Interaction-Round-a-Face Models with Fixed Boundary Conditions: The ABF Fusion Hierarchy

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Abstract

We use boundary weights and reflection equations to obtain families of commuting double-row transfer matrices for interaction-round-a-face (IRF) models with fixed boundary conditions. In particular, we consider the fusion hierarchy of the Andrews-Baxter-Forrester (ABF) models, for which we obtain diagonal, elliptic solutions to the reflection equations, and find that the double-row transfer matrices satisfy functional equations with the same form as in the case of periodic boundary conditions.

KEY WORDS: Andrews-Baxter-Forrester models, exactly solvable lattice models, fixed boundary conditions, reflection equations, Yang-Baxter equation

1. Introduction

Two-dimensional lattice spin models in statistical mechanics have traditionally been solved by imposing periodic boundary conditions on the rows of the lattice. The Yang-Baxter equation, together with such boundary conditions, then leads to families of commuting row transfer matrices and hence solvability [1]. However, it has been shown that by using boundary weights and reflection equations, it is also possible to construct commuting double-row transfer matrices for vertex models with open boundary conditions [2].

Although the bulk properties of physical interest are independent of the boundary conditions in the thermodynamic limit, there are many surface quantities, such as the boundary free energy, which are also important. Moreover, at criticality, the conformal spectra of lattice models do depend on the boundary conditions [3]. For these reasons it is of interest to study lattice models with non-periodic boundary conditions.

In this paper, we begin in section 2 with a brief outline of a procedure for obtaining commuting double-row transfer matrices for vertex models with open boundary conditions, based on that of [2] and its generalisations [4, 5, 6, 7].

In section 3, we present a scheme for applying fixed boundary conditions to interaction-round-a-face (IRF) models, motivated by the preceding scheme for vertex models and by

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various correspondences between IRF and vertex models [8, 9, 10, 11]. We note that very recently, reflection equations for IRF models similar to those obtained here, were obtained independently in [12].

In section 4, we specialise to an important example of IRF models, the Andrews-Baxter-Forrester (ABF) models [13], and present diagonal, elliptic solutions to their reflection equations.

In section 5, we extend our general formalism to accommodate IRF models which have a fusion hierarchy. Our approach to fusion is similar to that of [14] in which level 2 fusion of vertex models with open boundary conditions was considered.

In section 6, we return to the ABF models and consider their fusion hierarchy [15, 16, 17, 18]. We find that the double-row transfer matrices of this fusion hierarchy satisfy various symmetry properties, as well as functional equations and a generalised inversion identity with the same structure as in the case of periodic boundary conditions [19, 20].

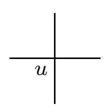
In section 7, we discuss future work, including that in which we plan to use the double-row transfer matrix functional equations to derive the boundary free energy and conformal spectra of the ABF models with fixed boundaries.

We conclude with three appendices, in which we prove some of the results used in the main text.

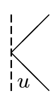
2. Vertex Models with Open Boundary Conditions

We now schematically outline the formalism for vertex models on which our treatment of IRF models is based. We note that the main differences between our formalism and that originally presented in [2] are that here only the first inversion relation is assumed, no R matrix symmetries are assumed, and the top row of the transfer matrix has the form $\boldsymbol{T}(\mu-u)^{t_0}$ rather than $\boldsymbol{T}(-u)^{-1}$, where $\boldsymbol{T}(u)$ is the form of the bottom row, μ is arbitrary and t_0 is transposition on the auxiliary space.

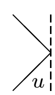
We are considering a vertex model with Boltzmann vertex weights



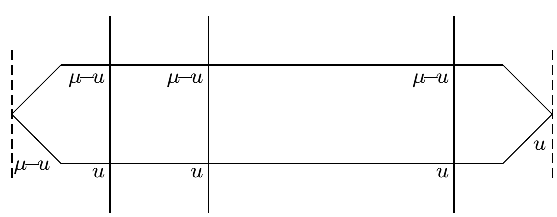
and left and right boundary weights



and

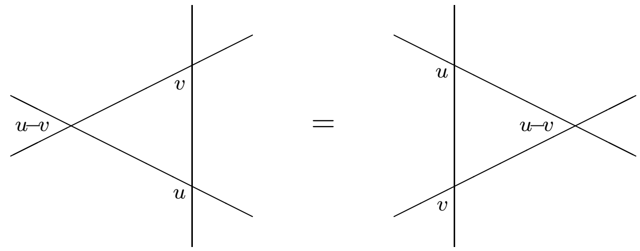


where u is a spectral parameter. Using these weights, we define a double-row transfer matrix as

$\boldsymbol{D}(u) \quad =$ 

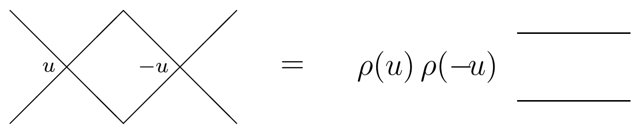
(2.1)

where μ is an arbitrary fixed parameter. If we now assume that the vertex and boundary weights satisfy the Yang-Baxter equation



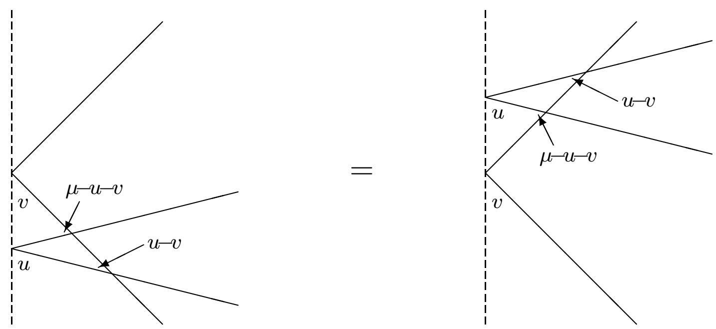
(2.2)

the inverse relation



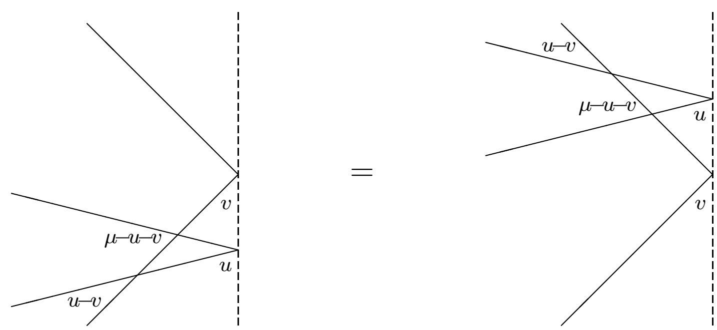
(2.3)

the left reflection equation



(2.4)

and the right reflection equation



(2.5)

where ρ is a model-dependent function, then it can be shown that the double-row transfer matrices form a commuting family,

$$\boldsymbol{D}(u)\,\boldsymbol{D}(v) = \boldsymbol{D}(v)\,\boldsymbol{D}(u) \tag{2.6}$$

We also note that if we regard the spectral parameter u as an effective angle $\pi u/\mu$, then the geometric angles in the diagrammatic Yang-Baxter and reflection equations are equal to the effective angles given by the corresponding values of the spectral parameter.

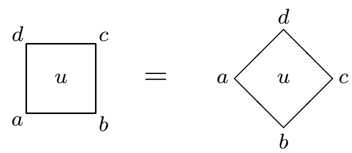
3. IRF Models with Fixed Boundary Conditions

3.1 Boltzmann Weights and Transfer Matrices

We now present our formalism for interaction-round-a-face (IRF) models. This was obtained from the preceding formalism for vertex models, together with the boundary crossing equations presented in [7], using various vertex-face correspondences [8, 9, 10, 11].

We are considering an IRF model with Boltzmann face weights

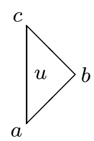
$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right) \quad = \quad$$



Here, the spins a, b, c, d take values from a discrete set and the spectral parameter u is a complex variable.

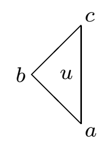
In order to accommodate fixed boundary conditions, we introduce left and right boundary weights, each of which depends on three spins,

$$B_{\rm L}\left(\begin{matrix} c \\ a \end{matrix}\; b \middle| u\right) \quad = \quad$$



and

$$B_{\rm R}\left(b\; \begin{matrix} c \\ a \end{matrix} \middle| u\right) \quad = \quad$$

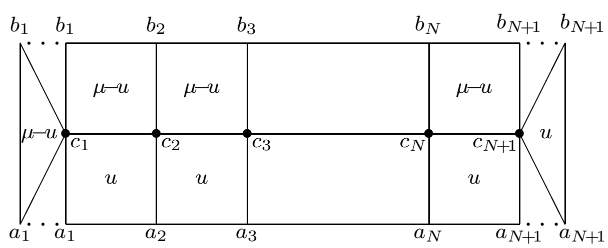


We now consider a lattice of width N and use these weights to construct a double-row transfer matrix. If $a_1, \ldots, a_{N+1}$ and $b_1, \ldots, b_{N+1}$ are two rows of spins, and μ is an arbitrary fixed parameter, then the corresponding entry of the double-row transfer matrix is defined by

$$\langle a_1, \ldots, a_{N+1} | \boldsymbol{D}(u) | b_1, \ldots, b_{N+1} \rangle \; = $$

$$\sum_{c_1 \ldots c_{N+1}} B_{\rm L}\left(\begin{matrix} b_1 \\ a_1 \end{matrix}\; c_1 \middle| \mu - u\right) \left[\prod_{j=1}^{N} W\left(\begin{matrix} c_j & c_{j+1} \\ a_j & a_{j+1} \end{matrix}\middle| u\right) W\left(\begin{matrix} b_j & b_{j+1} \\ c_j & c_{j+1} \end{matrix}\middle| \mu - u\right)\right] B_{\rm R}\left(c_{N+1}\; \begin{matrix} b_{N+1} \\ a_{N+1} \end{matrix} \middle| u\right)$$

$$= \qquad (3.1)$$



In this and all subsequent diagrams, we use solid circles to indicate spins which are summed over and dotted lines to connect identical spins.

We assume that the IRF model is associated with an adjacency condition, as specified by a symmetric adjacency matrix A each of whose entries is 0 or 1, and that the face and boundary weights satisfy the adjacency condition as follows:

$$A_{ab}\,A_{bc}\,A_{cd}\,A_{da}=0 \quad \text{implies} \quad W\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right)=0 \tag{3.2}$$

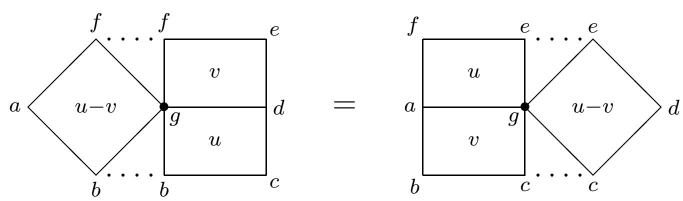
$$A_{ab}\,A_{bc}=0 \quad \text{implies} \quad B_{\rm L}\!\left(\begin{matrix} c & \\ & b \\ a & \end{matrix}\,\middle|\, u\right)=B_{\rm R}\!\left(\begin{matrix} & c \\ b & \\ & a \end{matrix}\,\middle|\, u\right)=0 \tag{3.3}$$

3.2 Local Relations

The face weights and boundary weights are assumed to satisfy the following local relations: the Yang-Baxter equation

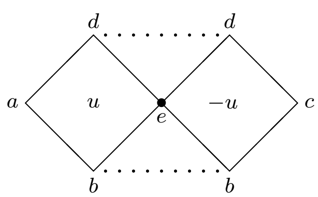
$$\sum_g W\!\left(\begin{matrix} f & g \\ a & b \end{matrix}\,\middle|\, u-v\right) W\!\left(\begin{matrix} g & d \\ b & c \end{matrix}\,\middle|\, u\right) W\!\left(\begin{matrix} f & e \\ g & d \end{matrix}\,\middle|\, v\right)=$$

$$\sum_g W\!\left(\begin{matrix} a & g \\ b & c \end{matrix}\,\middle|\, v\right) W\!\left(\begin{matrix} f & e \\ a & g \end{matrix}\,\middle|\, u\right) W\!\left(\begin{matrix} e & d \\ g & c \end{matrix}\,\middle|\, u-v\right) \tag{3.4}$$



the inversion relation

$$\sum_e W\!\left(\begin{matrix} d & e \\ a & b \end{matrix}\,\middle|\, u\right) W\!\left(\begin{matrix} d & c \\ e & b \end{matrix}\,\middle|\, -u\right)=\rho(u)\,\rho(-u)\,\delta_{ac}\,A_{ab}\,A_{ad} \tag{3.5}$$

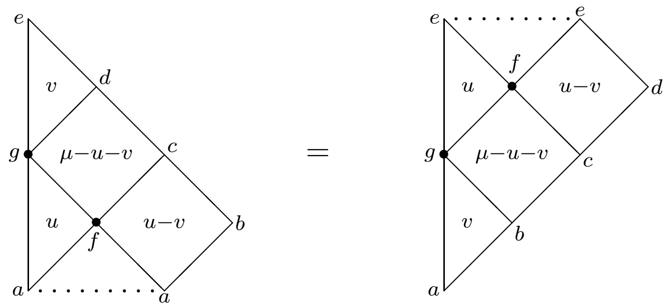


$$= \quad \rho(u)\,\rho(-u)\,\delta_{ac}\,A_{ab}\,A_{ad}$$

the left reflection equation

$$\sum_{fg} W\!\left(\begin{matrix} c & b \\ f & a \end{matrix}\,\middle|\, u-v\right) W\!\left(\begin{matrix} d & c \\ g & f \end{matrix}\,\middle|\, \mu-u-v\right) B_{\rm L}\!\left(\begin{matrix} g & \\ & f \\ a & \end{matrix}\,\middle|\, u\right) B_{\rm L}\!\left(\begin{matrix} e & \\ & d \\ g & \end{matrix}\,\middle|\, v\right)=$$

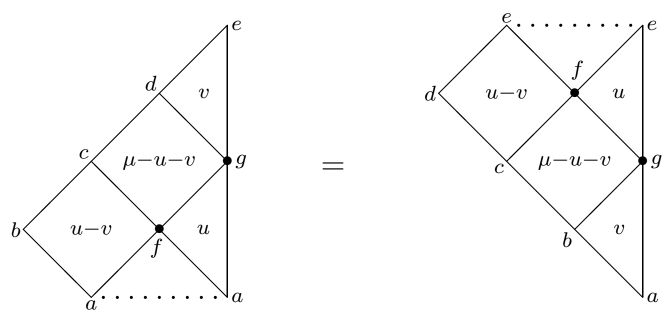
$$\sum_{fg} W\!\left(\begin{matrix} e & d \\ f & c \end{matrix}\,\middle|\, u-v\right) W\!\left(\begin{matrix} f & c \\ g & b \end{matrix}\,\middle|\, \mu-u-v\right) B_{\rm L}\!\left(\begin{matrix} e & \\ & f \\ g & \end{matrix}\,\middle|\, u\right) B_{\rm L}\!\left(\begin{matrix} g & \\ & b \\ a & \end{matrix}\,\middle|\, v\right) \tag{3.6}$$



the right reflection equation

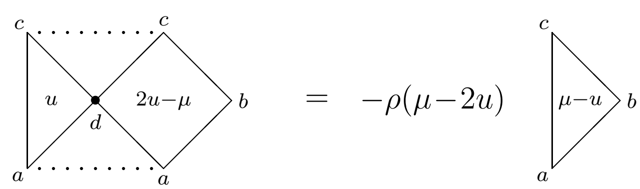
$$\sum_{fg} W\left(\begin{matrix} c & f \\ b & a \end{matrix}\,\middle|\, u-v\right)\; W\left(\begin{matrix} d & g \\ c & f \end{matrix}\,\middle|\, \mu-u-v\right)\; B_{\mathrm{R}}\left(f\;\begin{matrix} g \\ a \end{matrix}\,\middle|\, u\right)\; B_{\mathrm{R}}\left(d\;\begin{matrix} e \\ g \end{matrix}\,\middle|\, v\right) =$$

$$\sum_{fg} W\left(\begin{matrix} e & f \\ d & c \end{matrix}\,\middle|\, u-v\right)\; W\left(\begin{matrix} f & g \\ c & b \end{matrix}\,\middle|\, \mu-u-v\right)\; B_{\mathrm{R}}\left(f\;\begin{matrix} e \\ g \end{matrix}\,\middle|\, u\right)\; B_{\mathrm{R}}\left(b\;\begin{matrix} g \\ a \end{matrix}\,\middle|\, v\right) \qquad (3.7)$$



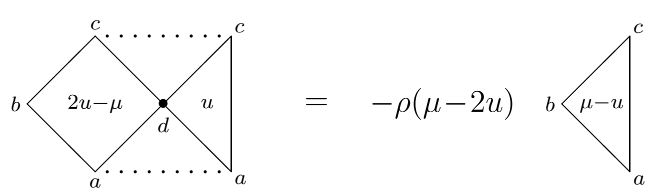
the left boundary crossing equation

$$\sum_{d} B_{\mathrm{L}}\left(\begin{matrix} c \\ a \end{matrix}\; d\,\middle|\, u\right)\; W\left(\begin{matrix} c & b \\ d & a \end{matrix}\,\middle|\, 2u-\mu\right) = -\rho(\mu-2u)\; B_{\mathrm{L}}\left(\begin{matrix} c \\ a \end{matrix}\; b\,\middle|\, \mu-u\right) \qquad (3.8)$$



and the right boundary crossing equation

$$\sum_{d} W\left(\begin{matrix} c & d \\ b & a \end{matrix}\,\middle|\, 2u-\mu\right)\; B_{\mathrm{R}}\left(d\;\begin{matrix} c \\ a \end{matrix}\,\middle|\, u\right) = -\rho(\mu-2u)\; B_{\mathrm{R}}\left(b\;\begin{matrix} c \\ a \end{matrix}\,\middle|\, \mu-u\right) \qquad (3.9)$$



These equations are to be satisfied for all values of the external spins and all values of the spectral parameters. The function ρ is model-dependent and μ is the same parameter as in (3.1).

We note that the local relations (3.4)–(3.9) are consistent with the initial condition

$$W\begin{pmatrix} d & c \\ a & b \end{pmatrix}\!\!\left|\,0\right) = -\rho(0)\,\delta_{ac}\,A_{ab}\,A_{ad} \tag{3.10}$$

(The minus sign is used here, and in the boundary crossing equations, in order to provide consistency with subsequent fusion equations.) More specifically, with this initial condition we see that (3.4) holds for $u = v$ or $v = 0$, that (3.5) holds for $u = 0$, that (3.6) and (3.7) hold for $u = v$, and that (3.8) and (3.9) hold for $u = \mu/2$. Furthermore, we find that (3.4) holds for $u = 0$, due to (3.5), while (3.6) and (3.7) hold for $v = \mu - u$ due to (3.8) and (3.9). Indeed, the inversion relation can be motivated by the Yang-Baxter equation together with the initial condition, while the boundary crossing equations can be motivated by the reflection equations together with the initial condition and the inversion relation.

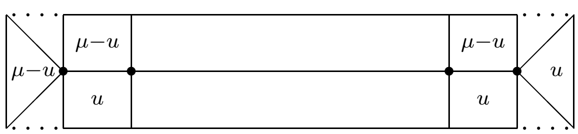
3.3 Crossing Symmetry of Double-Row Transfer Matrices

The double-row transfer matrices satisfy crossing symmetry,

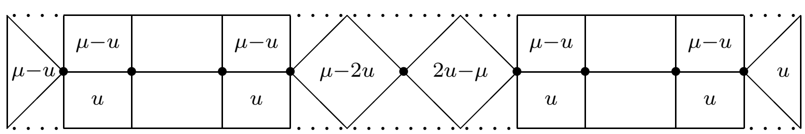
$$\boldsymbol{D}(u) = \boldsymbol{D}(\mu - u) \tag{3.11}$$

We prove this by considering an entry of $\boldsymbol{D}(u)$, applying the inversion relation at an arbitrary point, then using the Yang-Baxter equation N times, and finally applying both boundary crossing equations:

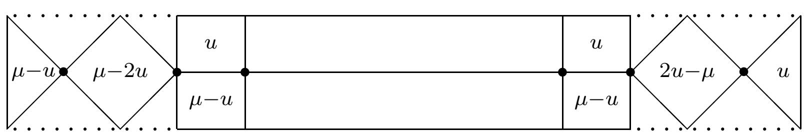
$\boldsymbol{D}(u) \;=$



$= \dfrac{1}{\eta(u)}$



$= \dfrac{1}{\eta(u)}$



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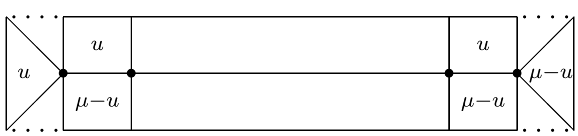


$= \boldsymbol{D}(\mu - u)$

where $\eta(u) = \rho(\mu - 2u)\rho(2u - \mu)$.

3.4 Commutation of Double-Row Transfer Matrices

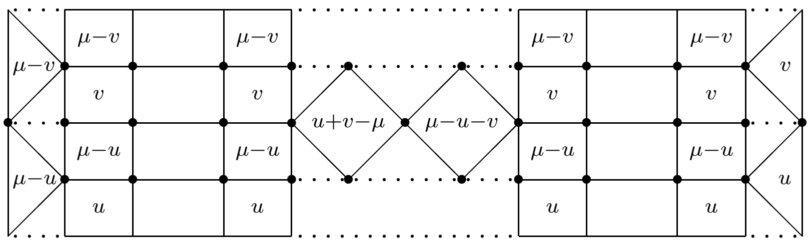
The double-row transfer matrices form a commuting family,

$$\boldsymbol{D}(u)\,\boldsymbol{D}(v) = \boldsymbol{D}(v)\,\boldsymbol{D}(u) \tag{3.12}$$

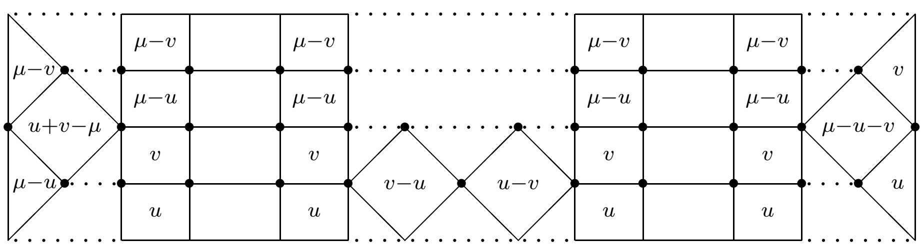
We prove this by the following steps, in each of which we use either the inversion relation, the Yang-Baxter equation N times, or the reflection equations:

$\boldsymbol{D}(u)\,\boldsymbol{D}(v)$

$=$ 

$= \dfrac{1}{\eta(u,v)}$ 

$= \dfrac{1}{\eta(u,v)}$ 

$= \dfrac{1}{\tilde{\eta}(u,v)}$ 

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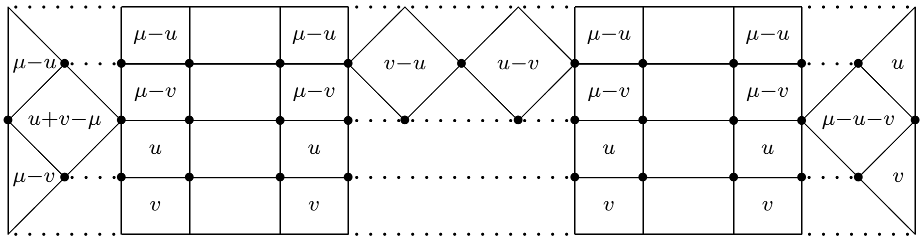

$= \dfrac{1}{\eta(u,v)}$

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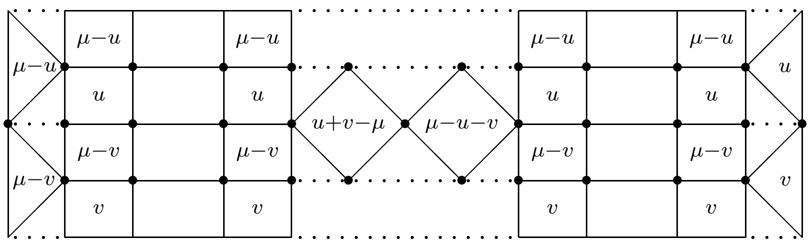

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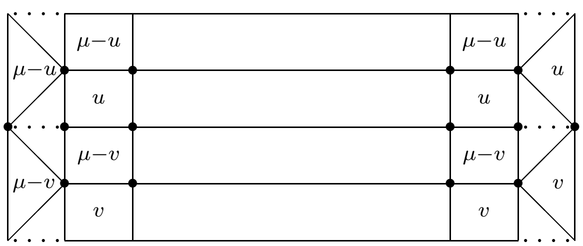


$= \quad \boldsymbol{D}(v)\,\boldsymbol{D}(u)$

where $\eta(u,v) = \rho(u+v-\mu)\rho(\mu-u-v)$ and $\tilde\eta(u,v) = \rho(v-u)\rho(u-v)\rho(u+v-\mu)\rho(\mu-u-v)$.

4. ABF Models

We now consider the particular case of Andrews-Baxter-Forrester (ABF) models [13]. There is one such model for each integer $L \geq 3$. The spins a—sometimes known also as heights—in this model take the values

$$a \in \{1, 2, \ldots, L\} \tag{4.1}$$

and adjacent spins must differ by 1,

$$A_{ab} = \delta_{a,b-1} + \delta_{a,b+1} \tag{4.2}$$

There is a fixed crossing parameter

$$\lambda = \frac{\pi}{L+1} \tag{4.3}$$

and the non-zero face weights are given by

$$\begin{aligned}
W\!\left(\begin{matrix} a\pm1 & a \\ a & a\mp1 \end{matrix}\,\middle|\, u\right) &= \frac{\theta(\lambda-u)}{\theta(\lambda)} \\
W\!\left(\begin{matrix} a & a\pm1 \\ a\mp1 & a \end{matrix}\,\middle|\, u\right) &= \sqrt{\frac{\theta((a-1)\lambda)\,\theta((a+1)\lambda)}{\theta(a\lambda)^2}}\;\frac{\theta(u)}{\theta(\lambda)} \\
W\!\left(\begin{matrix} a & a\pm1 \\ a\pm1 & a \end{matrix}\,\middle|\, u\right) &= \frac{\theta(a\lambda\pm u)}{\theta(a\lambda)}
\end{aligned} \tag{4.4}$$

Here θ is the standard elliptic theta-1 function of fixed nome $\hat{q}$

$$\theta(u) = \theta_1(u,\hat{q}) = 2\hat{q}^{1/4}\sin u \prod_{n=1}^{\infty}\left(1-2\hat{q}^{2n}\cos 2u + \hat{q}^{4n}\right)\left(1-\hat{q}^{2n}\right) \tag{4.5}$$

which satisfies the identity

$$\begin{aligned}
\theta(s+x)\,\theta(s-x)\,\theta(t+y)\,\theta(t-y) - \theta(s+y)\,\theta(s-y)\,\theta(t+x)\,\theta(t-x) \\
= \;\theta(s+t)\,\theta(s-t)\,\theta(x+y)\,\theta(x-y)
\end{aligned} \tag{4.6}$$

It can be seen that the ABF face weights satisfy various simple relations: reflection symmetries

$$W\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = W\!\left(\begin{matrix} b & c \\ a & d \end{matrix}\,\middle|\, u\right) \tag{4.7}$$

and

$$W\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = W\!\left(\begin{matrix} d & a \\ c & b \end{matrix}\,\middle|\, u\right) \tag{4.8}$$

crossing symmetry

$$W\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = \sqrt{\frac{\theta(a\lambda)\,\theta(c\lambda)}{\theta(b\lambda)\,\theta(d\lambda)}}\; W\!\left(\begin{matrix} a & b \\ d & c \end{matrix}\,\middle|\, \lambda-u\right) \tag{4.9}$$

full height reversal symmetry

$$W\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = W\!\left(\begin{matrix} L+1-d & L+1-c \\ L+1-a & L+1-b \end{matrix}\,\middle|\, u\right) \tag{4.10}$$

and the initial condition

$$W\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, 0\right) = \delta_{ac}\,A_{ab}\,A_{ad} \tag{4.11}$$

It is well-known that, essentially due to (4.6), the face weights also satisfy the Yang-Baxter equation, (3.4), and inversion relation, (3.5), with the function ρ given by

$$\rho(u) = \frac{\theta(u-\lambda)}{\theta(\lambda)} \tag{4.12}$$

We now define, as the only non-zero ABF boundary weights,

$$\begin{aligned} B_{\mathrm{L}}\left(\begin{matrix} a \\ a \end{matrix}\; a\mp1 \,\middle|\, u\right) &= \sqrt{\frac{\theta((a\mp1)\lambda)}{\theta(a\lambda)}}\;\frac{\theta(u+\frac{\lambda-\mu}{2}\mp\xi_{\mathrm{L}}(a))\,\theta(u\pm a\lambda+\frac{\lambda-\mu}{2}\pm\xi_{\mathrm{L}}(a))}{\theta(\lambda)^2} \\ B_{\mathrm{R}}\left(a\mp1\; \begin{matrix} a \\ a \end{matrix} \,\middle|\, u\right) &= \sqrt{\frac{\theta((a\mp1)\lambda)}{\theta(a\lambda)}}\;\frac{\theta(u+\frac{\lambda-\mu}{2}\mp\xi_{\mathrm{R}}(a))\,\theta(u\pm a\lambda+\frac{\lambda-\mu}{2}\pm\xi_{\mathrm{R}}(a))}{\theta(\lambda)^2} \end{aligned} \tag{4.13}$$

where $\xi_{\mathrm{L}}(a)$ and $\xi_{\mathrm{R}}(a)$ are arbitrary parameters independent of u. In Appendix A, we show that the reflection equations, (3.6) and (3.7), are satisfied by the ABF face and boundary weights. We also show that the ABF weights, together with ρ given by (4.12), satisfy the boundary crossing equations, (3.8) and (3.9).

If ξ_{L} and ξ_{R} satisfy

$$\xi_{\mathrm{L}}(L+1-a) = -\xi_{\mathrm{L}}(a)\,, \qquad \xi_{\mathrm{R}}(L+1-a) = -\xi_{\mathrm{R}}(a) \tag{4.14}$$

then the ABF boundary weights satisfy full height reversal symmetry

$$\begin{aligned} B_{\mathrm{L}}\left(\begin{matrix} c \\ a \end{matrix}\; b \,\middle|\, u\right) &= -B_{\mathrm{L}}\left(\begin{matrix} L+1-c \\ L+1-a \end{matrix}\; L+1-b \,\middle|\, u\right) \\ B_{\mathrm{R}}\left(b\; \begin{matrix} c \\ a \end{matrix} \,\middle|\, u\right) &= -B_{\mathrm{R}}\left(L+1-b\; \begin{matrix} L+1-c \\ L+1-a \end{matrix} \,\middle|\, u\right) \end{aligned} \tag{4.15}$$

The boundary weights (4.13) have the diagonal form

$$a \neq c \quad \text{implies} \quad B_{\mathrm{L}}\left(\begin{matrix} c \\ a \end{matrix}\; b \,\middle|\, u\right) = B_{\mathrm{R}}\left(b\; \begin{matrix} c \\ a \end{matrix} \,\middle|\, u\right) = 0 \tag{4.16}$$

This form leads to identical boundary spins at each end of the entire lattice. It is therefore convenient to regard these spins as labels for the boundaries and only the internal spins as matrix indices, and accordingly we define the ABF double-row transfer matrix with fixed left and right boundary spins a_{L} and a_{R} by

$$\langle a_2,\ldots,a_N \,|\, \boldsymbol{D}(a_{\mathrm{L}}a_{\mathrm{R}}|u) \,|\, b_2,\ldots,b_N\rangle = \langle a_{\mathrm{L}},a_2,\ldots,a_N,a_{\mathrm{R}} \,|\, \boldsymbol{D}(u) \,|\, a_{\mathrm{L}},b_2,\ldots,b_N,a_{\mathrm{R}}\rangle \tag{4.17}$$

It is natural in these models to take μ as

$$\mu = \lambda \tag{4.18}$$

With this choice, crossing symmetry of the face weights, (4.9), implies that $\boldsymbol{D}(a_{\rm L}a_{\rm R}|u)$ is symmetric

$$\boldsymbol{D}(a_{\rm L}a_{\rm R}|u)=\boldsymbol{D}(a_{\rm L}a_{\rm R}|u)^t \tag{4.19}$$

Ultimately, we will be interested in the isotropic point, $u=\lambda/2$, at which we now show it is possible to achieve a completely homogeneous lattice, with pure, fixed boundary conditions. If we set $\xi_{\rm L}(a_{\rm L})=\pm\lambda/2$, $\xi_{\rm R}(a_{\rm R})=\pm\lambda/2$ and $\mu=\lambda$, then

$$B_{\rm L}\!\left(\begin{matrix}a_{\rm L}\\ a_{\rm L}\end{matrix}\;\, a_{\rm L}\mp1\,\middle|\,\lambda/2\right)=B_{\rm R}\!\left(a_{\rm R}\mp1\;\,\begin{matrix}a_{\rm R}\\ a_{\rm R}\end{matrix}\,\middle|\,\lambda/2\right)=0$$

so that the transfer matrix $\boldsymbol{D}(a_{\rm L}a_{\rm R}|\lambda/2)$ is simply proportional to the matrix product of two rows of face weights, all with spectral parameter $\lambda/2$, with the three spins on the left boundary fixed to $a_{\rm L}$, $a_{\rm L}{\pm}1$, $a_{\rm L}$ and the three spins on the right boundary fixed to $a_{\rm R}$, $a_{\rm R}{\pm}1$, $a_{\rm R}$. Similarly, if we set $\xi_{\rm L}(a_{\rm L})=\pm\lambda/2$ and $\xi_{\rm R}(a_{\rm R})=\mp\lambda/2$, then $\boldsymbol{D}(a_{\rm L}a_{\rm R}|\lambda/2)$ has the spins on the left boundary fixed to $a_{\rm L}$, $a_{\rm L}{\pm}1$, $a_{\rm L}$ and the spins on the right boundary fixed to $a_{\rm R}$, $a_{\rm R}{\mp}1$, $a_{\rm R}$.

5. Fused IRF Models with Fixed Boundary Conditions

We now extend our general formalism to cover models which have a fusion hierarchy. For these models there is a discrete set of fusion levels, and we assume that each of these is labelled by a single integer, with the original, unfused model corresponding to fusion level 1.

The fused face weights, W^{pq}, are associated with two fusion levels—a horizontal level, p, and a vertical level, q—and the fused boundary weights, $B^q_{\rm L}$ and $B^q_{\rm R}$, are associated with one fusion level, q. There is now an adjacency matrix, A^q, for each fusion level q, with the adjacency conditions on the fused weights being

$$A^p_{ab}\,A^q_{bc}\,A^p_{cd}\,A^q_{da}=0 \quad\text{implies}\quad W^{pq}\!\left(\begin{matrix}d & c\\ a & b\end{matrix}\,\middle|\,u\right)=0 \tag{5.1}$$

$$A^q_{ab}\,A^q_{bc}=0 \quad\text{implies}\quad B^q_{\rm L}\!\left(\begin{matrix}c\\ a\end{matrix}\;\, b\,\middle|\,u\right)=B^q_{\rm R}\!\left(b\;\,\begin{matrix}c\\ a\end{matrix}\,\middle|\,u\right)=0 \tag{5.2}$$

The fused double-row transfer matrices are also associated with two fusion levels, and are defined by

$$\begin{aligned}&\langle a_1,\ldots,a_{N+1}|\boldsymbol{D}^{pq}(u)|b_1,\ldots,b_{N+1}\rangle \;=\\ &\qquad\sum_{c_1\ldots c_{N+1}} B^q_{\rm L}\!\left(\begin{matrix}b_1\\ a_1\end{matrix}\;\, c_1\,\middle|\,-u-(q-1)\lambda+\mu\right)\;\times\\ &\qquad\quad\left[\prod_{j=1}^{N} W^{pq}\!\left(\begin{matrix}c_j & c_{j+1}\\ a_j & a_{j+1}\end{matrix}\,\middle|\,u\right) W^{pq}\!\left(\begin{matrix}b_j & b_{j+1}\\ c_j & c_{j+1}\end{matrix}\,\middle|\,-u-(q-1)\lambda+\mu\right)\right] B^q_{\rm R}\!\left(c_{N+1}\;\,\begin{matrix}b_{N+1}\\ a_{N+1}\end{matrix}\,\middle|\,u\right)\end{aligned} \tag{5.3}$$

where λ and μ are arbitrary fixed parameters. In this generalised framework, the fused Yang-Baxter equation is

$$\sum_g W^{rq}\!\left(\begin{matrix} f & g \\ a & b \end{matrix}\,\middle|\, u-v\right) W^{pq}\!\left(\begin{matrix} g & d \\ b & c \end{matrix}\,\middle|\, u\right) W^{pr}\!\left(\begin{matrix} f & e \\ g & d \end{matrix}\,\middle|\, v\right) = \sum_g W^{pr}\!\left(\begin{matrix} a & g \\ b & c \end{matrix}\,\middle|\, v\right) W^{pq}\!\left(\begin{matrix} f & e \\ a & g \end{matrix}\,\middle|\, u\right) W^{rq}\!\left(\begin{matrix} e & d \\ g & c \end{matrix}\,\middle|\, u-v\right) \tag{5.4}$$

the fused inversion relation is

$$\sum_e W^{qr}\!\left(\begin{matrix} d & e \\ a & b \end{matrix}\,\middle|\, u\right) W^{rq}\!\left(\begin{matrix} d & c \\ e & b \end{matrix}\,\middle|\, -u\right) = \rho^{qr}(u)\, \rho^{rq}(-u)\, \delta_{ac}\, A^q_{ab}\, A^r_{ad} \tag{5.5}$$

the fused left reflection equation is

$$\begin{aligned}
&\rho^{rq}(u-v)\, \rho^{rq}(-u-v-(q-1)\lambda+\mu)\, \times \\
&\sum_{fg} W^{qr}\!\left(\begin{matrix} c & b \\ f & a \end{matrix}\,\middle|\, u-v+(q-r)\lambda\right) W^{qr}\!\left(\begin{matrix} d & c \\ g & f \end{matrix}\,\middle|\, -u-v-(r-1)\lambda+\mu\right) B^q_{\mathrm{L}}\!\left(\begin{matrix} g \\ a \end{matrix}\; f \,\middle|\, u\right) B^r_{\mathrm{L}}\!\left(\begin{matrix} e \\ g \end{matrix}\; d \,\middle|\, v\right) \\
=\; &\rho^{qr}(u-v+(q-r)\lambda)\, \rho^{qr}(-u-v-(r-1)\lambda+\mu)\, \times \\
&\sum_{fg} W^{rq}\!\left(\begin{matrix} e & d \\ f & c \end{matrix}\,\middle|\, u-v\right) W^{rq}\!\left(\begin{matrix} f & c \\ g & b \end{matrix}\,\middle|\, -u-v-(q-1)\lambda+\mu\right) B^q_{\mathrm{L}}\!\left(\begin{matrix} e \\ g \end{matrix}\; f \,\middle|\, u\right) B^r_{\mathrm{L}}\!\left(\begin{matrix} g \\ a \end{matrix}\; b \,\middle|\, v\right)
\end{aligned} \tag{5.6}$$

the fused right reflection equation is

$$\begin{aligned}
&\rho^{qr}(u-v+(q-r)\lambda)\, \rho^{qr}(-u-v-(r-1)\lambda+\mu)\, \times \\
&\sum_{fg} W^{rq}\!\left(\begin{matrix} c & f \\ b & a \end{matrix}\,\middle|\, u-v\right) W^{rq}\!\left(\begin{matrix} d & g \\ c & f \end{matrix}\,\middle|\, -u-v-(q-1)\lambda+\mu\right) B^q_{\mathrm{R}}\!\left(f\; \begin{matrix} g \\ a \end{matrix} \,\middle|\, u\right) B^r_{\mathrm{R}}\!\left(d\; \begin{matrix} e \\ g \end{matrix} \,\middle|\, v\right) \\
=\; &\rho^{rq}(u-v)\, \rho^{rq}(-u-v-(q-1)\lambda+\mu)\, \times \\
&\sum_{fg} W^{qr}\!\left(\begin{matrix} e & f \\ d & c \end{matrix}\,\middle|\, u-v+(q-r)\lambda\right) W^{qr}\!\left(\begin{matrix} f & g \\ c & b \end{matrix}\,\middle|\, -u-v-(r-1)\lambda+\mu\right) B^q_{\mathrm{R}}\!\left(f\; \begin{matrix} e \\ g \end{matrix} \,\middle|\, u\right) B^r_{\mathrm{R}}\!\left(b\; \begin{matrix} g \\ a \end{matrix} \,\middle|\, v\right)
\end{aligned} \tag{5.7}$$

the fused left boundary crossing equation is

$$\sum_d B^q_{\mathrm{L}}\!\left(\begin{matrix} c \\ a \end{matrix}\; d \,\middle|\, u\right) W^{qq}\!\left(\begin{matrix} c & b \\ d & a \end{matrix}\,\middle|\, 2u+(q-1)\lambda-\mu\right) = (-1)^q\, \rho^{qq}(-2u-(q-1)\lambda+\mu)\, B^q_{\mathrm{L}}\!\left(\begin{matrix} c \\ a \end{matrix}\; b \,\middle|\, -u-(q-1)\lambda+\mu\right) \tag{5.8}$$

and the fused right boundary crossing equation is

$$\sum_d W^{qq}\!\left(\begin{matrix} c & d \\ b & a \end{matrix}\,\middle|\, 2u+(q-1)\lambda-\mu\right) B^q_{\mathrm{R}}\!\left(d\; \begin{matrix} c \\ a \end{matrix} \,\middle|\, u\right) = (-1)^q\, \rho^{qq}(-2u-(q-1)\lambda+\mu)\, B^q_{\mathrm{R}}\!\left(b\; \begin{matrix} c \\ a \end{matrix} \,\middle|\, -u-(q-1)\lambda+\mu\right) \tag{5.9}$$

where ρ^{rq} are model-dependent functions. The fused local relations are consistent with the fused initial condition

$$W^{qq}\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle|\, 0\right) = (-1)^q\ \rho^{qq}(0)\ \delta_{ac}\ A^q_{ab}\ A^q_{ad} \tag{5.10}$$

It can be seen that the fused adjacency conditions, double row transfer matrix, local relations and initial condition reduce to (3.1)–(3.10) for $p = q = r = 1$.

By following a parallel sequence of steps to those of section 3.3, but now including the fusion levels p and q, it can be shown that the fused inversion relation and boundary crossing equations, (5.5), (5.8) and (5.9), imply that the fused double-row transfer matrices satisfy crossing symmetry

$$\boldsymbol{D}^{pq}(u) = \boldsymbol{D}^{pq}(-u-(q-1)\lambda+\mu) \tag{5.11}$$

Similarly, by following a parallel sequence of steps to those of section 3.4, it can be shown that the fused Yang-Baxter equation, inversion relation, and reflection equations, (5.4)–(5.7), imply that the fused double-row transfer matrices form a commuting family

$$\boldsymbol{D}^{pq}(u)\ \boldsymbol{D}^{pr}(v) = \boldsymbol{D}^{pr}(v)\ \boldsymbol{D}^{pq}(u) \tag{5.12}$$

6. ABF Fusion Hierarchy

6.1 Adjacency Conditions

We now return to the case of ABF models and consider their fusion hierarchy [15, 16, 17, 18]. For each L, we have $L+2$ fusion levels, labelled $-1, 0, \ldots, L$. The level q adjacency matrix, A^q, is defined by the condition that $A^q_{ab} = 1$ if and only if

$$a - b\ \in\ \{-q,\ -q+2,\ \ldots\ ,\ q-2,\ q\} \tag{6.1}$$

and

$$a + b\ \in\ \{q+2,\ q+4,\ \ldots\ ,\ 2L-q-2,\ 2L-q\} \tag{6.2}$$

It can be seen that

$$A^{-1} = 0\ , \qquad A^0 = I\ , \qquad A^1 = A\ , \qquad A^{L-2} = AY\ , \qquad A^{L-1} = Y\ , \qquad A^L = 0 \tag{6.3}$$

where I is the $L \times L$ identity matrix, A is given by (4.2), and Y is the $L \times L$ height reversal matrix

$$Y_{ab} = \delta_{L+1-a\,,\,b} \tag{6.4}$$

It can be shown that the fused adjacency matrices satisfy full height reversal symmetry

$$A^q = Y A^q\, Y \tag{6.5}$$

partial height reversal symmetry

$$A^q = Y A^{L-1-q} \tag{6.6}$$

and the fusion rules

$$A^q A \;=\; A^{q-1} + A^{q+1} \tag{6.7}$$

$$(A^q)^2 \;=\; I + A^{q-1}\,A^{q+1} \tag{6.8}$$

and

$$(\tilde{A}^q)^2 \;=\; (I + \tilde{A}^{q-1})\,(I + \tilde{A}^{q+1}) \tag{6.9}$$

where

$$\tilde{A}^q = A^{q-1}\,A^{q+1} \tag{6.10}$$

For what follows, it is useful to define a set, P^q_{ab}, of $q-1$-point paths between a and b, as

$$P^q_{ab} = \begin{cases} \{1,\ldots,L\}^{q-1} & ,\ A^q_{ab}=0 \\ \Big\{(c_1,\ldots,c_{q-1}) \in \{1,\ldots,L\}^{q-1} \,\Big|\, A_{ac_1}\,A_{c_1c_2}\,\ldots\,A_{c_{q-2}c_{q-1}}\,A_{c_{q-1}b}=1\Big\}, & A^q_{ab}=1 \end{cases} \tag{6.11}$$

6.2 Face and Boundary Weights

We now define ABF fused face weights, W^{pq}, and fused boundary weights, $B^q_{\rm L}$ and $B^q_{\rm R}$. These definitions will involve the fusion normalisation function

$$\theta^q_k(u) \quad = \quad \frac{\prod_{j=0}^{q-1}\theta(u+k\lambda-j\lambda)}{\theta(\lambda)^q} \tag{6.12}$$

and the fusion gauge factors

$$X^q_{ab} \quad = \quad \begin{cases} 1 & ,\quad A^q_{ab}=0 \\ \dfrac{\prod\limits_{j=\frac{a+b-q}{2}}^{\frac{a+b+q}{2}}\theta(j\lambda)\;\prod\limits_{j=2}^{\frac{a-b+q}{2}}\theta(j\lambda)\;\prod\limits_{j=2}^{\frac{b-a+q}{2}}\theta(j\lambda)}{\theta(\lambda)^{2q-1}} & ,\quad A^q_{ab}=1 \end{cases} \tag{6.13}$$

and

$$G^q_{a_0,a_1,\ldots,a_{q-1},a_q} \quad = \quad X^q_{a_0a_q}\,\frac{\theta(\lambda)^{q+1}}{\prod_{j=0}^{q}\theta(a_j\lambda)} \tag{6.14}$$

where, as before, λ is given by (4.3) and θ is given by (4.5). Throughout this section, a product $\prod_{j=j'}^{j''} P(j)$ is taken to be 1 for $j'' < j'$.

For weights involving fusion level -1, we must, in order to satisfy the adjacency condition, define

$$W^{p,-1}\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = W^{-1,q}\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = B^{-1}_{\rm L}\!\left(\begin{matrix} c & \\ a & b \end{matrix}\,\middle|\, u\right) = B^{-1}_{\rm R}\!\left(\begin{matrix} & c \\ b & a \end{matrix}\,\middle|\, u\right) = 0 \tag{6.15}$$

For weights involving fusion level 0, we define

$$
\begin{aligned}
W^{p,0}\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) &= \theta^p_{-1}(u)\,\delta_{ad}\,\delta_{bc}\,A^p_{ab} \\
W^{0,q}\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) &= \delta_{ab}\,\delta_{cd}\,A^q_{ad}
\end{aligned}
\tag{6.16}
$$

$$
B^0_{\rm L}\!\left(\begin{matrix} c & \\ & b \\ a & \end{matrix}\,\middle|\, u\right) = B^0_{\rm R}\!\left(\begin{matrix} & c \\ b & \\ & a \end{matrix}\,\middle|\, u\right) = \delta_{ab}\,\delta_{bc}
$$

For fusion level 1, the non-zero ABF weights are defined as

$$
\begin{aligned}
W^{11}\!\left(\begin{matrix} a\pm1 & a \\ a & a\mp1 \end{matrix}\,\middle|\, u\right) &= \frac{\theta(\lambda-u)}{\theta(\lambda)} \\
W^{11}\!\left(\begin{matrix} a & a\pm1 \\ a\mp1 & a \end{matrix}\,\middle|\, u\right) &= -\,\frac{\theta((a\pm1)\lambda)\,\theta(u)}{\theta(a\lambda)\,\theta(\lambda)} \\
W^{11}\!\left(\begin{matrix} a & a\pm1 \\ a\pm1 & a \end{matrix}\,\middle|\, u\right) &= \frac{\theta(a\lambda\pm u)}{\theta(a\lambda)} \\
B^1_{\rm L}\!\left(\begin{matrix} a & \\ & a\mp1 \\ a & \end{matrix}\,\middle|\, u\right) &= \mp\,\frac{\theta((a\mp1)\lambda)\,\theta(u+\frac{\lambda-\mu}{2}\mp\xi_{\rm L}(a))\,\theta(u\pm a\lambda+\frac{\lambda-\mu}{2}\pm\xi_{\rm L}(a))}{\theta(\lambda)^3} \\
B^1_{\rm R}\!\left(\begin{matrix} & a \\ a\mp1 & \\ & a \end{matrix}\,\middle|\, u\right) &= \mp\,\frac{\theta(u+\frac{\lambda-\mu}{2}\mp\xi_{\rm R}(a))\,\theta(u\pm a\lambda+\frac{\lambda-\mu}{2}\pm\xi_{\rm R}(a))}{\theta(a\lambda)\,\theta(\lambda)}
\end{aligned}
\tag{6.17}
$$

where, as before, $\xi_{\rm L}(a)$ and $\xi_{\rm R}(a)$ are arbitrary constants. These weights are related to the standard ABF weights, (4.4) and (4.13), by the gauge transformation

$$
\begin{aligned}
W^{11}\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) &= \epsilon_a\,\epsilon_c\,\sqrt{\frac{\theta(c\lambda)}{\theta(a\lambda)}}\;W\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) \\
B^1_{\rm L}\!\left(\begin{matrix} c & \\ & b \\ a & \end{matrix}\,\middle|\, u\right) &= \epsilon_{a+1}\,\epsilon_b\,\frac{\sqrt{\theta(a\lambda)\,\theta(b\lambda)}}{\theta(\lambda)}\;B_{\rm L}\!\left(\begin{matrix} c & \\ & b \\ a & \end{matrix}\,\middle|\, u\right) \\
B^1_{\rm R}\!\left(\begin{matrix} & c \\ b & \\ & a \end{matrix}\,\middle|\, u\right) &= \epsilon_{a+1}\,\epsilon_b\,\frac{\theta(\lambda)}{\sqrt{\theta(a\lambda)\,\theta(b\lambda)}}\;B_{\rm R}\!\left(\begin{matrix} & c \\ b & \\ & a \end{matrix}\,\middle|\, u\right)
\end{aligned}
\tag{6.18}
$$

where ϵ_a are factors whose required properties are†

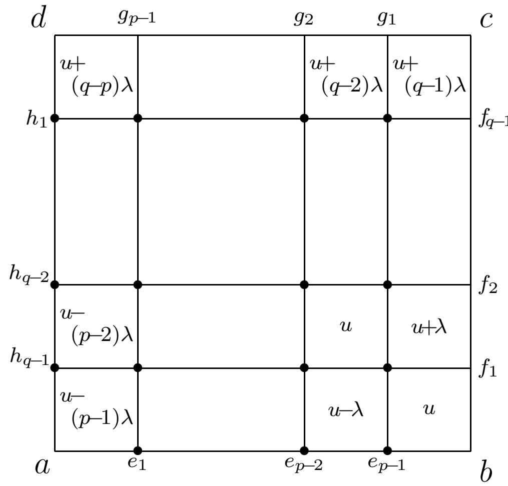
$$
(\epsilon_a)^2 = 1\,, \qquad\qquad \epsilon_a\,\epsilon_{a+2} = -1
\tag{6.19}
$$

†For example, one of the (four possible) choices for ϵ is $\epsilon_a = \begin{cases} 1, & a = 0 \text{ or } 1 \pmod 4 \\ -1, & a = 2 \text{ or } 3 \pmod 4 \end{cases}$

The Yang-Baxter, inversion, reflection and boundary crossing equations, (3.4)–(3.9), are still satisfied by these level 1 weights since the gauge factors corresponding to the internal spins of these equations cancel, while the gauge factors corresponding to the external spins are the same on both sides of each equation. However, the face weights no longer satisfy the reflection symmetry (4.8). We note that the face and boundary weights which appear in all subsequent diagrams are the level 1 ABF weights of (6.17).

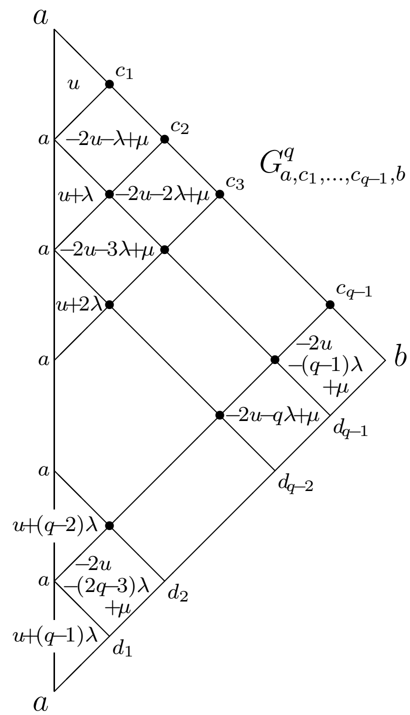
We now proceed to ABF weights involving higher fusion levels, which are defined in terms of sums of products of the level 1 weights of (6.17) as follows:

$$W^{pq}\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = \frac{A^p_{ab}\,A^q_{ad}}{\prod_{j=0}^{q-2}\theta^p_j(u)} \sum_{e_1\dots e_{p-1}} \sum_{h_1\dots h_{q-1}} \tag{6.20}$$



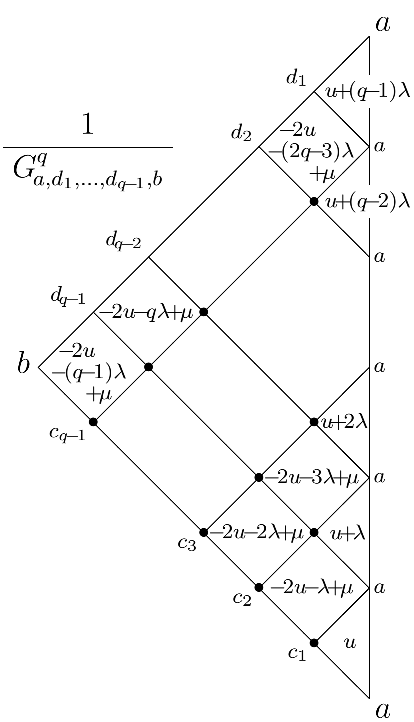
where $(f_1,\ldots,f_{q-1})\in P^q_{bc}$ and $(g_1,\ldots,g_{p-1})\in P^p_{cd}$,

$$B^q_{\mathrm{L}}\!\left(\begin{matrix} c & \\ a & b\end{matrix}\,\middle|\, u\right) \;=\; \frac{\delta_{ac}\ A^q_{ab}}{\prod_{j=0}^{q-2}\theta^{j+1}_{2j+1}(2u-\mu)} \sum_{c_1\ldots c_{q-1}} \tag{6.21}$$



where $(d_1,\ldots,d_{q-1})\in P^q_{ab}$, and

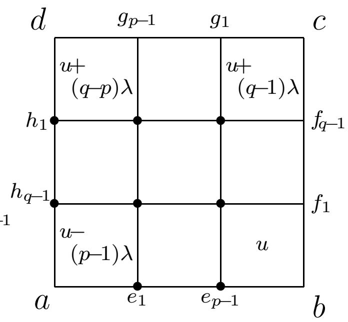
$$B^q_{\mathrm{R}}\!\left(\begin{matrix} & c \\ b & a\end{matrix}\,\middle|\, u\right) \;=\; \frac{\delta_{ac}\ A^q_{ab}}{\prod_{j=0}^{q-2}\theta^{j+1}_{2j+1}(2u-\mu)} \sum_{c_1\ldots c_{q-1}} \frac{1}{G^q_{a,d_1,\ldots,d_{q-1},b}} \tag{6.22}$$



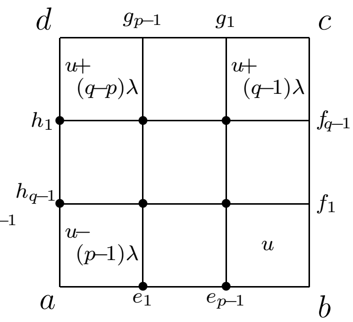
where $(d_1,\ldots,d_{q-1}) \in P^q_{ab}$. It is shown in [17] that $W^{pq}\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right)$ is independent of the choice of $(f_1,\ldots,f_{q-1}) \in P^q_{bc}$ and $(g_1,\ldots,g_{p-1}) \in P^p_{cd}$, and it can be shown similarly that $B^q_{\rm L}\left(\begin{matrix} a & \\ & b \\ a & \end{matrix}\,\middle|\, u\right)$ and $B^q_{\rm R}\left(\begin{matrix} & a \\ b & \\ & a \end{matrix}\,\middle|\, u\right)$ are each independent of the choice of $(d_1,\ldots,d_{q-1}) \in P^q_{ab}$.

It can also be shown that $W^{pq}\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right)$ satisfies (5.1) even though the adjacency condition is only explicitly applied on the edges being summed. This can be regarded as a push-through property of the entries of the adjacency matrix,

$$A^p_{ab}\,A^q_{bc}\,A^p_{cd}\,A^q_{da} \times \sum_{e_1\ldots e_{p-1}}\;\sum_{h_1\ldots h_{q-1}}$$



$$= \; A^p_{ab}\,A^q_{da} \times \sum_{e_1\ldots e_{p-1}}\;\sum_{h_1\ldots h_{q-1}}$$

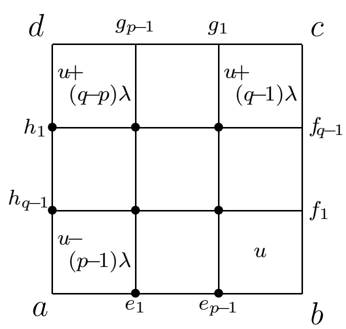


(6.23)

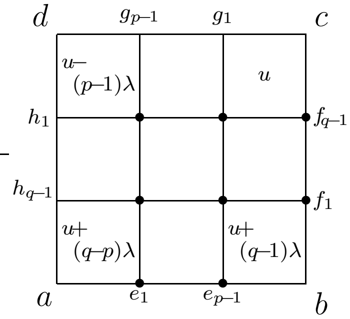
Furthermore, it can be shown that the configuration of level 1 weights in $W^{pq}\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right)$ can be re-oriented, as follows:

$$\left(\prod_{j=0}^{q-2}\theta^p_j(u)\right)\quad W^{pq}\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right)$$

$$= \quad A^p_{ab}\,A^q_{da}\;\sum_{e_1\ldots e_{p-1}}\;\sum_{h_1\ldots h_{q-1}} \tag{6.24}$$



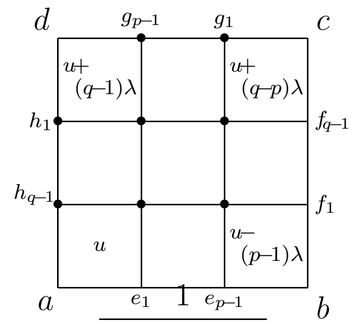
$$= \; A^p_{ab}\, A^q_{bc} \sum_{e_1 \ldots e_{p-1}} \; \sum_{f_1 \ldots f_{q-1}} \frac{1}{G^q_{d,h_1,\ldots,h_{q-1},a}}$$



$$G^q_{b,f_1,\ldots,f_{q-1},c} \qquad (6.25)$$

$$G^p_{c,g_1,\ldots,g_{p-1},d}$$

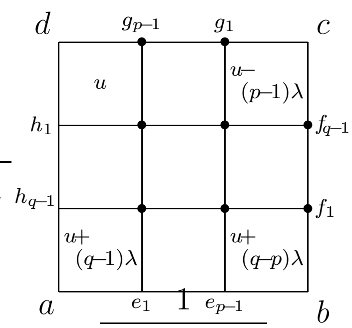
$$= \; A^p_{cd}\, A^q_{da} \sum_{g_1 \ldots g_{p-1}} \; \sum_{h_1 \ldots h_{q-1}} \qquad (6.26)$$



$$\frac{1}{G^p_{a,e_1,\ldots,e_{p-1},b}}$$

$$G^p_{c,g_1,\ldots,g_{p-1},d}$$

$$= \; A^q_{bc}\, A^p_{cd} \sum_{f_1 \ldots f_{q-1}} \; \sum_{g_1 \ldots g_{p-1}} \frac{1}{G^q_{d,h_1,\ldots,h_{q-1},a}}$$



$$G^q_{b,f_1,\ldots,f_{q-1},c} \qquad (6.27)$$

$$\frac{1}{G^p_{a,e_1,\ldots,e_{p-1},b}}$$

In these expressions, the external edge spins which are not summed are arbitrary, as long as we have $(e_1, \ldots, e_{p-1}) \in P^p_{ab}$ in (6.26) and (6.27), $(f_1, \ldots, f_{q-1}) \in P^q_{bc}$ in (6.24) and (6.26), $(g_1, \ldots, g_{p-1}) \in P^p_{cd}$ in (6.24) and (6.25), and $(h_1, \ldots, h_{q-1}) \in P^q_{da}$ in (6.25) and (6.27). One way to prove (6.23) and (6.25)–(6.27) is to use the fusion projection operators of [18], which satisfy a push-through property relative to the fused face weights and whose entries are proportional to the gauge factors G.

In [17], it is shown that for the ABF fused face weights, the summation over multiple spins in (6.20) can always be reduced either to a single term or to a summation over a single index, and that $\prod_{j=0}^{q-2}\theta_j^p(u)$ always arises as a common factor. The resulting expressions for the weights are presented, and from these we find that we have crossing symmetry

$$W^{pq}\begin{pmatrix} d & c \\ a & b \end{pmatrix}\!\!\left|\, u\right) = (-1)^{p(q-1)}\;\epsilon_a\,\epsilon_b\,\epsilon_c\,\epsilon_d\;\frac{\theta(a\lambda)}{\theta(d\lambda)}\;\frac{X^p_{cd}}{X^p_{ab}}\;W^{pq}\begin{pmatrix} a & b \\ d & c \end{pmatrix}\!\!\left|\, -u+(p-q+1)\lambda\right) \tag{6.28}$$

partial height reversal symmetry

$$W^{pq}\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = (-1)^{pL}\;\epsilon_{a+d}\,\epsilon_{b+c}\;\frac{X^q_{bc}}{X^q_{ad}}\;W^{p,L-1-q}\left(\begin{matrix} d & c \\ L+1-a & L+1-b \end{matrix}\,\middle|\, u+(q+1)\lambda\right) \tag{6.29}$$

full height reversal symmetry

$$W^{pq}\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = W^{pq}\left(\begin{matrix} L+1-d & L+1-c \\ L+1-a & L+1-b \end{matrix}\,\middle|\, u\right) \tag{6.30}$$

and an initial condition

$$W^{qq}\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, 0\right) = \theta^q_q(0)\;\delta_{ac}\;A^q_{ab}\;A^q_{ad} \tag{6.31}$$

Properties (6.28) and (6.30) can be proved alternatively by applying the corresponding properties of the level 1 weights directly in (6.20).

Using techniques similar to those used in [17] to derive explicit formulae for the ABF fused face weights, it can be also shown that, for the boundary weights, the summations over multiple spins in (6.21) and (6.22) always reduce to a single term, with $\prod_{j=0}^{q-2}\theta^{j+1}_{2j+1}(2u-\mu)$ as a factor. This gives, for $A^q_{ab}=1$,

$$\begin{aligned} B^q_{\mathrm{L}}\left(\begin{matrix} a \\ a \end{matrix}\; b\,\middle|\, u\right) \;=\; & \\ \prod_{j=1}^{\frac{a-b+q}{2}} & \frac{\theta(j\lambda)\;\theta((a-j)\lambda)\;\theta(-u-(q-j)\lambda-\frac{\lambda-\mu}{2}+\xi_{\mathrm{L}}(a))\;\theta(u+(q-j+a)\lambda+\frac{\lambda-\mu}{2}+\xi_{\mathrm{L}}(a))}{\theta(\lambda)^4} \\ \times \prod_{j=1}^{\frac{b-a+q}{2}} & \frac{\theta(j\lambda)\;\theta((a+j)\lambda)\;\theta(u+(q-j)\lambda+\frac{\lambda-\mu}{2}+\xi_{\mathrm{L}}(a))\;\theta(u+(q-j-a)\lambda+\frac{\lambda-\mu}{2}-\xi_{\mathrm{L}}(a))}{\theta(\lambda)^4} \end{aligned} \tag{6.32}$$

and

$$\begin{aligned} B^q_{\mathrm{R}}\left(b\;\begin{matrix} a \\ a \end{matrix}\,\middle|\, u\right) \;=\; & \prod_{j=1}^{\frac{a-b+q}{2}} \frac{\theta(-u-(q-j)\lambda-\frac{\lambda-\mu}{2}+\xi_{\mathrm{R}}(a))\;\theta(u+(q-j+a)\lambda+\frac{\lambda-\mu}{2}+\xi_{\mathrm{R}}(a))}{\theta(j\lambda)\;\theta((b+j)\lambda)} \\ & \times \prod_{j=1}^{\frac{b-a+q}{2}} \frac{\theta(u+(q-j)\lambda+\frac{\lambda-\mu}{2}+\xi_{\mathrm{R}}(a))\;\theta(u+(q-j-a)\lambda+\frac{\lambda-\mu}{2}-\xi_{\mathrm{R}}(a))}{\theta(j\lambda)\;\theta((b-j)\lambda)} \end{aligned} \tag{6.33}$$

It follows from these expressions that the ABF fused boundary weights satisfy partial height reversal symmetry

$$\begin{aligned}
&\theta^{L-q}_{-2}(u+\tfrac{\lambda-\mu}{2}-\xi_{\rm L}(a))\;\theta^{L-q}_{-2}(u+\tfrac{\lambda-\mu}{2}+\xi_{\rm L}(a))\;B^q_{\rm L}\!\left(\begin{matrix}a\\a\end{matrix}\;b\,\middle|\,u\right) \quad = \\
&\quad -\left(\frac{X^q_{ab}}{\theta^L_L(0)}\right)^2\;\theta^{q+1}_{q-1-a}(u+\tfrac{\lambda-\mu}{2}-\xi_{\rm L}(a))\;\theta^{q+1}_{q-1+a}(u+\tfrac{\lambda-\mu}{2}+\xi_{\rm L}(a))\;B^{L-1-q}_{\rm L}\!\left(\begin{matrix}a\\a\end{matrix}\;L+1-b\,\middle|\,u+(q+1)\lambda\right)
\end{aligned} \tag{6.34}$$

$$\begin{aligned}
&\theta^{L-q}_{-2}(u+\tfrac{\lambda-\mu}{2}-\xi_{\rm R}(a))\;\theta^{L-q}_{-2}(u+\tfrac{\lambda-\mu}{2}+\xi_{\rm R}(a))\;B^q_{\rm R}\!\left(b\;\begin{matrix}a\\a\end{matrix}\,\middle|\,u\right) \quad = \\
&\quad -\left(\frac{\theta^L_L(0)}{X^q_{ab}}\right)^2\;\theta^{q+1}_{q-1-a}(u+\tfrac{\lambda-\mu}{2}-\xi_{\rm R}(a))\;\theta^{q+1}_{q-1+a}(u+\tfrac{\lambda-\mu}{2}+\xi_{\rm R}(a))\;B^{L-1-q}_{\rm R}\!\left(L+1-b\;\begin{matrix}a\\a\end{matrix}\,\middle|\,u+(q+1)\lambda\right)
\end{aligned}$$

and, provided that (4.14) is satisfied, full height reversal symmetry

$$\begin{aligned}
B^q_{\rm L}\!\left(\begin{matrix}c\\a\end{matrix}\;b\,\middle|\,u\right) &= B^q_{\rm L}\!\left(\begin{matrix}L+1-c\\L+1-a\end{matrix}\;L+1-b\,\middle|\,u\right)\\
B^q_{\rm R}\!\left(b\;\begin{matrix}c\\a\end{matrix}\,\middle|\,u\right) &= B^q_{\rm R}\!\left(L+1-b\;\begin{matrix}L+1-c\\L+1-a\end{matrix}\,\middle|\,u\right)
\end{aligned} \tag{6.35}$$

6.3 Local Relations

We now consider the fused local relations, (5.4)–(5.9). It is shown in [17] that the ABF fused face weights satisfy the fused Yang-Baxter equation (5.4). A proof proceeds as follows: if fusion level -1 is involved, then each side of (5.4) is zero, if fusion level 0 is involved, then each side of (5.4) immediately reduces to a product of the same terms, and if higher fusion levels only are involved, then (5.4) can be verified by setting internal arbitrary spins equal to adjoining summed spins, using (6.23) to push all explicit occurrences of the fused adjacency condition to external edges, and applying the original Yang Baxter equation, (3.4), pqr times.

It can also be shown that the ABF fused face weights satisfy the fused inversion relation (5.5) with

$$\rho^{qr}(u) = \theta^q_{-1}(u) \tag{6.36}$$

Again, if fusion level -1 is involved, then each side of (5.5) is zero, if fusion level 0 is involved, then the left side of (5.5) immediately reduces to the same product of terms as the right side, and if higher fusion levels only are involved, then (5.5) can be verified by setting internal arbitrary spins equal to adjoining summed spins, using (6.23) to push all explicit occurrences of the fused adjacency condition to external edges, and applying the original inversion relation, (3.5), qr times.

Finally, in Appendix B we show that the ABF fused face and boundary weights, together with ρ^{qr} given by (6.36), also satisfy the fused reflection equations, (5.6) and (5.7), and fused boundary crossing equations, (5.8) and (5.9), where λ in these equations is taken as the crossing parameter (4.3) and μ is arbitrary.

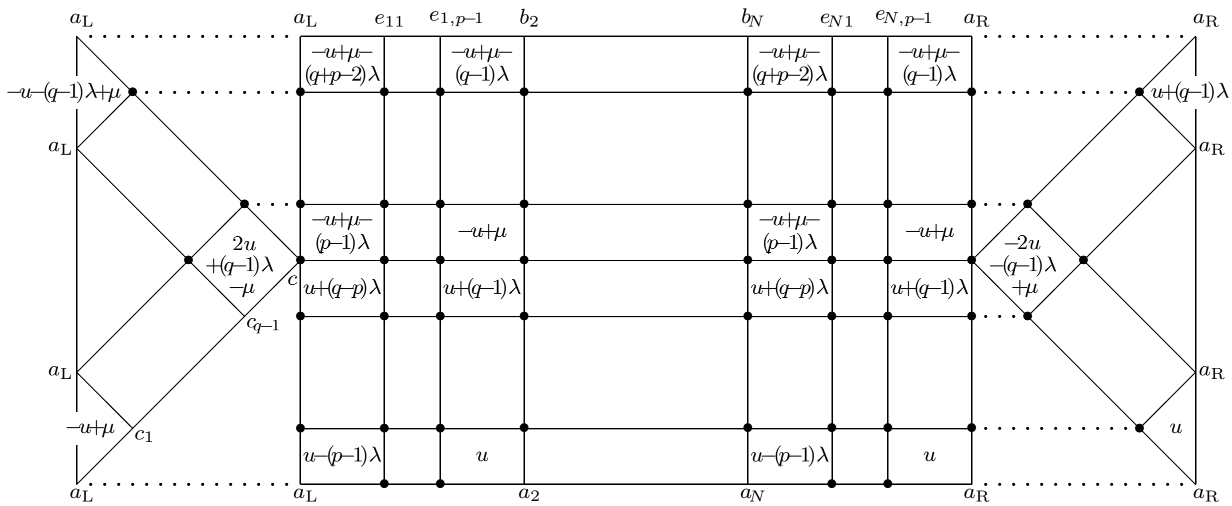
6.4 Double-Row Transfer Matrices

We now consider the ABF fused double-row transfer matrices $\boldsymbol{D}^{pq}(u)$, which are defined by (5.3), with λ given by (4.3), and the ABF fused double-row transfer matrices with fixed left and right boundary spins $a_{\rm L}$ and $a_{\rm R}$, $\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)$, given by

$$\langle a_2,\ldots,a_N\,|\,\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)\,|\,b_2,\ldots,b_N\rangle = \langle a_{\rm L},a_2,\ldots,a_N,a_{\rm R}\,|\,\boldsymbol{D}^{pq}(u)\,|\,a_{\rm L},b_2,\ldots,b_N,a_{\rm R}\rangle \quad (6.37)$$

By re-configuring the fused face weights in the top row according to (6.25), using (6.23) repeatedly to push all explicit entries of A^p to the lower edges, and all explicit entries of A^q to a single internal edge, setting as many internal arbitrary spins as possible equal to adjoining summed spins, and cancelling all of the gauge factors G which appear along the top row, we find that $\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)$ can be written as

$$\langle a_2,\ldots,a_N \,|\, \boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u) \,|\, b_2,\ldots,b_N\rangle \quad = \quad \frac{A^p_{a_{\rm L}a_2}\ldots A^p_{a_N a_{\rm R}}}{K^{pq}(u)} \;\sum_c A^q_{a_{\rm L}c}\,\times \tag{6.38}$$



where

$$K^{pq}(u)=\prod_{j=0}^{q-2}\Big(\theta^p_j(u)\;\theta^p_{-j-1}(-u+\mu)\Big)^N\;\theta^{j+1}_{2j+1}(2u-\mu)\;\theta^{j+1}_{-j-1}(-2u+\mu) \tag{6.39}$$

and we must have

$$(e_{11},\ldots,e_{1,p-1})\in P^p_{a_{\rm L}b_2},\;\ldots\;,(e_{N1},\ldots,e_{N,p-1})\in P^p_{b_N a_{\rm R}}$$

and, for each c in the sum,

$$(c_1,\ldots,c_{q-1})\in P^q_{a_{\rm L}c}$$

We note that the spins $c_1,\ldots,c_{q-1}$ can not be set equal to the adjoining summed spins, as they only become arbitrary *after* the summation has occurred.

Since the required fused local relations are satisfied, we have, from (5.12), commutativity

$$\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)\;\boldsymbol{D}^{pr}(a_{\rm L}a_{\rm R}|v)=\boldsymbol{D}^{pr}(a_{\rm L}a_{\rm R}|v)\;\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u) \tag{6.40}$$

and, from (5.11), crossing symmetry

$$\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)=\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|-u-(q-1)\lambda+\mu) \tag{6.41}$$

It follows, from (6.29) and (6.34), that the ABF fused double-row transfer matrices also satisfy partial height reversal symmetry

$$\alpha^{L-q}_{-2}(a_{\rm L}a_{\rm R}|u)\;\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)\;=\;(-1)^{pN}\;\beta^{q+1}_{q-1}(a_{\rm L}a_{\rm R}|u)\;\boldsymbol{D}^{p,L-1-q}(a_{\rm L}a_{\rm R}|u+(q+1)\lambda) \tag{6.42}$$

or, equivalently,

$$\beta^{L-q}_{-2}(a_{\rm L}a_{\rm R}|u)\;\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)\;=\;(-1)^{pN}\;\alpha^{q+1}_{q-1}(a_{\rm L}a_{\rm R}|u)\;\boldsymbol{D}^{p,L-1-q}(a_{\rm L}a_{\rm R}|u+(q+1)\lambda) \tag{6.43}$$

where

$$\alpha^r_k(a_{\rm L}a_{\rm R}|u) = \theta^r_k(u+\tfrac{\lambda-\mu}{2}-\xi_{\rm L}(a_{\rm L}))\ \theta^r_k(u+\tfrac{\lambda-\mu}{2}+\xi_{\rm L}(a_{\rm L})) \times \theta^r_k(u+\tfrac{\lambda-\mu}{2}-\xi_{\rm R}(a_{\rm R}))\ \theta^r_k(u+\tfrac{\lambda-\mu}{2}+\xi_{\rm R}(a_{\rm R})) \tag{6.44}$$

$$\beta^r_k(a_{\rm L}a_{\rm R}|u) = \theta^r_{k-a_{\rm L}}(u+\tfrac{\lambda-\mu}{2}-\xi_{\rm L}(a_{\rm L}))\ \theta^r_{k+a_{\rm L}}(u+\tfrac{\lambda-\mu}{2}+\xi_{\rm L}(a_{\rm L})) \times \theta^r_{k-a_{\rm R}}(u+\tfrac{\lambda-\mu}{2}-\xi_{\rm R}(a_{\rm R}))\ \theta^r_{k+a_{\rm R}}(u+\tfrac{\lambda-\mu}{2}+\xi_{\rm R}(a_{\rm R})) \tag{6.45}$$

Considering $q=-1$, 0, $L-1$ and L in (6.42) or (6.43), we have, using (6.15) and (6.16),

$$\boldsymbol{D}^{p,-1}(a_{\rm L}a_{\rm R}|u) = \boldsymbol{0} \tag{6.46}$$

$$\boldsymbol{D}^{p,L}(a_{\rm L}a_{\rm R}|u) = \boldsymbol{0} \tag{6.47}$$

$$\boldsymbol{D}^{p,0}(a_{\rm L}a_{\rm R}|u) = f^p_{-1}(u)\ \boldsymbol{I}^p(a_{\rm L}a_{\rm R}) \tag{6.48}$$

$$\boldsymbol{D}^{p,L-1}(a_{\rm L}a_{\rm R}|u) = (-1)^{pN}\,\frac{\alpha^L_{L-2}(a_{\rm L}a_{\rm R}|u)}{\beta^1_{-2}(a_{\rm L}a_{\rm R}|u)}\ f^p_{-2}(u)\ \boldsymbol{I}^p(a_{\rm L}a_{\rm R}) = (-1)^{pN}\,\frac{\beta^L_{L-2}(a_{\rm L}a_{\rm R}|u)}{\alpha^1_{-2}(a_{\rm L}a_{\rm R}|u)}\ f^p_{-2}(u)\ \boldsymbol{I}^p(a_{\rm L}a_{\rm R}) \tag{6.49}$$

where

$$f^p_k(u) = \Big(\theta^p_k(u)\ \theta^p_{-k-1}(-u+\mu)\Big)^N \tag{6.50}$$

and $\boldsymbol{I}^p(a_{\rm L}a_{\rm R})$ is the adjacency-inclusive identity

$$\langle a_2,a_3,\ldots,a_{N-1},a_N\,|\,\boldsymbol{I}^p(a_{\rm L}a_{\rm R})\,|\,b_2,b_3,\ldots,b_{N-1},b_N\rangle = \delta_{a_2b_2}\ldots\delta_{a_Nb_N}\ A^p_{a_{\rm L}a_2}\,A^p_{a_2a_3}\ldots A^p_{a_{N-1}a_N}\,A^p_{a_Na_{\rm R}} \tag{6.51}$$

It follows from (6.30) and (6.35), with (4.14), that the ABF fused double-row transfer matrices satisfy full height reversal symmetry

$$\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u) = \boldsymbol{Y}\ \boldsymbol{D}^{pq}(L+1-a_{\rm L},L+1-a_{\rm R}|u)\,\boldsymbol{Y} \tag{6.52}$$

where

$$\langle a_2,\ldots,a_N\,|\,\boldsymbol{Y}\,|\,b_2,\ldots,b_N\rangle = \delta_{L+1-a_2\,,\,b_2}\ \ldots\ \delta_{L+1-a_N\,,\,b_N} \tag{6.53}$$

For the ABF fused models, a natural choice for μ in $\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)$ is

$$\mu = p\lambda \tag{6.54}$$

We note that p-dependence, as opposed to q-dependence, of μ in $\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)$ does not destroy commutativity or crossing symmetry. With this choice, crossing symmetry of the fused face weights, (6.28), implies that $\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)$ is similar to its transpose

$$\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u) = \boldsymbol{S}^p(a_{\rm L}a_{\rm R})^{-1}\ \boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)^t\ \boldsymbol{S}^p(a_{\rm L}a_{\rm R}) \tag{6.55}$$

where

$$\langle a_2, a_3, \ldots, a_{N-1}, a_N \,|\, \boldsymbol{S}^p(a_{\rm L} a_{\rm R}) \,|\, b_2, b_3, \ldots, b_{N-1}, b_N \rangle \quad = \quad \delta_{a_2 b_2} \ldots \delta_{a_N b_N}\; X^p_{a_{\rm L} a_2}\, X^p_{a_2 a_3} \ldots X^p_{a_{N-1} a_N}\, X^p_{a_N a_{\rm R}}\, \frac{\theta(\lambda)^{N-1}}{\theta(a_2\lambda)\ldots\theta(a_N\lambda)} \tag{6.56}$$

6.5 Functional Equations

In Appendix C, we show that the ABF fused double-row transfer matrices satisfy functional equations whose structure reflects that of the fusion rule (6.7) satisfied by the adjacency matrices. There are two families of functional equations,

$$\begin{aligned} g^0_q(2u-\lambda)\; \boldsymbol{D}^{pq}(a_{\rm L} a_{\rm R}|u)\; \boldsymbol{D}^{p1}(a_{\rm L} a_{\rm R}|u-\lambda) \quad = \quad & \alpha^1_{-1}(a_{\rm L} a_{\rm R}|u)\; \beta^1_{-1}(a_{\rm L} a_{\rm R}|u)\; g^{-1}_q(2u-\lambda)\; f^p_{-2}(u)\; \boldsymbol{D}^{p,q-1}(a_{\rm L} a_{\rm R}|u+\lambda) \\ & + \; g^1_q(2u-\lambda)\; f^p_{-1}(u)\; \boldsymbol{D}^{p,q+1}(a_{\rm L} a_{\rm R}|u-\lambda) \end{aligned} \tag{6.57}$$

and

$$\begin{aligned} g^0_q(2u+q\lambda)\; \boldsymbol{D}^{pq}(a_{\rm L} a_{\rm R}|u)\; \boldsymbol{D}^{p1}(a_{\rm L} a_{\rm R}|u+q\lambda) \quad = \quad & \alpha^1_{q-1}(a_{\rm L} a_{\rm R}|u)\; \beta^1_{q-1}(a_{\rm L} a_{\rm R}|u)\; g^{-1}_q(2u+q\lambda)\; f^p_q(u)\; \boldsymbol{D}^{p,q-1}(a_{\rm L} a_{\rm R}|u) \\ & + \; g^1_q(2u+q\lambda)\; f^p_{q-1}(u)\; \boldsymbol{D}^{p,q+1}(a_{\rm L} a_{\rm R}|u) \end{aligned} \tag{6.58}$$

where

$$g^k_q(u) = \theta^1_{k-1}(u-\mu)\; \theta^1_{q-k}(u-\mu) \tag{6.59}$$

The importance of these equations is that they describe the essential content of the fusion hierarchy, since we see that either family, together with $\boldsymbol{D}^{p0}(a_{\rm L} a_{\rm R}|u)$ and $\boldsymbol{D}^{p1}(a_{\rm L} a_{\rm R}|u)$, can be used to determine recursively the higher fusion level double-row transfer matrices.

It can be shown, using induction as done in [20] for the periodic-boundary case, that (6.46), (6.48), and either (6.57) or (6.58), imply that the fused double-row transfer matrices also satisfy a generalised inversion identity which corresponds to (6.8),

$$\begin{aligned} \theta^1_{q-1}(2u-\mu)\; \theta^1_{q+1}(2u-\mu)\; \boldsymbol{D}^{pq}(a_{\rm L} a_{\rm R}|u)\; \boldsymbol{D}^{pq}(a_{\rm L} a_{\rm R}|u+\lambda) \quad = \quad & \alpha^q_{q-1}(a_{\rm L} a_{\rm R}|u)\; \beta^q_{q-1}(a_{\rm L} a_{\rm R}|u)\; \theta^1_{-1}(2u-\mu)\; \theta^1_{2q+1}(2u-\mu)\; f^p_{-1}(u)\; f^p_q(u)\; \boldsymbol{I}^p(a_{\rm L} a_{\rm R}) \\ & + \; \theta^1_q(2u-\mu)^2\; \boldsymbol{D}^{p,q-1}(a_{\rm L} a_{\rm R}|u+\lambda)\; \boldsymbol{D}^{p,q+1}(a_{\rm L} a_{\rm R}|u) \end{aligned} \tag{6.60}$$

Finally, if we define

$$\boldsymbol{d}^{pq}(a_{\rm L} a_{\rm R}|u) \quad = \quad \frac{\theta^1_q(2u-\mu)^2\; \boldsymbol{D}^{p,q-1}(a_{\rm L} a_{\rm R}|u+\lambda)\; \boldsymbol{D}^{p,q+1}(a_{\rm L} a_{\rm R}|u)}{\alpha^q_{q-1}(a_{\rm L} a_{\rm R}|u)\; \beta^q_{q-1}(a_{\rm L} a_{\rm R}|u)\; \theta^1_{-1}(2u-\mu)\; \theta^1_{2q+1}(2u-\mu)\; f^p_{-1}(u)\; f^p_q(u)} \tag{6.61}$$

then we obtain an identity which corresponds to (6.9),

$$\boldsymbol{d}^{pq}(a_{\rm L}a_{\rm R}|u)\;\boldsymbol{d}^{pq}(a_{\rm L}a_{\rm R}|u{+}\lambda) \;=\; \Big(\boldsymbol{I}^p(a_{\rm L}a_{\rm R})+\boldsymbol{d}^{p,q-1}(a_{\rm L}a_{\rm R}|u{+}\lambda)\Big) \times \Big(\boldsymbol{I}^p(a_{\rm L}a_{\rm R})+\boldsymbol{d}^{p,q+1}(a_{\rm L}a_{\rm R}|u)\Big) \tag{6.62}$$

7. Discussion

We have presented a general formalism for applying fixed boundary conditions to IRF models and have specialised to the case of ABF models and their fusion hierarchy. In future work, we intend both to continue our study of ABF models and to proceed with the application of fixed boundary conditions to other IRF models.

With regard to the ABF models, we note that the functional equations (6.57) and (6.58) have the same structure as those satisfied by the ABF row transfer matrices with periodic boundary conditions [19]. We therefore plan to use the same approach as in [19], to obtain Bethe ansatz equations for the eigenspectra of the double-row transfer matrices. We also hope to calculate the boundary free energy of these models, and to use the technique of [20] to calculate analytically the central charges and conformal weights. Other directions in which our treatment of ABF models could be developed further would be to investigate the existence of boundary weights of a non-diagonal form, and to explore the connection between ABF boundary weights and known boundary weights for the related eight-vertex model.

With regard to other models, the ABF models correspond to the A series within the standard A–D–E models, and we plan to study the other members of this group. We also hope to consider the dilute A–D–E models. In these models the unfused adjacency matrices allow identical spins to be adjacent, which would enable us to consider the important case of fixed boundaries of the form $a, a, a, \ldots$, whereas for the level 1 ABF models only the form $a, a\pm 1, a, \ldots$ is possible.

Appendix A: Derivation of ABF Boundary Weights

In this appendix, we find boundary weights which, together with the ABF face weights (4.4), satisfy the reflection equations (3.6) and (3.7). We then show that these weights also satisfy the boundary crossing equations (3.8) and (3.9).

We observe from the spectral parameter dependence of the reflection equations, that it suffices to solve the equations for the case $\mu=\lambda$, since if we then replace u by $u{+}\frac{\lambda-\mu}{2}$ in the resulting boundary weights, this will give a solution for the case of arbitrary μ. Furthermore, we see from the spin dependence of the reflection equations that, since the ABF face weights satisfy the symmetry (4.8), the left and right equations are effectively the same, so that it suffices also to solve them together.

We now assume that there are solutions which have the diagonal form

$$B_{\rm L}\!\left(\begin{matrix}c\\a\end{matrix}\;b\,\middle|\,u\right)=B_{\rm R}\!\left(b\;\begin{matrix}c\\a\end{matrix}\,\middle|\,u\right)=B(a,b|u)\;\delta_{ac} \tag{A.1}$$

The reflection equations, with $\mu=\lambda$, then become

$$\sum_f W\begin{pmatrix} c & f \\ b & a \end{pmatrix}\!\bigg|\, u{-}v\bigg)\; W\begin{pmatrix} d & a \\ c & f \end{pmatrix}\!\bigg|\,\lambda{-}u{-}v\bigg)\; B(a,f|u)\; B(a,d|v) =$$

$$\sum_f W\begin{pmatrix} a & f \\ d & c \end{pmatrix}\!\bigg|\, u{-}v\bigg)\; W\begin{pmatrix} f & a \\ c & b \end{pmatrix}\!\bigg|\,\lambda{-}u{-}v\bigg)\; B(a,f|u)\; B(a,b|v) \qquad \text{(A.2)}$$

This equation is trivially satisfied if $A_{ab}\,A_{bc}\,A_{cd}\,A_{da}=0$. Proceeding to $A_{ab}\,A_{bc}\,A_{cd}\,A_{da}=1$, we see that if $b=d=a\pm1$ and $c=a$ or $c=a\pm2$, then both sides of (A.2) are automatically equal, since the ABF weights satisfy the symmetry (4.7). The only set of spin assignments remaining is $b=a\pm1$, $c=a$ and $d=a\mp1$, where $a=2,\dots,L{-}1$, which gives $L{-}2$ pairs of identical equations,

$$\begin{aligned} &\sqrt{\frac{\theta((a{+}1)\lambda)}{\theta((a{-}1)\lambda)}}\;\theta(u{-}v)\;\theta(u{+}v{-}a\lambda)\;B(a,a{-}1|u)\;B(a,a{-}1|v)\\ &\qquad -\;\theta(u{+}v)\;\theta(u{-}v{-}a\lambda)\;B(a,a{-}1|u)\;B(a,a{+}1|v)\\ &\qquad -\;\theta(u{+}v)\;\theta(u{-}v{+}a\lambda)\;B(a,a{+}1|u)\;B(a,a{-}1|v)\\ &+\;\sqrt{\frac{\theta((a{-}1)\lambda)}{\theta((a{+}1)\lambda)}}\;\theta(u{-}v)\;\theta(u{+}v{+}a\lambda)\;B(a,a{+}1|u)\;B(a,a{+}1|v)\;=\;0 \end{aligned} \qquad \text{(A.3)}$$

We note that the boundary weights $B(1,2|u)$ and $B(L,L{-}1|u)$ do not appear in any of these equations and can therefore be set to arbitrary functions, $g_1(u)$ and $g_L(u)$. Returning to $a=2,\dots,L{-}1$, we now assume that there are constants $\xi(a)$ for which $B(a,a{-}1|\xi(a))=0$ but $B(a,a{+}1|\xi(a))\neq0$. Taking $v=\xi(a)$ in (A.3), we find that solutions must have the form

$$\begin{aligned} B(a,a{-}1|u) &= \sqrt{\theta((a{-}1)\lambda)}\;\theta(u{-}\xi(a))\;\theta(u{+}a\lambda{+}\xi(a))\;g_a(u)\\ B(a,a{+}1|u) &= \sqrt{\theta((a{+}1)\lambda)}\;\theta(u{+}\xi(a))\;\theta(u{-}a\lambda{-}\xi(a))\;g_a(u) \end{aligned} \qquad \text{(A.4)}$$

for some functions g_a.

We now verify that these are in fact solutions for arbitrary constants $\xi(a)$ and arbitrary functions g_a. Substituting (A.4) into the left side of (A.3) gives

$$\sqrt{\theta((a{+}1)\lambda)\,\theta((a{-}1)\lambda)}\;g_a(u)\;g_a(v)\;\Big(Q_a(u,v)-Q_a(u,-v)-Q_a(-u,v)+Q_a(-u,-v)\Big)$$

where

$$Q_a(u,v)=\theta(u{-}v)\;\theta(u{+}v{-}a\lambda)\;\theta(u{-}\xi(a))\;\theta(u{+}a\lambda{+}\xi(a))\;\theta(v{-}\xi(a))\;\theta(v{+}a\lambda{+}\xi(a))$$

Using the identity (4.6), we now find that

$$Q_a(u,v)-Q_a(u,-v)=\theta(a\lambda)\;\theta(2v)\;\theta(u{-}\xi(a))\;\theta(u{+}\xi(a))\;\theta(u{-}a\lambda{-}\xi(a))\;\theta(u{+}a\lambda{+}\xi(a))$$

which we can see is even in u, thus implying that the left side of (A.3) vanishes as required. The boundary weights (A.4) obtained here match those of (4.13) once we replace u by $u{+}\frac{\lambda-\mu}{2}$, make appropriate choices for g_a and set $\xi\mapsto\xi_{\mathrm{L/R}}$.

Finally, we consider the boundary crossing equations, (3.8) and (3.9), with the ABF face weights, the boundary weights found here, and ρ given by (4.12). These equations are satisfied since if $\delta_{ac}\, A_{ab}=0$, then both sides of the equations are zero, if $a=c=1$ or$L$ and $b=2$ or $L-1$, then the left sides are single terms which we immediately find are equal to the terms on the right side, and if $2\leq a=c\leq L-1$ and $b=a\pm 1$, then the left sides are sums of two terms which we find can be reduced to the terms on the right side using a single application of (4.6).

Appendix B: ABF Fused Reflection and Boundary Crossing Equations

In this appendix, we show that the fused right reflection and boundary crossing equations, (5.7) and (5.9), are satisfied by the ABF fused weights. The proofs for the fused left reflection and boundary crossing equations, (5.6) and (5.8), are similar.

We begin with (5.7). If $q=-1$ or $r=-1$ then, due to (6.15), each side of (5.7) is zero, and if $q=0$ or $r=0$ then, using (6.15) and (6.36), we find that each side of (5.7) reduces to a product of the same terms.

We now proceed to the case $q\geq 1$ and $r\geq 1$. Having substituted the ABF fused weights, (6.20) and (6.22), and ρ given by (6.36), into (5.7), we then re-configure the central fused face weights on each side according to (6.25) and the upper fused face weight on the right side according to (6.27), set internal arbitrary spins equal to adjoining summed spins, use (6.23) to push all explicit occurrences of the fused adjacency condition to external edges, cancel internal gauge factors G, and take external arbitrary spins to be the same on each side of the equation. After these steps, we find that the left side of (5.7) is given by

$$\frac{\theta^q_{-1}(u-v+(q-r)\lambda)\;\theta^q_{-1}(-u-v-(r-1)\lambda+\mu)}{\prod_{j=0}^{q-2}\theta^r_j(u-v)\;\theta^r_j(-u-v-(q-1)\lambda+\mu)\;\theta^{j+1}_{2j+1}(2u-\mu)\;\prod_{j=0}^{r-2}\theta^{j+1}_{2j+1}(2v-\mu)}\times$$

$$\frac{\delta_{ae}\,A^r_{ab}\,A^q_{bc}}{G^q_{c,h_1,\ldots,h_{q-1},d}\;G^r_{d,i_1,\ldots,i_{r-1},a}}\sum_{f_1\ldots f_{r-1}}\sum_{g_1\cdots g_{q-1}}\mathcal{L}^{qr}(u,v)_{a,b,c,d,a,f_1,\ldots,f_{r-1},g_1,\ldots,g_{q-1},h_1,\ldots,h_{q-1},i_1,\ldots,i_{r-1}}$$

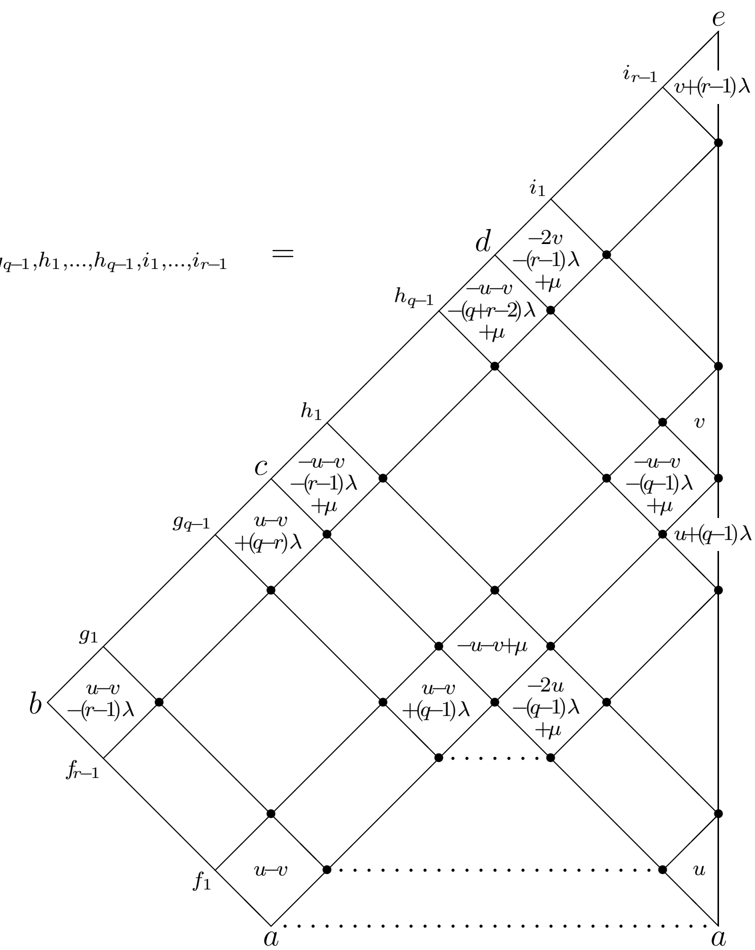
and that the right side of (5.7) is given by

$$\frac{\theta^r_{-1}(u-v)\;\theta^r_{-1}(-u-v-(q-1)\lambda+\mu)}{\prod_{j=0}^{r-2}\theta^q_j(u-v+(q-r)\lambda)\;\theta^q_j(-u-v-(r-1)\lambda+\mu)\;\theta^{j+1}_{2j+1}(2v-\mu)\;\prod_{j=0}^{q-2}\theta^{j+1}_{2j+1}(2u-\mu)}\times$$

$$\frac{\delta_{ae}\,A^r_{ab}\,A^q_{bc}}{G^q_{c,h_1,\ldots,h_{q-1},d}\;G^r_{d,i_1,\ldots,i_{r-1},a}}\sum_{f_1\ldots f_{r-1}}\sum_{g_1\cdots g_{q-1}}\mathcal{R}^{qr}(u,v)_{a,b,c,d,a,f_1,\ldots,f_{r-1},g_1,\ldots,g_{q-1},h_1,\ldots,h_{q-1},i_1,\ldots,i_{r-1}}$$

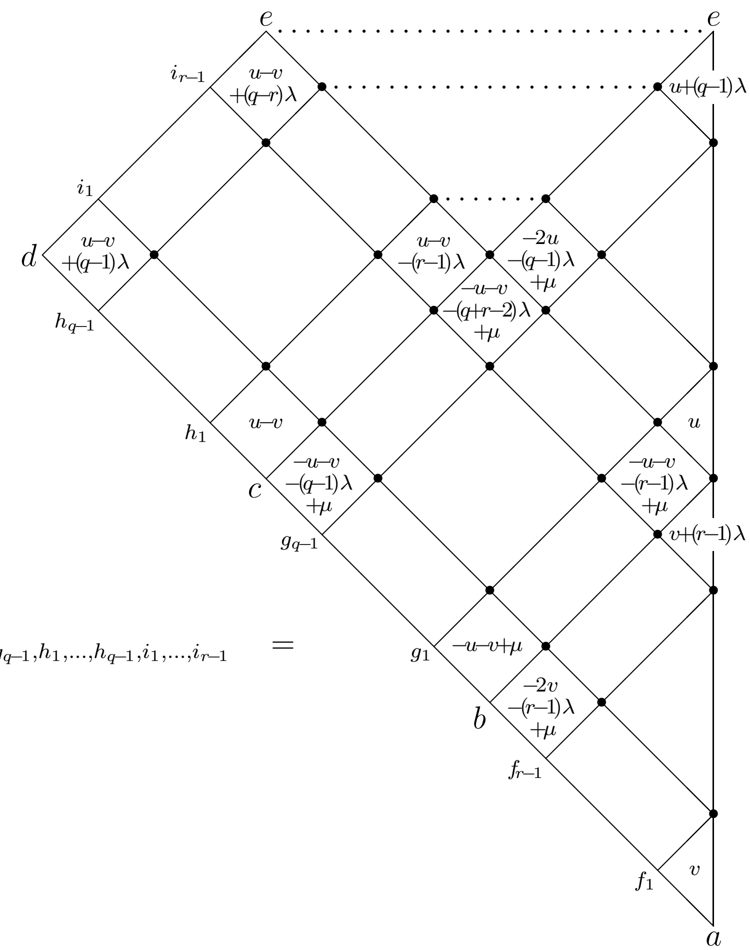
where we must have $(h_1,\ldots,h_{q-1})\in P^q_{cd}$ and $(i_1,\ldots,i_{r-1})\in P^r_{da}$, and where

$$\mathcal{L}^{qr}(u,v)_{a,b,c,d,e,f_1,\ldots,f_{r-1},g_1,\ldots,g_{q-1},h_1,\ldots,h_{q-1},i_1,\ldots,i_{r-1}} \quad =$$



and

$$\mathcal{R}^{qr}(u,v)_{a,b,c,d,e,f_1,\ldots,f_{r-1},g_1,\ldots,g_{q-1},h_1,\ldots,h_{q-1},i_1,\ldots,i_{r-1}} \quad =$$



We now claim that, for any model in which the original Yang Baxter equation, (3.4), and right reflection equation, (3.7), are satisfied, and for arbitrary λ, we in fact have

$$\mathcal{L}^{qr}(u,v)_{a,b,c,d,e,f_1,\ldots,f_{r-1},g_1,\ldots,g_{q-1},h_1,\ldots,h_{q-1},i_1,\ldots,i_{r-1}} = \mathcal{R}^{qr}(u,v)_{a,b,c,d,e,f_1,\ldots,f_{r-1},g_1,\ldots,g_{q-1},h_1,\ldots,h_{q-1},i_1,\ldots,i_{r-1}} \tag{B.1}$$

This can be proved by induction, which consists of showing that

$$\mathcal{L}^{1,1}(u,v) = \mathcal{R}^{1,1}(u,v)$$

that

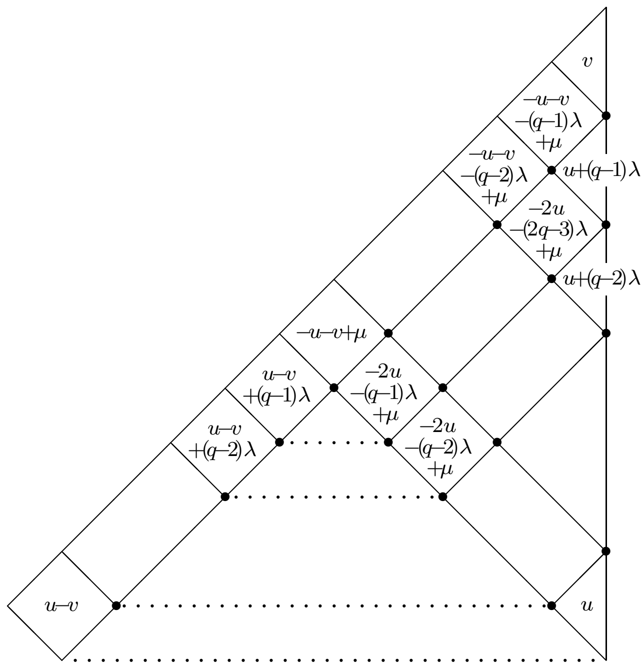
$$\mathcal{L}^{1,1}(u,v) = \mathcal{R}^{1,1}(u,v) \quad \text{and} \quad \mathcal{L}^{q-1,1}(u,v) = \mathcal{R}^{q-1,1}(u,v) \quad \text{imply that} \quad \mathcal{L}^{q,1}(u,v) = \mathcal{R}^{q,1}(u,v)$$

and, finally, that

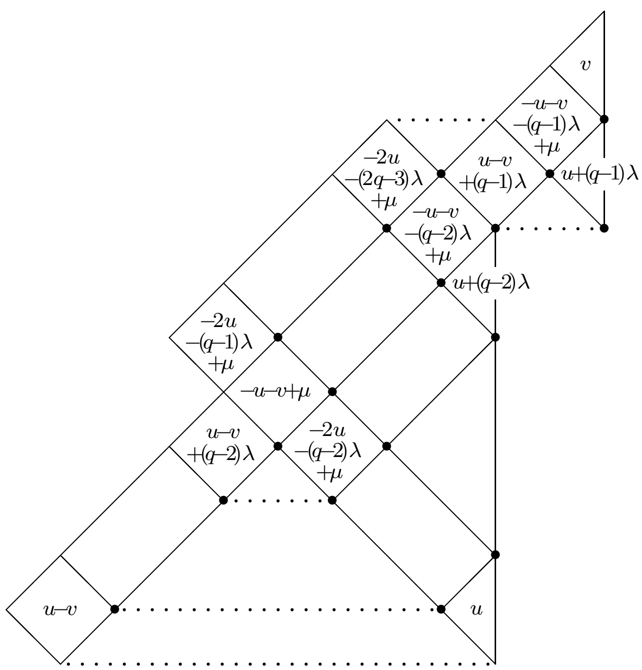
$$\mathcal{L}^{q,1}(u,v) = \mathcal{R}^{q,1}(u,v) \quad \text{and} \quad \mathcal{L}^{q,r-1}(u,v) = \mathcal{R}^{q,r-1}(u,v) \quad \text{imply that} \quad \mathcal{L}^{q,r}(u,v) = \mathcal{R}^{q,r}(u,v)$$

We know the first statement holds, since it is simply the original right reflection equation (3.7). We shall only explicitly demonstrate the second statement, since the third can be demonstrated similarly. We have, for $q \geq 2$,

$\mathcal{L}^{q,1}(u,v) \quad =$

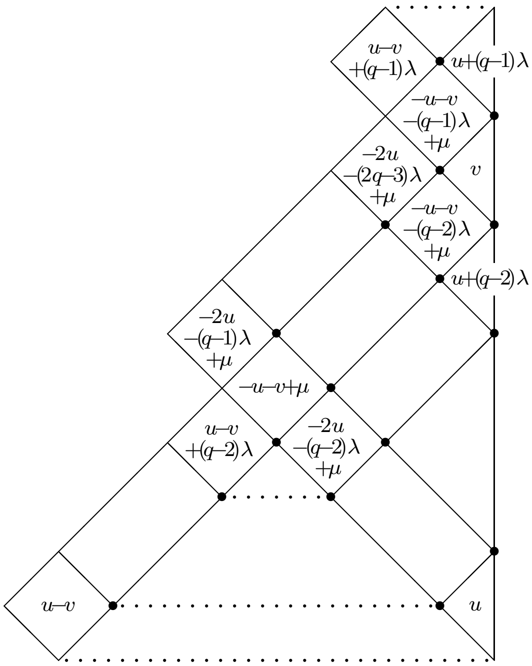


$=$



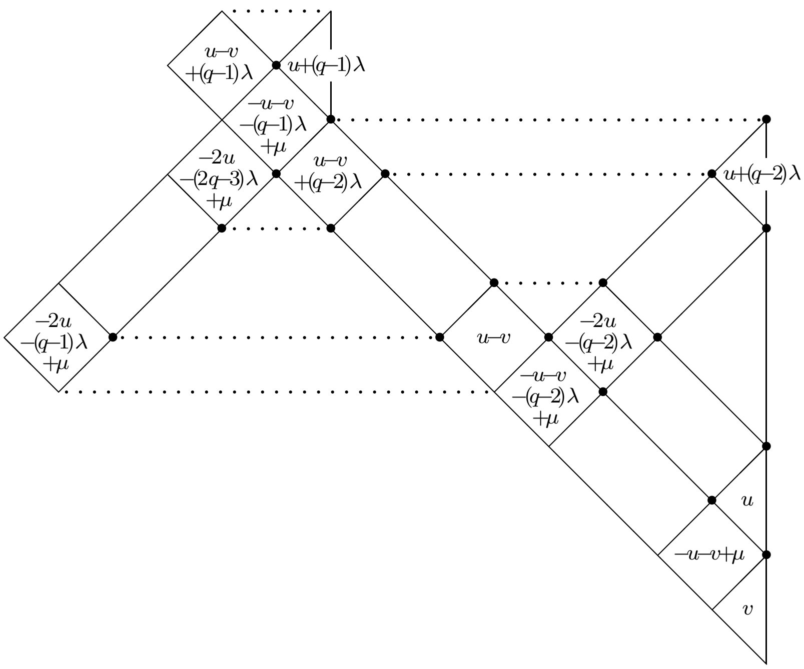
using the Yang-Baxter equation $q-1$ times

$=$

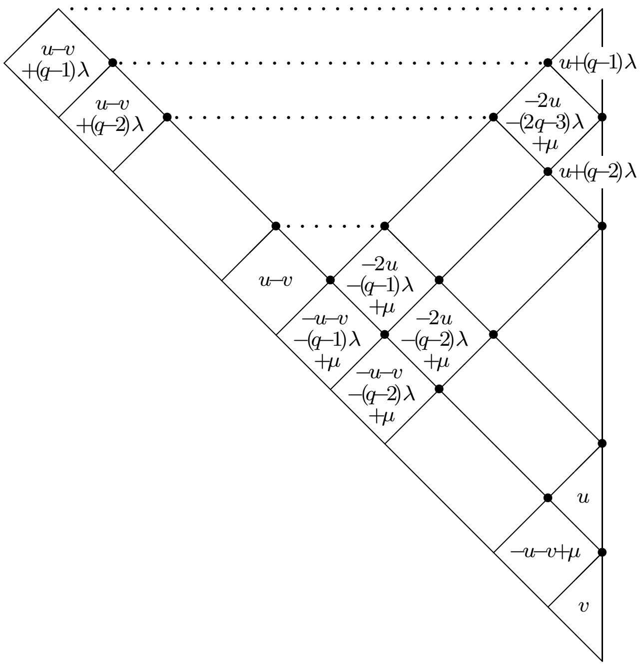


applying
$\mathcal{L}^{1,1}(u,v) =$
$\mathcal{R}^{1,1}(u,v)$

$=$



applying
$\mathcal{L}^{q-1,1}(u,v) =$
$\mathcal{R}^{q-1,1}(u,v)$



$=$ using the Yang-Baxter equation $q-1$ times

$$= \qquad \mathcal{R}^{q,1}(u,v)$$

Having established (B.1), it is straightforward to verify that the u, v dependent factors on each side of (5.7) are equal, which completes our proof that (5.7) is satisfied.

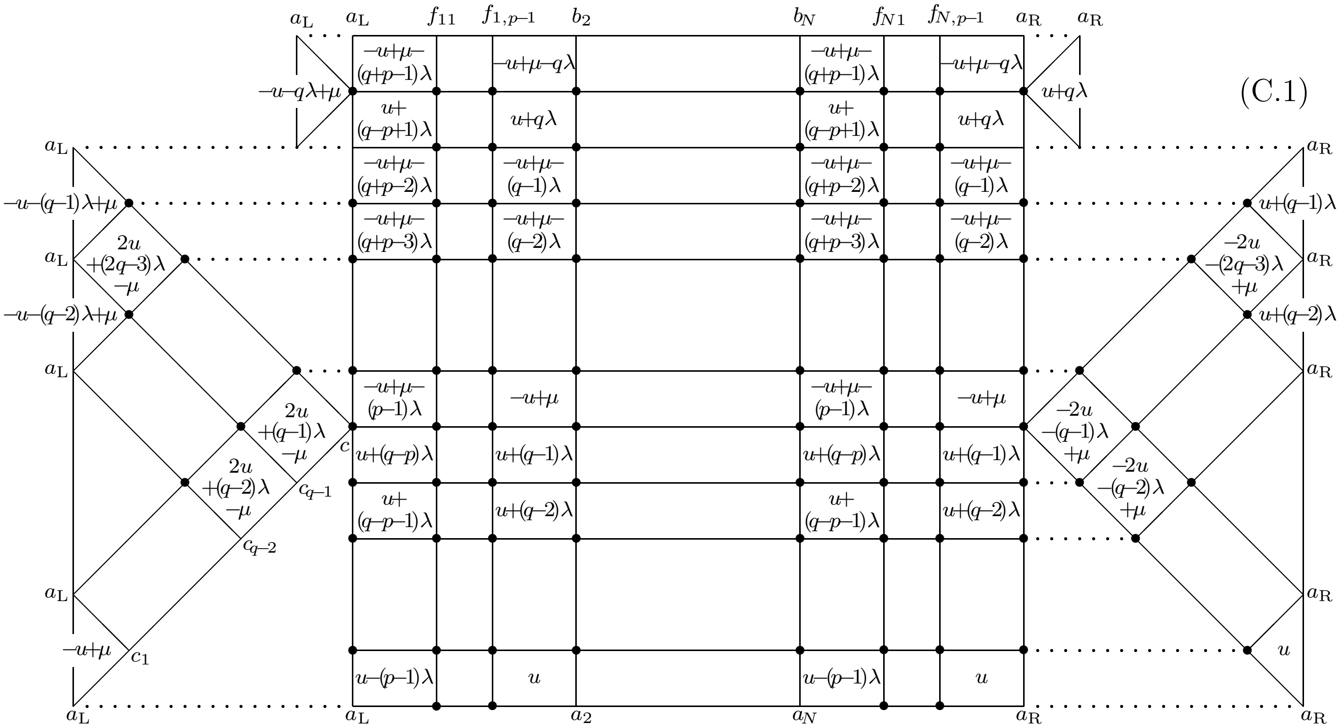
The proof that the fused right boundary crossing equation, (5.9), is also satisfied by the ABF weights corresponds closely to that for the fused right reflection equation. If $q=-1$ then each side of (5.9) is zero, and if $q=0$, then each side of (5.9) is given by $\delta_{ab}\,\delta_{bc}$. For $q\geq 1$, we re-configure the fused face weight on the left side of (5.9) according to (6.26), set internal arbitrary spins equal to adjoining summed spins, use (6.23) to push explicit occurrences of the fused adjacency condition to external edges, cancel internal gauge factors G, and take the external arbitrary spins to be the same on each side of the equation. After these steps, we find that each side of (5.7) is proportional to a sum of products of level 1 face and boundary weights, and that the proof can be completed by using induction on q to show that face and boundary weight components of each side are proportional, and then verifying that the overall proportionality factors on each side are the same. The induction argument here is valid for any model in which the original inversion relation, (3.5), and right boundary crossing equation, (3.9), are satisfied.

Appendix C: Proof of ABF Functional Equations

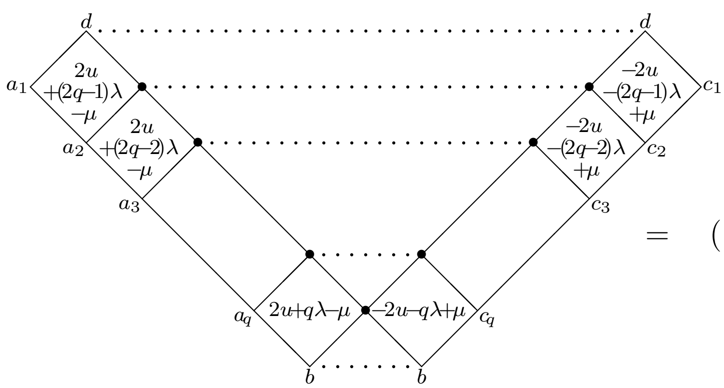
In this appendix, we prove that the ABF fused double-row transfer matrices satisfy the functional equations (6.58). The proof of (6.57) is similar.

We note that for $q=0$, (6.58) is immediately satisfied due to (6.46) and (6.48). We therefore proceed to the case $q\geq 1$ and begin by considering an entry of $\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)$ $\boldsymbol{D}^{p1}(a_{\rm L}a_{\rm R}|u+q\lambda)$. Using, (6.38), we find that

$$\langle a_2,\ldots,a_N \,|\, \boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)\ \boldsymbol{D}^{p1}(a_{\rm L}a_{\rm R}|u+q\lambda)\,|\, b_2,\ldots,b_N\rangle \;=\; \frac{A^p_{a_{\rm L}a_2}\ldots A^p_{a_N a_{\rm R}}}{K^{pq}(u)}\ \sum_c A^q_{a_{\rm L}c}\ \times \tag{C.1}$$



where we must have $(c_1,\ldots,c_{q-1})\in P^q_{a_{\rm L}c}$, for each c in the sum, and $(f_{11},\ldots,f_{1,p-1})\in P^p_{a_{\rm L}b_2}$, $\ldots$, $(f_{N1},\ldots,f_{N,p-1})\in P^p_{b_N a_{\rm R}}$. We now use the identity



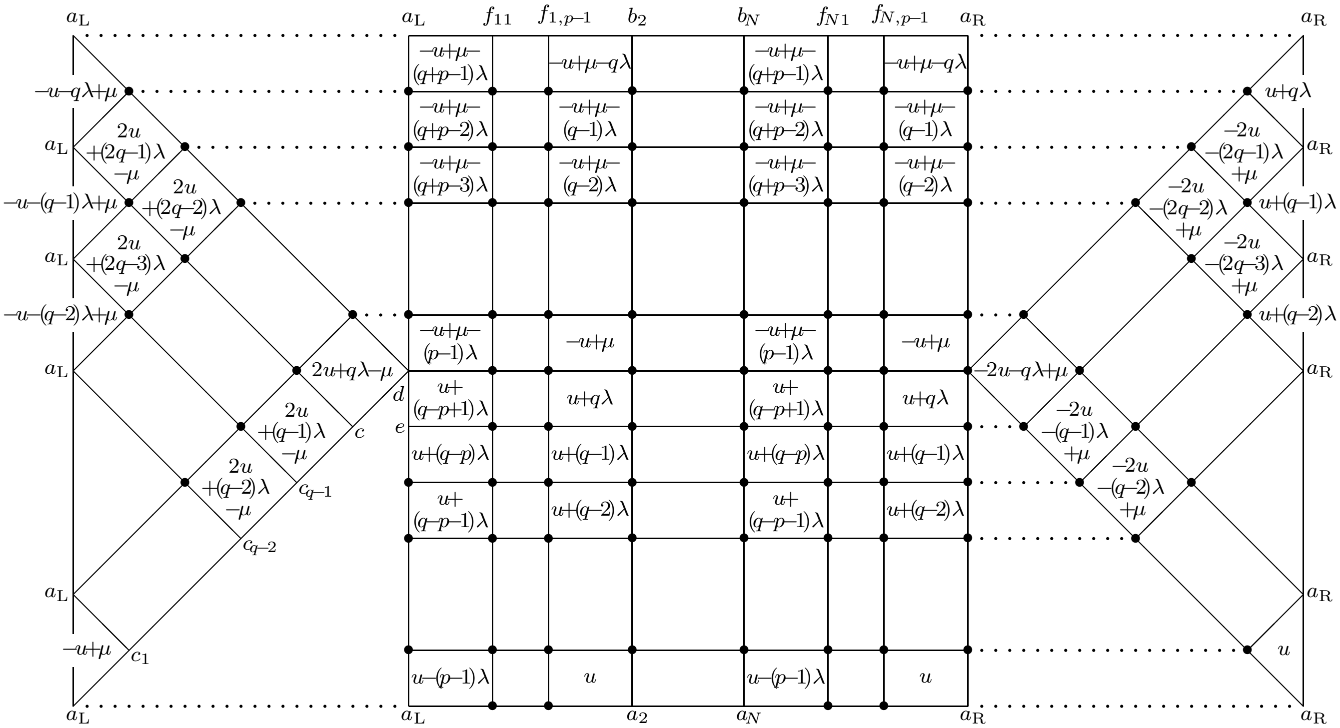
$$= \quad (-1)^q\,\theta^q_{2q-2}(2u-\mu)\,\theta^q_{2q}(2u-\mu) \times \delta_{a_1c_1}\ldots\delta_{a_qc_q}\ A_{da_1}\ldots A_{a_qb} \tag{C.2}$$

which can be obtained by using the inversion relation q times. By inserting (C.2) just below the top row of faces in (C.1), and then using the Yang-Baxter equation qpN times, we find that

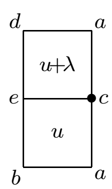
$$(-1)^q\,\theta^q_{2q-2}(2u-\mu)\,\theta^q_{2q}(2u-\mu)\,\langle a_2,\ldots,a_N \,|\, \boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)\ \boldsymbol{D}^{p1}(a_{\rm L}a_{\rm R}|u+q\lambda)\,|\, b_2,\ldots,b_N\rangle \;=$$
$$\frac{1}{K^{pq}(u)}\ \sum_d\ \sum_c\ A^q_{a_{\rm L}c}\ \mathcal{D}(a_{\rm L},c_1,\ldots,c_{q-2},c_{q-1},c,d,c)\Big|_{(c_1,\ldots,c_{q-1})\in P^q_{a_{\rm L}c}} \tag{C.3}$$

where we define

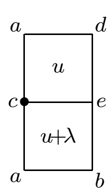
$$\mathcal{D}(a_{\rm L}, c_1, \ldots, c_{q-2}, c_{q-1}, c, d, e) \quad = \quad A^p_{a_{\rm L}a_2} \ldots A^p_{a_N a_{\rm R}} \times \qquad \text{(C.4)}$$



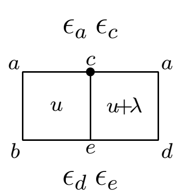
We note that the dependence of $\mathcal{D}$ on p, q, u and all of the external spins except $a_{\rm L}$ has been suppressed. The next step in the proof of (6.58), will be to decompose the sum over c, in (C.3), into antisymmetric and symmetric sums. However, in order to do so, we shall need several subsidiary results. We begin with the following local identities:

$$\sum_c \; \epsilon_b \, \epsilon_e$$ 

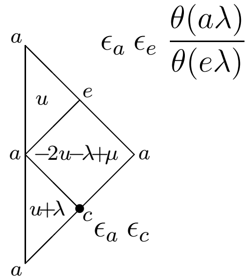
$$\epsilon_a \, \epsilon_c \quad = \quad \frac{\theta(a\lambda)}{\theta(b\lambda)} \, \frac{\theta(\lambda-u)\,\theta(\lambda+u)}{\theta(\lambda)^2} \, \delta_{bd} \, A_{ab} \, A_{be} \quad \text{(C.5a)}$$

$$\sum_c \; \epsilon_a \, \epsilon_c \, \frac{\theta(c\lambda)}{\theta(a\lambda)}$$ 

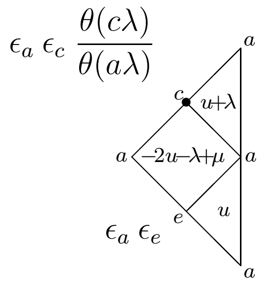
$$\epsilon_d \, \epsilon_e \, \frac{\theta(d\lambda)}{\theta(e\lambda)} \quad = \quad \frac{\theta(d\lambda)}{\theta(a\lambda)} \, \frac{\theta(\lambda-u)\,\theta(\lambda+u)}{\theta(\lambda)^2} \, \delta_{bd} \, A_{ab} \, A_{be} \quad \text{(C.5b)}$$

$$\sum_c$$ 

$$= \quad \frac{\theta(a\lambda)}{\theta(d\lambda)} \, \frac{\theta(\lambda-u)\,\theta(\lambda+u)}{\theta(\lambda)^2} \, \delta_{bd} \, A_{ab} \, A_{be} \quad \text{(C.5c)}$$

$\sum_c$ 

$$= \left(\frac{\theta(a\lambda)}{\theta(\lambda)}\right)^2 \, \frac{\theta(2u+3\lambda-\mu)}{\theta(\lambda)} \, \gamma_{\rm L}(a|u) \, A_{ae} \qquad \text{(C.5d)}$$

$\sum_c$ 

$$= \left(\frac{\theta(\lambda)}{\theta(a\lambda)}\right)^2 \, \frac{\theta(2u+3\lambda-\mu)}{\theta(\lambda)} \, \gamma_{\rm R}(a|u) \, A_{ae} \qquad \text{(C.5e)}$$

where

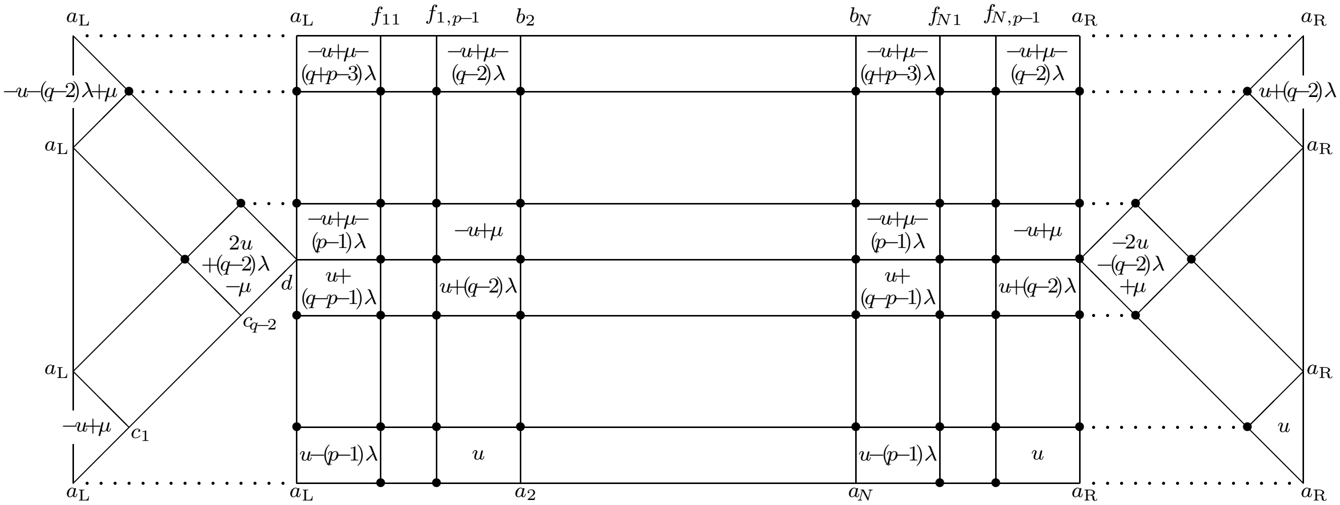
$$\gamma_{\rm L/R}(a|u) = \qquad \text{(C.6)}$$
$$\frac{\theta(u+\frac{\lambda-\mu}{2}-\xi_{\rm L/R}(a))\;\theta(u+\frac{\lambda-\mu}{2}+\xi_{\rm L/R}(a))\;\theta(u-a\lambda+\frac{\lambda-\mu}{2}-\xi_{\rm L/R}(a))\;\theta(u+a\lambda+\frac{\lambda-\mu}{2}+\xi_{\rm L/R}(a))}{\theta(\lambda)^4}$$

Identities (C.5a)– (C.5c) can each be proved as follows: if the external spins do not satisfy the adjacency conditions, then both sides of the equation are zero; if $b = d = a \pm 1$ and $e = a \pm 2$, or else $a = 1$ or L, then the left side is a single term, which we immediately find is equal to the term on the right side; if $b = a \pm 1$, $d = a \mp 1$ and $e = a$, then the left side is a sum of two terms, which immediately cancel, as required by the delta function on the right; finally, if $2 \le a \le L-1$, $b = d = a \pm 1$ and $e = a$, then the left side is a sum of two terms which can be reduced to the term on the right side using a single application of (4.6).

The proofs of (C.5d) and (C.5e) are similar: if a and e do not satisfy the adjacency condition, then both sides of the equation are zero; if $a = 1$ or L then the left side is a single term; and, if $2 \le a \le L-1$ and $e = a \pm 1$, then the left side is a sum of two terms which can be reduced to the term on the right side using (4.6).

We now use these identities repeatedly in (C.4). By starting at c and proceeding in a clockwise loop, using (C.5a) $q-1$ times, (C.5d) once, (C.5b) pN times, (C.5e) once, (C.5c) $q-1$ times, and (C.5a) pN times, we find that, for $q \ge 2$,

$$\epsilon_e \sum_c \epsilon_c\, \mathcal{D}(a_{\rm L}, c_1, \ldots, c_{q-2}, d, c, d, e) \quad = \quad A_{de}\, M^{pq}(a_{\rm L}a_{\rm R}|u)\, \times \tag{C.7a}$$



while, for $q = 1$,

$$\epsilon_e \sum_c \epsilon_c\, \mathcal{D}(a_{\rm L}, c, a_{\rm L}, e) = A_{a_{\rm L}e}\, M^{p1}(a_{\rm L}a_{\rm R}|u)\, \langle a_2, \ldots, a_N \,|\, \boldsymbol{I}^p(a_{\rm L}a_{\rm R}) \,|\, b_2, \ldots, b_N\rangle \tag{C.7b}$$

where

$$\begin{aligned} M^{pq}(a_{\rm L}a_{\rm R}|u) \quad &= \\ &\theta^{q-1}_{-1}(2u+(2q-3)\lambda-\mu)\; \theta^{q-1}_{1}(2u+(2q-3)\lambda-\mu)\; \theta^{1}_{0}(-2u-(2q-3)\lambda+\mu)\; \gamma_{\rm L}(a_{\rm L}|-u-q\lambda+\mu) \\ &\times \Big(\theta^{p}_{-1}(-u-q\lambda+\mu)\; \theta^{p}_{1}(-u-q\lambda+\mu)\Big)^N \\ &\times \theta^{1}_{0}(2u+(2q+1)\lambda-\mu)\; \gamma_{\rm R}(a_{\rm R}|u+(q-1)\lambda)\; \theta^{q-1}_{-1}(-2u-q\lambda+\mu)\; \theta^{q-1}_{1}(-2u-q\lambda+\mu) \\ &\times \Big(\theta^{p}_{-1}(u+(q-1)\lambda)\; \theta^{p}_{1}(u+(q-1)\lambda)\Big)^N \end{aligned} \tag{C.8}$$

We now assert the following properties of $\mathcal{D}$:

$$\begin{aligned} A^q_{a_{\rm L}e}\, \mathcal{D}(a_{\rm L}, c_1, \ldots, c_{q-2}, c_{q-1}, c, d, e) \quad &= \quad A^q_{a_{\rm L}e} A^q_{a_{\rm L}c}\, \mathcal{D}(a_{\rm L}, c_1, \ldots, c_{q-2}, c_{q-1}, c, d, e), \\ &\text{for } (c_1, \ldots, c_{q-1}) \in P^q_{a_{\rm L}c} \end{aligned} \tag{C.9a}$$

$$A^q_{a_{\rm L}e}\, \mathcal{D}(a_{\rm L}, c_1, \ldots, c_{q-2}, c_{q-1}, c, d, e) \text{ is independent of } (c_1, \ldots, c_{q-1}) \in P^q_{a_{\rm L}c} \tag{C.9b}$$

$$\epsilon_e \sum_c \epsilon_c\, \mathcal{D}(a_{\rm L}, c'_1, \ldots, c'_{q-2}, d, c, d, e) \text{ is independent of } e, \text{ for } A_{de} = 1 \tag{C.9c}$$

$$A^{q-1}_{a_{\rm L}d}\;\; \epsilon_e \sum_c \epsilon_c\, \mathcal{D}(a_{\rm L}, c'_1, \ldots, c'_{q-2}, d, c, d, e) \text{ is independent of } (c'_1, \ldots, c'_{q-2}) \in P^{q-1}_{a_{\rm L}d} \tag{C.9d}$$

$$A^{q+1}_{a_{\rm L}d} \sum_e \mathcal{D}(a_{\rm L}, c''_1, \ldots, c''_{q-2}, c''_{q-1}, c''_q, d, e) \text{ is independent of } (c''_1, \ldots, c''_q) \in P^{q+1}_{a_{\rm L}d} \tag{C.9e}$$

Properties (C.9a) and (C.9b) follow by considering $A^q_{a_\mathrm{L}e}\ \mathcal{D}(a_\mathrm{L},c_1,\ldots,c_{q-2},c_{q-1},c,d,e)$, with $(c_1,\ldots,c_{q-1})\in P^q_{a_\mathrm{L}c}$, as a linear combination of terms of the form

$$
\begin{aligned}
&W^{pq}\!\left(\begin{matrix} e & g_2\\ a_\mathrm{L} & a_2\end{matrix}\,\middle|\,u\right)\ \ldots\ W^{pq}\!\left(\begin{matrix} g_N & g_{N+1}\\ a_N & a_\mathrm{R}\end{matrix}\,\middle|\,u\right)\ B^q_\mathrm{R}\!\left(g_{N+1}\ \begin{matrix} a_\mathrm{R}\\ a_\mathrm{R}\end{matrix}\,\middle|\,u\right)\\
&\times\ W^{1q}\!\left(\begin{matrix} i_{N+1} & a_\mathrm{R}\\ h_{N+1} & g_{N+1}\end{matrix}\,\middle|\,-2u-(2q-1)\lambda+\mu\right)\\
&\times\ W^{pq}\!\left(\begin{matrix} i_N & i_{N+1}\\ h_N & h_{N+1}\end{matrix}\,\middle|\,-u-(q-1)\lambda+\mu\right)\ \ldots\ W^{pq}\!\left(\begin{matrix} i_1 & i_2\\ d & h_2\end{matrix}\,\middle|\,-u-(q-1)\lambda+\mu\right)\\
&\times\ W^{q1}\!\left(\begin{matrix} i_1 & d\\ a_\mathrm{L} & c\end{matrix}\,\middle|\,2u+(2q-1)\lambda-\mu\right)\ B^q_\mathrm{L}\!\left(\begin{matrix} a_\mathrm{L}\\ a_\mathrm{L}\end{matrix}\ c\,\middle|\,-u-(q-1)\lambda+\mu\right)
\end{aligned}
$$

Property (C.9c) follows immediately from (C.7a) and (C.7b), while property (C.9d) follows from (C.7a) by considering $A^{q-1}_{a_\mathrm{L}d}\ \ \epsilon_e\ \sum_c\ \epsilon_c\ \mathcal{D}(a_\mathrm{L},c'_1,\ldots,c'_{q-2},d,c,d,e)$, with $A_{de}=1$ and $(c'_1,\ldots,c'_{q-2})\in P^{q-1}_{a_\mathrm{L}d}$, as proportional to a sum of terms of the form

$$
\begin{aligned}
&W^{p,q-1}\!\left(\begin{matrix} g_1 & g_2\\ a_\mathrm{L} & a_2\end{matrix}\,\middle|\,u\right)\ \ldots\ W^{p,q-1}\!\left(\begin{matrix} g_N & g_{N+1}\\ a_N & a_\mathrm{R}\end{matrix}\,\middle|\,u\right)\ B^{q-1}_\mathrm{R}\!\left(g_{N+1}\ \begin{matrix} a_\mathrm{R}\\ a_\mathrm{R}\end{matrix}\,\middle|\,u\right)\\
&\times\ W^{p,q-1}\!\left(\begin{matrix} b_N & a_\mathrm{R}\\ g_N & g_{N+1}\end{matrix}\,\middle|\,-u-(q-2)\lambda+\mu\right)\ \ldots\ W^{p,q-1}\!\left(\begin{matrix} a_\mathrm{L} & b_2\\ g_1 & g_2\end{matrix}\,\middle|\,-u-(q-2)\lambda+\mu\right)\\
&\times\ B^{q-1}_\mathrm{L}\!\left(\begin{matrix} a_\mathrm{L}\\ a_\mathrm{L}\end{matrix}\ g_1\,\middle|\,-u-(q-2)\lambda+\mu\right)
\end{aligned}
$$

Finally, property (C.9e) follows by considering $A^{q+1}_{a_\mathrm{L}d}\ \sum_e\ \mathcal{D}(a_\mathrm{L},c''_1,\ldots,c''_{q-2},c''_{q-1},c''_q,d,e)$, with $(c''_1,\ldots,c''_q)\in P^{q+1}_{a_\mathrm{L}d}$, as a sum of terms of the form

$$
\begin{aligned}
&W^{p,q+1}\!\left(\begin{matrix} g_1 & g_2\\ a_\mathrm{L} & a_2\end{matrix}\,\middle|\,u\right)\ \ldots\ W^{p,q+1}\!\left(\begin{matrix} g_N & g_{N+1}\\ a_N & a_\mathrm{R}\end{matrix}\,\middle|\,u\right)\ B^{q+1}_\mathrm{R}\!\left(g_{N+1}\ \begin{matrix} a_\mathrm{R}\\ a_\mathrm{R}\end{matrix}\,\middle|\,u\right)\\
&\times\ W^{p,q+1}\!\left(\begin{matrix} b_N & a_\mathrm{R}\\ g_N & g_{N+1}\end{matrix}\,\middle|\,-u-q\lambda+\mu\right)\ \ldots\ W^{p,q+1}\!\left(\begin{matrix} a_\mathrm{L} & b_2\\ g_1 & g_2\end{matrix}\,\middle|\,-u-q\lambda+\mu\right)\\
&\times\ B^{q+1}_\mathrm{L}\!\left(\begin{matrix} a_\mathrm{L}\\ a_\mathrm{L}\end{matrix}\ g_1\,\middle|\,-u-q\lambda+\mu\right)
\end{aligned}
$$

We now return to the sum over c in (C.3), which we claim can be decomposed into antisymmetric and symmetric sums,

$$
\begin{aligned}
&\sum_c A^q_{a_\mathrm{L}c}\ \mathcal{D}(a_\mathrm{L},c_1,\ldots,c_{q-2},c_{q-1},c,d,c)\quad = \qquad\qquad \text{(C.10)}\\
&\qquad A^{q-1}_{a_\mathrm{L}d}\ \ \epsilon_e\ \sum_c\ \epsilon_c\ \mathcal{D}(a_\mathrm{L},c'_1,\ldots,c'_{q-2},d,c,d,e)\quad+\quad A^{q+1}_{a_\mathrm{L}d}\ \sum_e\ \mathcal{D}(a_\mathrm{L},c''_1,\ldots,c''_{q-2},c''_{q-1},c''_q,d,e)
\end{aligned}
$$

In this decomposition we assume the following: that $(c_1,\ldots,c_{q-1}) \in P^q_{a_{\rm L}c}$ for each c in the sum of the left side; that e satisfies $A_{de}=1$ and that $(c'_1,\ldots,c'_{q-2}) \in P^{q-1}_{a_{\rm L}d}$ in the antisymmetric sum; and that $(c''_1,\ldots,c''_q) \in P^{q+1}_{a_{\rm L}d}$ in the symmetric sum. Therefore, due to (C.9b)–(C.9e), all of these spins are arbitrary.

We now proceed to prove (C.10). We begin by constructing the following table of values of the adjacency matrix entries which appear in (C.10) (as well as in the fusion rule (6.7)):

	$d{-}a\notin\{-q{-}1,-q{+}1,-q{+}3,\ldots,q{-}3,q{-}1,q{+}1\}$			$d{-}a=-q{-}1$		$d{-}a=q{+}1$		$d{-}a\in\{-q{+}1,-q{+}3,\ldots,q{-}3,q{-}1\}$								
								$d{+}a\le q{-}1$		$d{+}a=q{+}1$		$q{+}3\le d{+}a\le 2L{-}q{-}1$	$d{+}a=2L{-}q{+}1$		$d{+}a\ge 2L{-}q{+}3$	
	$d=1$	$2\le d\le L{-}1$	$d=L$	$d=1$	$2\le d\le L{-}1$	$2\le d\le L{-}1$	$d=L$	$d=1$	$2\le d\le L{-}1$	$d=1$	$2\le d\le L{-}1$	$2\le d\le L{-}1$	$2\le d\le L{-}1$	$d=L$	$2\le d\le L{-}1$	$d=L$
$A^q_{a,d-1}$	–	0	0	–	0	1	1	–	0	–	0	1	1	1	0	0
$A^q_{a,d+1}$	0	0	–	1	1	0	–	0	0	1	1	1	0	–	0	–
$A^{q-1}_{a,d}$	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0
$A^{q+1}_{a,d}$	0	0	0	1	1	1	1	0	0	0	0	1	0	0	0	0

The entries in this table can all be obtained directly from the fused adjacency conditions, (6.1) and (6.2). We now denote the left side of (C.10) by $\mathcal{L}$ and the right side of (C.10) by $\mathcal{R}$, and consider cases corresponding to those listed in the table.

(I) $d-a_{\rm L} \notin \{-q-1,-q+1,\ldots,q-1,q+1\}$, $d+a_{\rm L}\le q-1$ or $d+a_{\rm L}\ge 2L-q+3$

In these cases, $\mathcal{L}$ and $\mathcal{R}$ are each zero.

(II) $d-a_{\rm L} = \pm(q+1)$

In these cases, $\mathcal{L}$ and $\mathcal{R}$ are each given by the single term

$$\mathcal{L}=\mathcal{R}=\mathcal{D}(a_{\rm L},a_{\rm L}\pm1,\ldots,a_{\rm L}\pm q,a_{\rm L}\pm(q+1),a_{\rm L}\pm q)$$

(III) $d-a_{\rm L}\in\{-q+1,\ldots,q-1\}$

In these cases, we can satisfy $(c_1,\ldots,c_{q-1})\in P^q_{a_{\rm L}c}$, for each c in the sum in $\mathcal{L}$, by taking $c_{q-1}=d$ and $(c_1,\ldots,c_{q-2})\in P^{q-1}_{a_{\rm L},d}$. We use this choice in each of the following subcases:

(i) $d+a_{\rm L}=q+1$

In this case, $\mathcal{L}$ is comprised of the single term

$$\mathcal{L}=\mathcal{D}(a_{\rm L},c_1,\ldots,c_{q-2},d,d+1,d,d+1)$$

Meanwhile, $\mathcal{R}$ is comprised of the antisymmetric sum only, for which we choose $e=d+1$. If $d=1$, we have a single term, which immediately matches $\mathcal{L}$. For $d\ge2$, we have

$$\mathcal{R}=\mathcal{D}(a_{\rm L},c'_1,\ldots,c'_{q-2},d,d+1,d,d+1)-\mathcal{D}(a_{\rm L},c'_1,\ldots,c'_{q-2},d,d-1,d,d+1)$$

but, by taking $c=d-1$ and $e=d+1$ in (C.9a), we find that the second of these terms vanishes and, therefore, we again have a single term which matches $\mathcal{L}$.

(ii) $d+a_{\rm L}=2L-q+1$

This case is similar to the previous one, with $\mathcal{L}$ now comprised of the single term

$$\mathcal{L}=\mathcal{D}(a_{\rm L},c_1,\ldots,c_{q-2},d,d-1,d,d-1)$$

Again, $\mathcal{R}$ is comprised of the antisymmetric sum only, for which we now choose $e=d-1$. If $d=L$ we immediately have a term which matches $\mathcal{L}$, while for $d\le L-1$, we have

$$\mathcal{R}=\mathcal{D}(a_{\rm L},c'_1,\ldots,c'_{q-2},d,d-1,d,d-1)-\mathcal{D}(a_{\rm L},c'_1,\ldots,c'_{q-2},d,d+1,d,d-1)$$

but, as before, we find that the second of these terms vanishes by taking $c=d+1$ and $e=d-1$ in (C.9a).

(iii) $q+3\le d+a_{\rm L}\le 2L-q-1$

In this case, we have

$$\mathcal{L}=\mathcal{D}(a_{\rm L},c_1,\ldots,c_{q-2},d,d-1,d,d-1)+\mathcal{D}(a_{\rm L},c_1,\ldots,c_{q-2},d,d+1,d,d+1)$$

For $\mathcal{R}$, we choose, in the antisymmetric sum, $e=d\pm1$, and, in the symmetric sum, $c''_q=d\mp1$, $c''_{q-1}=d$, and $(c''_1,\ldots,c''_{q-2})\in P^{q-1}_{a_{\rm L}d}$, giving

$$\begin{aligned}\mathcal{R}\ &=\ \mathcal{D}(a_{\rm L},c'_1,\ldots,c'_{q-2},d,d\pm1,d,d\pm1)-\mathcal{D}(a_{\rm L},c'_1,\ldots,c'_{q-2},d,d\mp1,d,d\pm1)\\ &\quad+\mathcal{D}(a_{\rm L},c''_1,\ldots,c''_{q-2},d,d\mp1,d,d\pm1)+\mathcal{D}(a_{\rm L},c''_1,\ldots,c''_{q-2},d,d\mp1,d,d\mp1)\end{aligned}$$

We see that the two middle terms of $\mathcal{R}$ cancel, while the two outer terms match those of $\mathcal{L}$. This completes our proof of (C.10)

We now substitute (C.10) and (C.7a) or (C.7b) into (C.3), and use (C.4), (6.38) and (6.39) to give

$$\begin{aligned}&(-1)^q\,\theta^q_{2q-2}(2u-\mu)\,\theta^q_{2q}(2u-\mu)\,\langle a_2,\ldots,a_N\,|\,\boldsymbol{D}^{pq}(a_{\rm L}a_{\rm R}|u)\,\boldsymbol{D}^{p1}(a_{\rm L}a_{\rm R}|u+q\lambda)\,|\,b_2,\ldots,b_N\rangle\ = \qquad \text{(C.11)}\\ &\frac{M^{pq}(a_{\rm L}a_{\rm R}|u)}{\Big(\theta^p_{q-2}(u)\,\theta^p_{-q+1}(-u+\mu)\Big)^N\,\theta^{q-1}_{2q-3}(2u-\mu)\,\theta^{q-1}_{-q+1}(-2u+\mu)}\ \langle a_2,\ldots,a_N\,|\,\boldsymbol{D}^{p,q-1}(a_{\rm L}a_{\rm R}|u)\,|\,b_2,\ldots,b_N\rangle\\ &+\Big(\theta^p_{q-1}(u)\,\theta^p_{-q}(-u+\mu)\Big)^N\,\theta^q_{2q-1}(2u-\mu)\,\theta^q_{-q}(-2u+\mu)\ \langle a_2,\ldots,a_N\,|\,\boldsymbol{D}^{p,q+1}(a_{\rm L}a_{\rm R}|u)\,|\,b_2,\ldots,b_N\rangle\end{aligned}$$

By using (C.8) and then cancelling the common factor $(-1)^q\,\theta^{q-1}_{2q-2}(2u-\mu)\,\theta^{q-1}_{2q-1}(2u-\mu)$ from each side of (C.11), it is relatively straightforward to show that the coefficients of each term in (C.11) match those in (6.58), which completes our proof of (6.58).

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