(a,a',b|\alpha) = \begin{cases} (-1)^{\delta_{a,a'}}A^{(1)}_{a_\alpha,a'}\left(\dfrac{S(a+1)}{S(a')}\right)^{1/2} & \text{if } a=b,\ \alpha=1 \text{ and } \mathrm{val}(a)=3\\ (-1)^{a-a'}\left(\dfrac{S(a)}{S(a')}\right)^{1/2} & \text{if } a=b \text{ and } \mathrm{val}(a)=2\\ \delta_{a',a_\alpha} & \text{otherwise.} \end{cases} \tag{4.4}$$

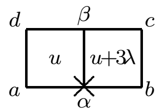
Above preliminary discussion defines the fused face weights for the symmetry $(1,1)$. As shown in the procedure it is exactly the 1×2 fusion of su(2) type. We refer [22] for the details. Here we emphasize it as the fusion $(1,1)$ is a member of su(2) type.

$u+(m+n-1)\lambda_2$	$u+(m+n-2)\lambda_2$	$\cdots\cdots$	$u+n\lambda_2$	$u+(n-1)\lambda_2$	$\cdots\cdots$	u
$u+(m+n)\lambda_2$	$u+(m+n-1)\lambda_2$	$\cdots\cdots$	$u+(n+1)\lambda_2$			

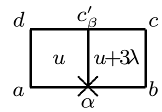
Figure 4. The sequence of the spectral parameter shift for $Y = (n, m)$ and here $\lambda_2 = 2\lambda$.

Lemma 4.1 *If (a, b) and (d, c) are admissible edges at fusion level $(1, 1)$, put*

$$W_{(1,1)}\begin{pmatrix} d & \beta & c \\ a & \alpha & b \end{pmatrix}\bigg|\, u \bigg) \quad = \quad r_2^1$$



$$= r_2^1$$



$$= r_2^1 \sum_{a'} \phi_{(1,1)}(a, a', b|\alpha)\, W\begin{pmatrix} d & c'_\beta \\ a & a' \end{pmatrix}\bigg|\, u\bigg) W\begin{pmatrix} c'_\beta & c \\ a' & b \end{pmatrix}\bigg|\, u+3\lambda\bigg) \qquad (4.5)$$

where the sum is over all allowed spins a', the bond variables take values $1 \leq \alpha \leq A_{a,b}^{(1,1)}$, $1 \leq \beta \leq A_{c,d}^{(1,1)}$, and where the coefficients $\phi_{(1,1)}(a, a', b|\alpha)$ are those of eq. (4.4). Furthermore, the value of c' on the RHS is chosen such that $(d, c', c) \in indpath_{(1,1)}[d, c]$, β labeling the elements of $indpath_{(1,1)}[d, c]$. The such defined weights satisfy the Yang-Baxter equation. In particular, we note that for all fixed $a, b, c, d, \alpha, \beta$ we have

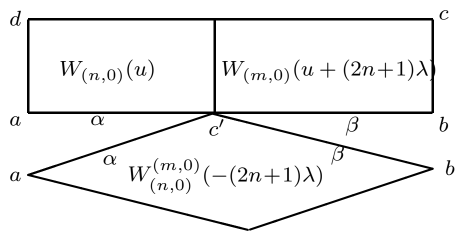
$$W_{(1,1)}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix}\;\bigg|\; u\right) = 0 \qquad \textit{if } u = 0, \lambda \textit{ and } -4\lambda.$$

4.2 Fusion at Level (n, m)

Fusion at Level (n,m) can be constructed by generalizing the fusion procedure of simple case $(1, 1)$. Firstly, let us consider the one row fusion. Let $Y = (n, m)$ be a Young diagram with $n + 2m$ nodes in Fig 4. We use $2m$ faces with spectral parameter shifts listed in the left half douible rows Young diagram of Fig 4. As the fusion (3.17) for the case $(0, m)$ we can construct

$$W_{(0,m)}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix}\;\bigg|\; u + 2n\lambda\right) \sim \left(\prod_{k=1}^{n}(-r_k^1)\right)\; W_{(m,0)}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix}\;\bigg|\; u + 2n\lambda + \lambda\right) \qquad (4.6)$$

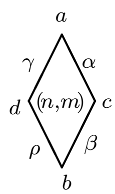
By faces with other spectral parameter shifts of Fig 4 we can construct the fusion $W_{(n,0)}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix}\;\Big|\; u\right)$. The symmetric $(1, 0) \times (n, m)$ fusion can be constructed by studying the product



(4.7)

The projector in (4.7) is given by (3.21) with the spectral parameter $u = -2n\lambda - \lambda$. Where the sum is over α, β, c'. Similar the discussion in previous sections, the fused face weight is found by introducing the coeffecients $\phi_{(n,m)}[a,b](j|\mu)$ and then splitting the projector from the fused faces. To do so we again need to know the symmetry about the paths according to the projector. For the simple case of fusion $(1,1)$ this is gained by studying the single face $X^{(a,d)}(-3\lambda)$ in (2.2) and thus we obtain the coeffecients (4.4).

Let $(a,b;n,m)$ be the set of paths of 2 steps from a to b through α, c, β so that

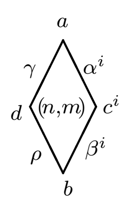


$$:= W^{(m,0)}_{(n,0)}\left(\begin{matrix} a & \alpha & c \\ \gamma & & \beta \\ d & \rho & b \end{matrix} \;\middle|\; -2n\lambda - \lambda\right) \neq 0 \tag{4.8}$$

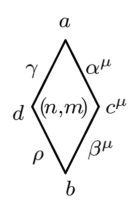
That means that

$$(a,b;n,m) = \{(a,\alpha,c,\beta,b) | 1 \le \alpha \le A^{(n,0)}_{a,c}, 1 \le \beta \le A^{(m,0)}_{c,b} \; and \; A^{(n,0)}_{a,c} A^{(m,0)}_{c,b} \neq 0\} \; . \tag{4.9}$$

It is obvious that the number $|(a,b;n,m)|$ of paths in the set $(a,b;n,m)$ is $[A^{(n,0)}A^{(m,0)}]_{a,b}$. Simialr to the elementary fusion procedure of case $(2,0)$ the fusion coefficients $\phi_{(n,m)}[a,b](i|\mu)$ are introduced by



$$= \sum_{\mu=1}^{|[a,b;(n,m)]|} \phi_{(n,m)}[a,b](i|\mu) \tag{4.10}$$



where the independent paths $(a, \alpha^\mu, c^\mu, \beta^\mu, b) = [a,b;(n,m)|\mu] \in \text{indpath}_{(n,m)}[a,b]$ and $|[a,b;(n,m)]|$ is the number of the independent paths. Obviously, the number of the independent paths is the difference of $|(a,b;n,m)|$ from the number of all independent equations (4.10)

$$|\text{indpath}_{(n,m)}[a,b]| = |[a,b;(n,m)]| = A^{(n,m)}_{a,b} \; . \tag{4.11}$$

Specially, (4.10) is over determined if $n+m \ge 2L$. This means that there is no projector for the fusion when $n+m \ge 2L$ and so no independent path. Therefore we have $A^{(n,m)} = 0$.

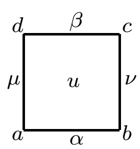
For calculating the (n,m) fused face weights it is necessary to find the coefficients $\phi_{(n,m)}[a,b](j|\mu)$ and $\text{indpath}_{(n,m)}[a,b]$. This can be done from the explicit weights $W^{(m,0)}_{(n,0)}$. The projector is the fused face weights of $(m,0)\times(n,0)$. The symmetry on the paths of the projector must follow from the property (3.25)-(3.26) and the face $X(-3\lambda)$, which also define the relation (4.10). One does not have simple or unified form for the coefficients. It is tedious but very straight forward calculation. Instead of doing so we formulate the the fused weights of symmetric $1\times(n,m)$ in terms of the coefficients as follows

$$
\begin{aligned}
&W_{(n,m)}\left(\begin{matrix} d & \nu & c \\ a & \mu & b \end{matrix}\,\middle|\, u\right) \\
&= \prod_{k=1}^{m}\left(-s^1_{n+k}s^1_{n+k-5/2}\right)\sum_{i=1}^{|(a,b;n,m)|}\phi_{(n,m)}[a,b](i|\mu) \\
&\times W_{(n,0)}\left(\begin{matrix} d & [d,c;(n,m)|\nu]_2 & [d,c;(n,m)|\nu]_3 \\ a & (a,b;n,m|i)_2 & (a,b;n,m|i)_3 \end{matrix}\,\middle|\, u\right) \\
&\times W_{(m,0)}\left(\begin{matrix} [d,c;(n,m)|\nu]_3 & [d,c;(n,m)|\nu]_4 & c \\ (a,b;n,m|i)_3 & (a,b;n,m|i)_4 & b \end{matrix}\,\middle|\, u+n\lambda_2+\lambda\right) \qquad (4.12)
\end{aligned}
$$

where ν,μ are the bond variables $1\le\nu\le A^{(n,m)}_{(d,c)}$ and $1\le\mu\le A^{(n,m)}_{(a,b)}$. The push-through property follows from the (4.7). To see that let us push the projector from the bottom through upto the top by using the Yang-Baxter equation (3.27). It is easy to see the path along the top from d to c satisfies the same property as the projector. So from (4.10) we have the push through property for $W_{(n,m)}$

Based on one row fusion $W_{(n,m)}$ we can easily build up the multi-row fusion. Let $m,n,\overline{m},\overline{n}$ be positive integers and define

$$
W^{(n,m)}_{(\overline{n},\overline{m})}\left(\begin{matrix} d & \beta & c \\ \mu & & \nu \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) =
$$



$$
\begin{aligned}
&= \prod_{k=1}^{\overline{n}}(-s^m_{k+1/2}s^m_{k-2})\prod_{k=1}^{\overline{m}}(-s^m_{k+\overline{n}+1}s^m_{k+\overline{n}-3/2}) \\
&\sum_{i=1}^{|(a,d;n,m)|}\phi_{(n,m)}[a,d](i|\mu)\sum_{\rho}W^{(m,0)}_{(\overline{n},\overline{m})}\left(\begin{matrix} d & \beta & c \\ (a,d;n,m|i)_4 & & [b,c;(n,m)|\nu]_4 \\ (a,d;n,m|i)_3 & \rho & [b,c;(n,m)|\nu]_3 \end{matrix}\,\middle|\, u\right) \\
&\times W^{(n,0)}_{(\overline{n},\overline{m})}\left(\begin{matrix} (a,d;n,m|i)_3 & \alpha(i) & [b,c;(n,m)|\nu]_3 \\ (a,d;n,m|i)_2 & & [b,c;(n,m)|\nu]_2 \\ a & \alpha & b \end{matrix}\,\middle|\, u-2m\lambda-\lambda\right), \qquad (4.13)
\end{aligned}
$$

where the sum over ρ is over $1,2,\cdots,|[(a,d;n,m|i)_3,[b,c;(n,m)|\nu]_3;(\overline{n},\overline{m})]|$. The push-through property of each rwo carries over to the fused multi-row fused faces. Therefore we have the following Lemma.

Lemma 4.2 *For the independent paths* $[d, c; (\overline{n}, \overline{m}))|\beta]$ *and their dependent path* $(d, c; \overline{n}, \overline{m}|j)$

$$W^{(n,m)}_{(\overline{n},\overline{m})}\left(\begin{matrix} d & j & c \\ \mu & & \nu \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) = \sum_{\beta=1}^{A^{(\overline{n},\overline{m})}_{(d,c)}} \phi_{(n,0)}[d,c](j|\beta)\; W^{(n,m)}_{(\overline{n},\overline{m})}\left(\begin{matrix} d & \beta & c \\ \mu & & \nu \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) \tag{4.14}$$

Similarly, for the independent paths $[b, c; (n, m)|\nu]$ *and their dependent path* $(b, c; n, m|j)$

$$W^{(n,m)}_{(\overline{n},\overline{m})}\left(\begin{matrix} d & \beta & c \\ \mu & & j \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) = \sum_{\nu=1}^{A^{(n,m)}_{(b,c)}} \phi_{(m,0)}[b,c](j|\nu)\; W^{(n,m)}_{(\overline{n},\overline{m})}\left(\begin{matrix} d & \beta & c \\ \mu & & \nu \\ a & \alpha & b \end{matrix}\,\middle|\, u\right). \tag{4.15}$$

Then the lemma implies that the fuse faces satisfy the Yang-Baxter relation.

Theorem 4.3 *For a group of positive inters* $m, n, l, \overline{m}, \overline{n}, \bar{l}$*, the fused face weights (4.14) satisfy the following Yang-Baxter equation*

$$\begin{aligned}
&\sum_{(\eta_1,\eta_2,\eta_3)} \sum_g W^{(m,\overline{m})}_{(n,\overline{n})}\left(\begin{matrix} e & \mu & d \\ \nu & & \eta_3 \\ f & \eta_1 & g \end{matrix}\,\middle|\, u\right) W^{(m,\overline{m})}_{(l,\bar{l})}\left(\begin{matrix} d & \gamma & c \\ \eta_3 & & \beta \\ g & \eta_2 & b \end{matrix}\,\middle|\, u-v\right) \times W^{(l,\bar{l})}_{(n,\overline{n})}\left(\begin{matrix} f & \eta_1 & g \\ \rho & & \eta_2 \\ a & \alpha & b \end{matrix}\,\middle|\, v\right) \\
&= \sum_{(\eta_1,\eta_2,\eta_3)} \sum_g W^{(m,\overline{m})}_{(n,\overline{n})}\left(\begin{matrix} g & \eta_1 & c \\ \eta_3 & & \beta \\ a & \alpha & b \end{matrix}\,\middle|\, u\right) \times W^{(m,\overline{m})}_{(l,\bar{l})}\left(\begin{matrix} e & \eta_2 & g \\ \nu & & \eta_3 \\ f & \rho & a \end{matrix}\,\middle|\, u-v\right) W^{(l,\bar{l})}_{(n,\overline{n})}\left(\begin{matrix} e & \mu & d \\ \eta_2 & & \gamma \\ g & \eta_1 & c \end{matrix}\,\middle|\, v\right)
\end{aligned} \tag{4.16}$$

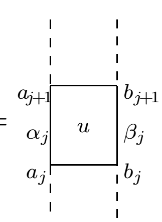
5 SU(3) fusion hierarchy

5.1 Functional equations

The fusion rule (2.8) is the relations for the adjacency matrices by fused models. We will see in this subsection that the theory carries over to the level of the row transfer matrix.

Suppose that $\mathbf{a}$ ($\boldsymbol{\alpha}$) and $\mathbf{b}$ ($\boldsymbol{\beta}$) are allowed spin (bond) configurations of two consecutive rows of an N (even) column lattice with periodic boundary conditions. The elements of the fused row transfer matrix $\mathbf{T}(u)$ are given by

$$\langle \boldsymbol{a}, \boldsymbol{\alpha} | \mathbf{T}^{(n,m)}_{(\overline{n},\overline{m})}(u) | \boldsymbol{b}, \boldsymbol{\beta}\rangle = \prod_{j=1}^{N} \sum_{\eta_j} W^{(n,m)}_{(\overline{n},\overline{m})}\left(\begin{matrix} a_{j+1} & \eta_{j+1} & b_{j+1} \\ \alpha_j & & \beta_j \\ a_j & \eta_j & b_j \end{matrix}\,\middle|\, u\right) = \tag{5.1}$$



where the periodic condition or $a_{N+1} = a_1$, $b_{N+1} = b_1$ and $\eta_{N+1} = \eta_1$ are taken. Obviously, the Yang-Baxter equations (4.16) imply that

$$[\mathbf{T}^{(n,m)}_{(\overline{n},\overline{m})}(u), \mathbf{T}^{(n,m)}_{\overline{(n',m')}}(v)] = 0\ . \tag{5.2}$$

Thus for each pair n, m fixed we obtained the hierarchy of commuting families of transfer matrices. Moreover, the fusion procedure implies various relations among these transfer matrices. Let us define

$$\begin{aligned} \boldsymbol{T}^{(n,m)}_k &= \boldsymbol{T}^{(n',m')}_{(n,m)}(u+2k\lambda) \\ \boldsymbol{T}^{(n,m)} &= 0 \quad \text{if } n<0 \text{ or } m<0 \\ \boldsymbol{T}^{(0,0)} &= \boldsymbol{I} \end{aligned} \tag{5.3}$$

and

$$s^Y_n = \prod_{j\in Y}\left[\frac{\vartheta_1(u-u_j+2n\lambda)}{\sqrt{\vartheta_1(2\lambda)\vartheta_1(3\lambda)}}\right] \tag{5.4}$$

$$f^Y_n = (-1)^{nN}\left[s^Y_{n-3/2}s^Y_{n-1}s^Y_{n-1/2}s^Y_{n+1}s^Y_{n+3/2}s^Y_{n+2}\right]^N \tag{5.5}$$

where the shifts u_i are listed in the Fig 4 with $u = 0$. We summarize them in following theorems.

Theorem 5.1 ($su(3)$ Fusion Hierarchy) *The functional relations are of su(3) type*

$$\boldsymbol{T}^{(n,0)}_0\boldsymbol{T}^{(1,0)}_n = \boldsymbol{T}^{(n-1,1)}_0 + \boldsymbol{T}^{(n+1,0)}_0 \tag{5.6}$$

$$\boldsymbol{T}^{(n,0)}_0\boldsymbol{T}^{(0,1)}_n = \boldsymbol{T}^{(n,1)}_0 + f^Y_{n-1}\boldsymbol{T}^{(n-1,0)}_0\ . \tag{5.7}$$

The fused transfer matrix vanishes for $m+n \geq 2L$. That means the closure condition

$$\mathbf{T}^{(n,m)} = 0 \quad \textit{if } m+n \geq 2L \tag{5.8}$$

Theorem 5.2 (Symmetry) *The fused transfer matrices $\boldsymbol{T}^{(n,m)}_0$ can be given by*

$$\boldsymbol{T}^{(n,0)}_0\boldsymbol{T}^{(0,m)}_n = \boldsymbol{T}^{(n,m)}_0 + f^Y_{n-1}\boldsymbol{T}^{(n-1,0)}_0\boldsymbol{T}^{(0,m-1)}_{n+1} \tag{5.9}$$

and satisfy the symmetry

$$\boldsymbol{T}^{(n,m)}(u)\left[\prod_{k=0}^{n-1} s^Y_{k-2}s^Y_{k+1/2}\right]^N = \left[\boldsymbol{T}^{(m,n)}(-u-2(n+m-2)\lambda)\right]^T(-1)^{Nn(m-n)}\left[\prod_{k=0}^{m-1} s^Y_{n+k-3/2}s^Y_{n+k+1}\right]^N \tag{5.10}$$

where subscript T means transpose.

The theorem 5.1 gives (1.21)–(1.23) for $m'=0$ and $n'=1$.

As we mentioned in subsection 2.1 that the fusion hierarchy equations (2.8) can be obtained from the functional relations of the fused transfer matrices. Let us consider the simpliest case of $n'=1, m'=0$ and system size $N=1$. Note the limit

$$\begin{aligned} A^{(n,m)} &= \lim_{iu\to\infty}\Big(\boldsymbol{T}^{(n,m)}(u)/[\rho_7(u)]^{n+2m}\Big) \\ &= \lim_{iu\to\infty}\Big(\boldsymbol{T}^{(n,m)}_{(n',m')}/[\rho_7(u)]^{n+2m}\Big) \end{aligned} \tag{5.11}$$

So the fusion hierarchy equations (5.6)-(5.9) also imply the valid adjacency equations (2.8).

5.2 Bethe ansatz of fusion hierarchies

The eigenvalues $\Lambda_{(1,0)}(u)$ and the Bethe ansatz equations of the row transfer matrices $\boldsymbol{T}(u)$ has been given in [23]. In subsection 1.3 we have shown the $su(3)$ structure of the fusion hierarchy (1.21)–(1.23) using the Bethe ansatz eigenvalues. With the knowledge of the $su(3)$ fusion we can generalize the Bethe ansatz eigenvalues to the fused transfer matrices. This has been studied for the $su(2)$ fusion hierarchies in [22] and here we study the the $su(3)$ case.

First let us consider the transfer matrix $\boldsymbol{T}^{(n',m')}_{(1,0)}(u) = \boldsymbol{T}^{Y}_{(1,0)}(u)$. The eigenvalues $\Lambda^{Y}_{(1,0)}(u)$ and the Bethe ansatz equations of this transfer matrix are given by

$$\begin{aligned} \Lambda^{Y}_{(1,0)}(u) &= \omega\,[s^{Y}_{-1}(u)s^{Y}_{-3/2}(u)]^N \frac{Q^{Y}(u+\lambda)}{Q^{(Y)}(u-\lambda)} + \omega^{-1}\,[s^{Y}_{0}(u)s^{Y}_{-1/2}(u)]^N \frac{Q^{Y}(u-4\lambda)}{Q^{Y}(u-2\lambda)} \\ &\quad + [(-1)^{n'} s^{Y}_{0}(u)s^{Y}_{-3/2}(u)]^N \frac{Q^{Y}(u)Q^{Y}(u-3\lambda)}{Q^{Y}(u-\lambda)Q^{Y}(u-2\lambda)} \end{aligned} \tag{5.12}$$

where

$$Q^{Y}(u) = \prod_{j=1}^{(n+2m)N} \vartheta_1(u-iu_j) \tag{5.13}$$

and the zeros $\{u_j\}$ satisfy the Bethe ansatz equations

$$\omega\left[\frac{s^{Y}_{1/2}(iu_j)}{s^{Y}_{-1/2}(iu_j)}\right]^N = (-1)^{n'N+1}\prod_{k=1}^{(n+2m)N} \frac{\vartheta_1(iu_j-iu_k+2\lambda)\vartheta_1(iu_j-iu_k-\lambda)}{\vartheta_1(iu_j-iu_k-2\lambda)\vartheta_1(iu_j-iu_k+\lambda)} \tag{5.14}$$

with $j=1,\cdots,N$ and $\omega=\exp(i\pi\ell/(L+1))$, $\ell=1,\cdots,L$.

Similar to the discussion in subsection 1.3 set

$$
\begin{aligned}
\boxed{1}^{\,k} &= \omega\,[s^Y_{k-1}(u)s^Y_{k-3/2}(u)]^N\,\frac{Q^Y(u+k\lambda_2+\lambda)}{Q^{(Y)}(u+k\lambda_2-\lambda)} \\
\boxed{2}^{\,k} &= [-s^Y_k(u)s^Y_{k-3/2}(u)]^N\,\frac{Q^Y(u+k\lambda_2)Q^Y(u+k\lambda_2-3\lambda)}{Q^Y(u+k\lambda_2-\lambda)Q^Y(u+k\lambda_2-2\lambda)} \\
\boxed{3}^{\,k} &= \omega^{-1}\,[s^Y_k(u)s^Y_{k-1/2}(u)]^N\,\frac{Q^Y(u+k\lambda_2-4\lambda)}{Q^Y(u+k\lambda_2-2\lambda)}
\end{aligned}
\tag{5.15}
$$

so that

$$
\Lambda^Y_{(1,0)}(u) \;=\; \boxed{1}^{\,0}+\boxed{2}^{\,0}+\boxed{3}^{\,0}=\sum\boxed{}^{\,0}
\tag{5.16}
$$

The $su(3)$ functional equations described in theorem 5.1 imply the eigenvalues of the fused row transfer matrix at level (n,m) can be written as

$$
\Lambda^Y_{(n,m)}(u) \;=\; \sum \left.\begin{array}{|c|c|c|c|c|c|}\hline & \cdots & & & \cdots & \\ \hline & \cdots & & \multicolumn{3}{c}{\underbrace{\hphantom{xxxxxxxx}}_{n}} \\ \cline{1-3} \multicolumn{3}{c}{\underbrace{\hphantom{xxxxxxxx}}_{m}} \end{array}\right.^{0}
\tag{5.17}
$$

where the number of terms in the sum is given by the dimension of the irreducible representations of $su(3)$

$$
(n+1)(m+1)(n+m+2)/2.
\tag{5.18}
$$

Such a Young tableau denotes the product of the $(n+2m)$ labeled boxes as given by (5.15) where it is understood that the relative shifts in the arguments are given by Figure **??**. These zeros $\{u_j\}$ satisfy the Bethe ansatz equations (5.14). Thus $\Lambda^Y_{(n,m)}(u)$ also can be represented by the determinantal form (1.33) where f is given by (5.5).

6 Discussion

We have presented the $su(3)$ fusion procedure for the dilute A_L models. The fusion of critical dilute D and E models can be implemented in a similar way. The fusion hierarchy and TBA-like equations for the models can be solved for the finite-size corrections and hence the central charges and scaling dimensions. These will be published elsewhere [27].

Acknowledgements

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