Reflection Equations and Exactly Solvable Lattice Spin Models

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Abstract

Reflection equations are used to obtain families of commuting double-row transfer matrices for interaction-round-a-face (IRF) models with fixed and free boundary conditions. We illustrate our methods for the Andrews-Baxter-Forrester (ABF) models which are L-state models associated with the quantum group $U_q(su(2))$ at a root of unity. We construct elliptic solutions to the reflection equations for the ABF models by a procedure which uses fusion to build the solutions starting from a trivial solution.

1 . Introduction

It is well known [1] that two-dimensional lattice spin models are exactly solvable if their Boltzmann weights satisfy the celebrated Yang-Baxter equations. In this paper we review some recent work [2, 3] extending integrability based on Yang-Baxter equations to face or Interaction-Round-a-Face (IRF) models with a boundary. This work was motivated by and builds on work by Sklyanin [4] and others on related vertex models. Although our methods are quite general we illustrate the methods for the L-state Restricted Solid-on-Solid models (RSOS) of Andrews, Baxter and Forrester [5] (ABF). At criticality, these models are related to the quantum group $U_q(su(2))$ with q a root of unity. In particular, the ABF models include the Ising model ($L = 3$) and the solvable hard square and hexagon models ($L = 4$) as special cases. It is our intention here to give a brief overview introducing the main ideas of this topic. For a full treatment of the material and more comprehensive references the reader is referred to the original articles. Some other recent relevant articles are listed in the bibliography [6–15].

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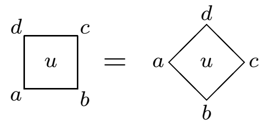
2. IRF Models with a Boundary

2.1 Boltzmann Weights and Transfer Matrices

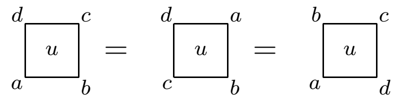
Consider an IRF model with Boltzmann face weights

$$W\begin{pmatrix} d & c \\ a & b \end{pmatrix}\!\!\bigg|\, u \bigg) = \tag{2.1}$$



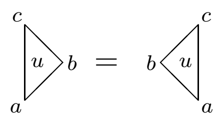
Here, the spins a, b, c, d take values from a discrete set and the face weights depend on a spectral parameter u. Graphically we represent the face in one of these two standard orientations. Suppose also that the face weights satisfy the symmetries

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = \tag{2.2}$$



To allow for a boundary, we introduce boundary weights which depend on three spins

$$B\left(\begin{matrix} c & \\ & b \\ a & \end{matrix}\,\middle|\, u\right) = B\left(\begin{matrix} & c \\ b & \\ & a \end{matrix}\,\middle|\, u\right) = \tag{2.3}$$



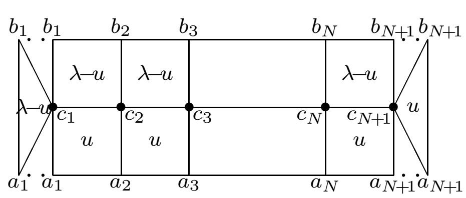
In general there will be an adjacency condition associated with the IRF model as specified by a symmetric adjacency matrix A whose entries are either 0 or 1. We assume that the face and boundary weights vanish unless the adjacency condition is satisfied for each nearest neighbour pair along an edge. For this purpose a and c in the boundary weights are considered as second neighbours. For the most part we will consider the diagonal case $a = c$.

For a lattice of N columns we use the face and boundary weights to construct a double-row transfer matrix. Let $a_1, \ldots, a_{N+1}$ and $b_1, \ldots, b_{N+1}$ be two consecutive rows of spins and let λ be an arbitrary fixed parameter, then the corresponding entry of the double-row transfer matrix is defined by

$$\langle a_1, \ldots, a_{N+1}|\boldsymbol{D}(u)|b_1, \ldots, b_{N+1}\rangle \;=$$

$$\sum_{c_1 \ldots c_{N+1}} B\left(\begin{matrix} b_1 & \\ & c_1 \\ a_1 & \end{matrix}\,\middle|\, \lambda - u\right)\left[\prod_{j=1}^{N} W\left(\begin{matrix} c_j & c_{j+1} \\ a_j & a_{j+1} \end{matrix}\,\middle|\, u\right) W\left(\begin{matrix} b_j & b_{j+1} \\ c_j & c_{j+1} \end{matrix}\,\middle|\, \lambda - u\right)\right] B\left(\begin{matrix} & b_{N+1} \\ c_{N+1} & \\ & a_{N+1} \end{matrix}\,\middle|\, u\right)$$

$$= \tag{2.4}$$



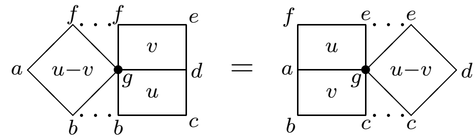
In this and all subsequent diagrams, we use solid circles to indicate spins which are summed over and dotted lines to connect identical spins.

2.2 Local Relations

We suppose the face and boundary weights satisfy the following three local relations:
Yang-Baxter equation

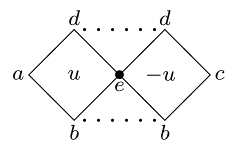
$$\sum_g W\left(\begin{matrix} f & g \\ a & b \end{matrix}\middle| u-v\right) W\left(\begin{matrix} g & d \\ b & c \end{matrix}\middle| u\right) W\left(\begin{matrix} f & e \\ g & d \end{matrix}\middle| v\right) =$$

$$\sum_g W\left(\begin{matrix} a & g \\ b & c \end{matrix}\middle| v\right) W\left(\begin{matrix} f & e \\ a & g \end{matrix}\middle| u\right) W\left(\begin{matrix} e & d \\ g & c \end{matrix}\middle| u-v\right) \tag{2.5}$$



Inversion relation

$$\sum_e W\left(\begin{matrix} d & e \\ a & b \end{matrix}\middle| u\right) W\left(\begin{matrix} d & c \\ e & b \end{matrix}\middle| -u\right) = \rho(u)\,\rho(-u)\,\delta_{ac}\,A_{ab}\,A_{ad} \tag{2.6}$$

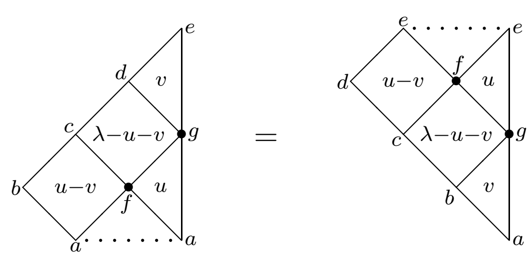


$= \rho(u)\,\rho(-u)\,\delta_{ac}\,A_{ab}\,A_{ad}$

Reflection equation

$$\sum_{fg} W\left(\begin{matrix} c & f \\ b & a \end{matrix}\middle| u-v\right) W\left(\begin{matrix} d & g \\ c & f \end{matrix}\middle| \lambda-u-v\right) B\left(f\ \begin{matrix} g \\ a \end{matrix}\middle| u\right) B\left(d\ \begin{matrix} e \\ g \end{matrix}\middle| v\right) =$$

$$\sum_{fg} W\left(\begin{matrix} e & f \\ d & c \end{matrix}\middle| u-v\right) W\left(\begin{matrix} f & g \\ c & b \end{matrix}\middle| \lambda-u-v\right) B\left(f\ \begin{matrix} e \\ g \end{matrix}\middle| u\right) B\left(b\ \begin{matrix} g \\ a \end{matrix}\middle| v\right) \tag{2.7}$$



These equations are to be satisfied for all values of the external spins and all values of the spectral parameters. The function ρ is model-dependent and λ is the same parameter as in (2.4).

2.3 Commutation of Double-Row Transfer Matrices

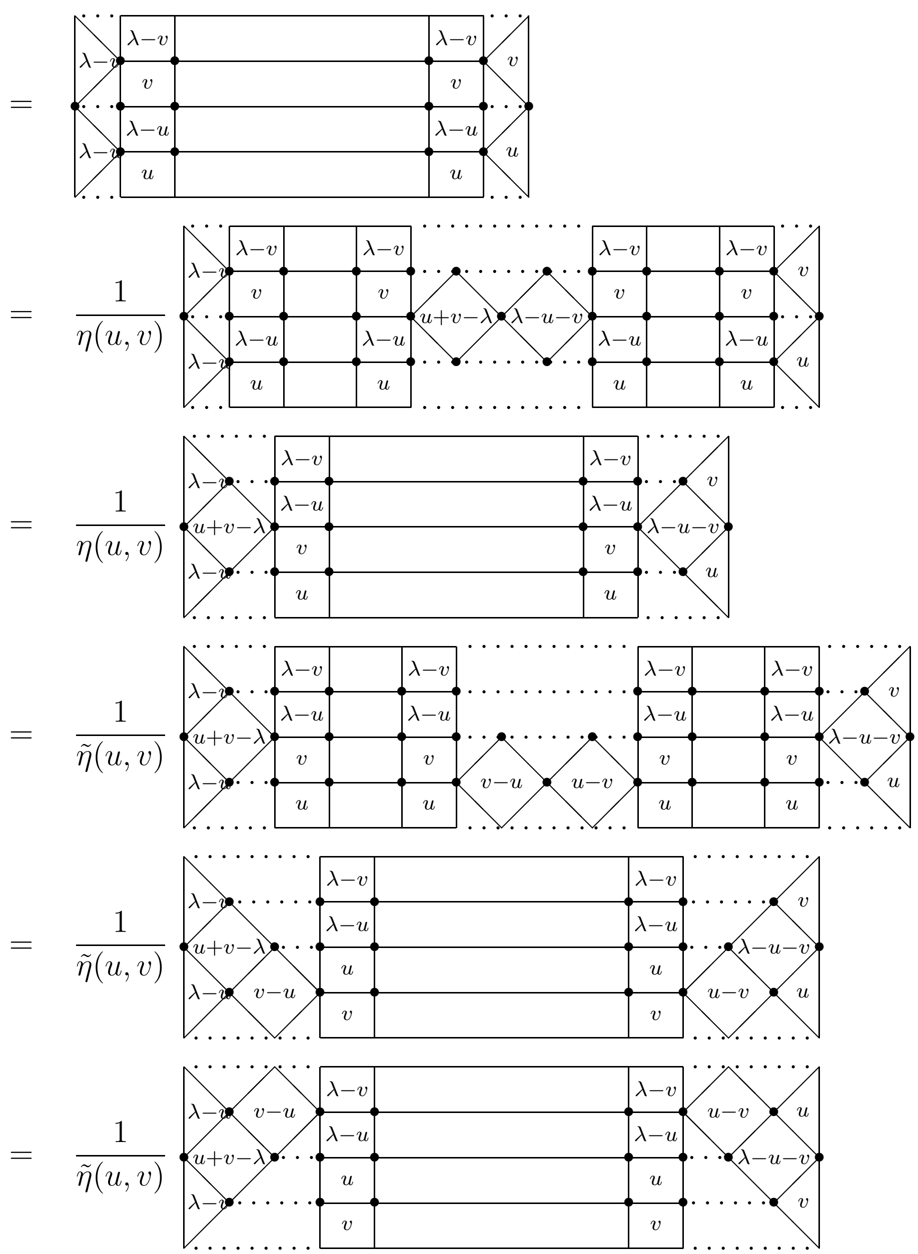
	YBE + Inversion + Reflection
⇒	Commuting Double Row Transfer Matrices
⇒	Solvability

The three local relations imply commutation of the double-row transfer matrices

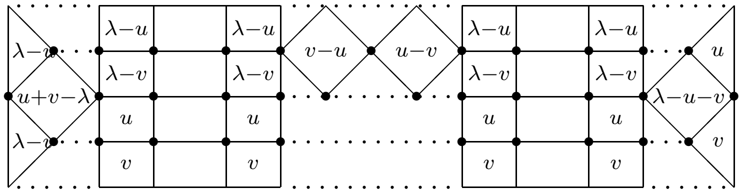
$$\boldsymbol{D}(u)\,\boldsymbol{D}(v)=\boldsymbol{D}(v)\,\boldsymbol{D}(u) \tag{2.8}$$

This follows by generalizing ideas of Baxter [1] for the periodic case. It can be proved graphically in a straightforward way:

$\boldsymbol{D}(u)\,\boldsymbol{D}(v)$



$$= \quad \frac{1}{\tilde{\eta}(u,v)}$$



$$= \quad \frac{1}{\eta(u,v)}$$

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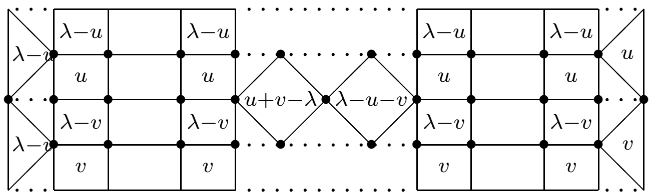


$$=$$

$$= \quad \boldsymbol{D}(v)\,\boldsymbol{D}(u)$$

where $\eta(u,v) = \rho(u+v-\lambda)\rho(\lambda-u-v)$ and $\tilde{\eta}(u,v) = \rho(v-u)\rho(u-v)\rho(u+v-\lambda)\rho(\lambda-u-v)$.

3. ABF Models

Let us consider the Andrews-Baxter-Forrester (ABF) models [5]. For each integer $L \geq 3$, the spins or heights a of this model are restricted to the values

$$a \in \{1, 2, \ldots, L\} \tag{3.1}$$

and adjacent spins must differ by 1, so the adjacency matrix is

$$A_{ab} = \delta_{a,b-1} + \delta_{a,b+1} \tag{3.2}$$

There is a fixed crossing parameter

$$\lambda = \frac{\pi}{L+1} \tag{3.3}$$

and the non-zero face weights are given by

$$W\begin{pmatrix} a\pm1 & a \\ a & a\mp1 \end{pmatrix}\!\!\left|\,u\right.\Bigg) \quad = \quad \frac{\theta(\lambda-u)}{\theta(\lambda)}$$

$$W\begin{pmatrix} a & a\pm1 \\ a\mp1 & a \end{pmatrix}\bigg| u \Bigg) = \sqrt{\frac{\theta((a-1)\lambda)\,\theta((a+1)\lambda)}{\theta(a\lambda)^2}}\,\frac{\theta(u)}{\theta(\lambda)} \tag{3.4}$$

$$W\begin{pmatrix} a & a\pm1 \\ a\pm1 & a \end{pmatrix}\bigg| u \Bigg) = \frac{\theta(a\lambda\pm u)}{\theta(a\lambda)}$$

Here θ is the standard elliptic theta-1 function of fixed nome $\hat{q}$

$$\theta(u) = \theta_1(u, \hat{q}) = 2\hat{q}^{1/4}\sin u \prod_{n=1}^{\infty}\left(1 - 2\hat{q}^{2n}\cos 2u + \hat{q}^{4n}\right)\left(1 - \hat{q}^{2n}\right) \tag{3.5}$$

At criticality, $\hat{q} = 0$ and we can take $\theta(u) = \sin u$. In this case the weights are associated with the fundamental representation of $U_q(su(2))$ with q a root of unity given by

$$q = e^{i\lambda}. \tag{3.6}$$

Off-criticality, there is an additional deformation parameter given by the nome $\hat{q}$ of the elliptic functions which plays the role of temperature in these models.

The ABF face weights (3.4) satisfy the Yang-Baxter equation (2.5) and inversion relation (2.6) with the function ρ given by

$$\rho(u) = \frac{\theta(u-\lambda)}{\theta(\lambda)}. \tag{3.7}$$

Let us define [2], as the only non-zero ABF boundary weights

$$B\left(a\mp1 \;\begin{matrix} a \\ a \end{matrix}\,\bigg|\, u\right) = \sqrt{\frac{\theta((a\mp1)\lambda)}{\theta(a\lambda)}}\;\frac{\theta(u\mp\xi(a))\,\theta(u\pm a\lambda\pm\xi(a))}{\theta(\lambda)^2} \tag{3.8}$$

where $\xi(a)$ is an arbitrary parameter independent of u that can be chosen to be different for the left and the right boundaries. Then the face and boundary weights also satisfy the reflection equation (2.7). It follows that the ABF models can be solved in the presence of a boundary.

In the next section, we will obtain the solution (3.8) to the reflection equations (2.7) by direct construction. Since these weights are diagonal in the sense that

$$a \neq c \quad \text{implies} \quad B\left(b \;\begin{matrix} c \\ a \end{matrix}\,\bigg|\, u\right) = 0 \tag{3.9}$$

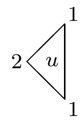
they are consistent with fixed boundary conditions at each end of the entire lattice. Ultimately, we are interested in the isotropic point, $u = \lambda/2$. At this point the lattice becomes homogeneous with pure, fixed boundary conditions for appropriate choices of $\xi(a_{\rm L}), \xi(a_{\rm R})$ on the left and right boundaries. If we set $\xi(a_{\rm L}) = \pm\lambda/2$, $\xi(a_{\rm R}) = \pm\lambda/2$, then

$$B\left(\begin{matrix} a_{\rm L} \\ a_{\rm L} \end{matrix}\; a_{\rm L}\mp1 \,\bigg|\, \lambda/2\right) = B\left(\begin{matrix} a_{\rm R} \\ a_{\rm R} \end{matrix}\; a_{\rm R}\mp1 \,\bigg|\, \lambda/2\right) = 0 \tag{3.10}$$

so that the double-row transfer matrix $\boldsymbol{D}(\lambda/2)$ is simply proportional to the matrix product of two rows of face weights, all with spectral parameter $\lambda/2$, with the three spins on the left boundary fixed to $a_{\rm L}$, $a_{\rm L}\pm1$, $a_{\rm L}$ and the three spins on the right boundary fixed to $a_{\rm R}$, $a_{\rm R}\pm1$, $a_{\rm R}$. The integrable boundary condition therefore contains within it the natural fixed boundary conditions you would apply to the isotropic ABF models.

4. Solutions to Reflection Equations

In this section we describe the construction of solutions to the reflection equations (2.7) for the ABF models (3.4). The two key ingredients are a trivial solution and fusion. Suppose the boundary heights in the reflection equation (2.7) are diagonal with $a = g = e = 1$. Then, because of the adjacency condition, we must have $b = d = f = 2$ for the adjacent spins. Using the symmetry relations (2.2) it follows that there is a trivial solution



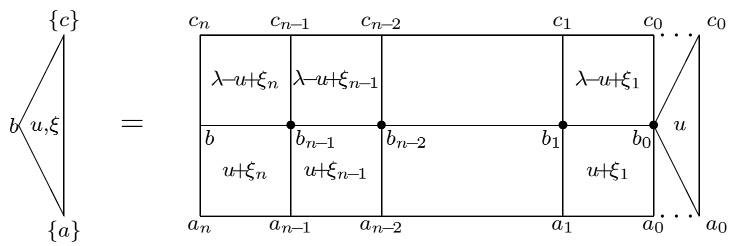
$$= g(u) = \text{arbitrary function of } u \tag{4.1}$$

Elementary or level-2 fusion [16, 17] asserts that

$$\sum_e W\left(\begin{matrix} d & f \\ a & e \end{matrix}\,\middle|\, u-\lambda\right) W\left(\begin{matrix} f & c \\ e & b \end{matrix}\,\middle|\, u\right) = \text{independent of } f. \tag{4.2}$$

Strictly speaking, this only holds in a particular gauge which breaks the diagonal reflection symmetries (2.2) but it is always possible to work in this gauge and then to restore the symmetry afterwards.

To obtain new solutions to the reflection equations from a given solution we introduce generalized reflection equations by replacing each boundary weight in the reflection equations (2.7) with the generalized boundary weight



(4.3)

These generalized weights depend on the n-step row configurations $\{a\} = \{a_n, a_{n-1}, \ldots, a_0\}$ and $\{c\} = \{c_n, c_{n-1}, \ldots, c_0\}$. In addition we add column inhomogeneities $\xi_1, \xi_2, \ldots, \xi_n$ to the spectral parameters in successive columns. Since this does not change any of the differences of spectral parameters in a given column, it is straightforward using the Yang-Baxter equation to show that the generalized reflection equations hold if the original reflection equations hold. Of course the sum on g in (2.7) is replaced by a sum over n-step paths.

To remove the dependence on the n-step row configurations we choose the column inhomogeneities to be

$$\xi_j = \xi - (j-1)\lambda, \quad j = 1, 2, \ldots, n \tag{4.4}$$

and use repeated elementary fusion to project out the dependence on the path by summing on the spins $\{a_{n-1}, a_{n-2}, \ldots, a_1\}$. This ensures that the result is independent of the spins $\{c_{n-1}, c_{n-2}, \ldots, c_1\}$.

Let $A^{(1)} = A$ be the adjacency matrix (3.2) of the ABF models and $A^{(0)} = I$ the identity matrix. Then the adjacency matrix $A^{(n)}$ giving the allowed neighbouring spins which lead to nonzero weights after level-n fusion is given by the fusion rule

$$A^{(n)}A^{(1)} = A^{(n+1)} + A^{(n-1)}. \tag{4.5}$$

Accordingly, with $a_0 = c_0 = 1$, the only paths which give a nonzero weight are those with $a_n = c_n = n+1$. In the generalized boundary weight on the left side of (4.3) we can therefore forget about the dependence on the paths and replace both $\{a\}$ and $\{c\}$ with the height $n+1$. Calculating the overall weight on the right side, putting in the trivial boundary weights, now gives a new solution to the original reflection equation. This argument gives a nontrivial solution for any height by choosing the proper fusion level. Moreover, we can make a different choice for the parameter ξ in each case. So finally, replacing the arbitrary height $n+1$ by a and choosing $g(u)$ appropriately we obtain precisely the boundary weights (3.8).

This process can be repeated, starting with the diagonal solution, to build up non-diagonal solutions. The non-diagonal solutions contain an additional parameter χ arising from the column inhomogeneities. In this case we choose

$$n = \begin{cases} (L-1)/2, & L \text{ odd} \\ L/2, & L \text{ even} \end{cases} \tag{4.6}$$

with $b_0 = c_0 = n$ or $n+1$. From the fusion rule it then follows that a_n and c_n are allowed to take on either all odd or all even heights. In this sense the non-diagonal boundary weights correspond to free boundary conditions. By this means the non-diagonal weights can be obtained explicitly. However, since the actual formulas [3] are somewhat cumbersome, we will not duplicate the formulas here.

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