THE DILUTE LATTICE MODELS AND WHAT THEY TELL US ABOUT CRITICAL BEHAVIOUR

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ABSTRACT

The dilute lattice models are families of exactly solvable two-dimensional lattice models which exhibit some new properties not found in previous exactly solvable lattice models. A general review is given of these models and their significance in understanding critical behaviour. The models are important for at least two reasons. First, the off-critical models yield solvable models in symmetry-breaking fields including a model in the universality class of the Ising model in a magnetic field. Second, these models provide lattice realizations of all the unitary minimal theories of conformal field theory with central charge $c < 1$. In other words, these models give exactly solvable representatives of all possible universality classes for $c < 1$.

1. Dilute A–D–E Lattice Models

The dilute A–D–E lattice models are families of exactly solvable two-dimensional lattice models. They were introduced independently by Warnaar, Nienhuis and Seaton[1] and by Roche[2] in 1992 before the StatPhys18 meeting in Berlin. At criticality, the models are parametrized by trigonometric functions. The dilute A models also admit an elliptic parameterization[1] and can be solved[3] off-criticality. These models are significant for a number of reasons. We summarize these reasons here and refer to the original literature for more in depth discussions.

Like the A–D–E models of Pasquier[4], the states $a, b, c, d = 1, 2, \ldots, L$ of the dilute A–D–E models take values on the A–D–E Dynkin diagrams of the classical A–D–E Lie algebras shown in Figure 1. Associated with each A–D–E graph is an adjacency matrix

$$A_{a,b} = \begin{cases} 1, & a,\, b \text{ connected} \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

Denote the Coxeter number of an A–D–E graph by h. The nonnegative elements S_a of the Perron-Frobenius eigenvector of the adjacency matrix are then given by

$$\sum_b A_{a,b} S_b = 2\cos\left(\frac{\pi}{h}\right) S_a, \qquad h = \begin{cases} L+1, & A_L \\ 2L-2, & D_L \\ 12, 18, 30, & E_{6,7,8}. \end{cases} \tag{1.2}$$

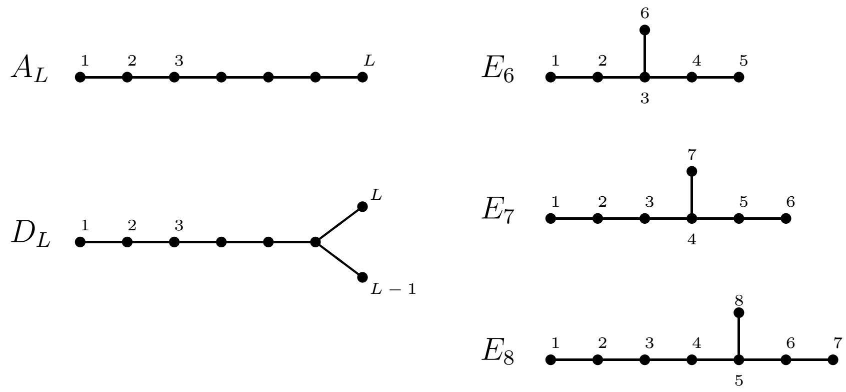


Figure 1. The adjacency graphs of the classical A–D–E models.

The face weights of the dilute A–D–E models are defined, in terms of the data of the A–D–E graph, by

$$\begin{aligned} W\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) &= \rho_1(u)\delta_{a,b,c,d} + \rho_2(u)\delta_{a,b,c}A_{a,d} + \rho_3(u)\delta_{a,c,d}A_{a,b} \\ &+ \sqrt{\frac{S_a}{S_b}}\rho_4(u)\delta_{b,c,d}A_{a,b} + \sqrt{\frac{S_c}{S_a}}\rho_5(u)\delta_{a,b,d}A_{a,c} + \rho_6(u)\delta_{a,b}\delta_{c,d}A_{a,c} \\ &+ \rho_7(u)\delta_{a,d}\delta_{c,b}A_{a,b} + \rho_8(u)\delta_{a,c}A_{a,b}A_{a,d} + \sqrt{\frac{S_aS_c}{S_bS_d}}\rho_9(u)\delta_{b,d}A_{a,b}A_{b,c}. \end{aligned} \tag{1.3}$$

The trigonometric functions are

$$\begin{aligned} \rho_1(u) &= 1 + \frac{\sin u \sin(3\lambda - u)}{\sin(2\lambda)\sin(3\lambda)}, & \rho_2(u) &= \rho_3(u) = \frac{\sin(3\lambda - u)}{\sin(3\lambda)} \\ \rho_4(u) &= \rho_5(u) = \frac{\sin u}{\sin(3\lambda)}, & \rho_6(u) &= \rho_7(u) = \frac{\sin u \sin(3\lambda - u)}{\sin(2\lambda)\sin(3\lambda)} \\ \rho_8(u) &= \frac{\sin(2\lambda - u)\sin(3\lambda - u)}{\sin(2\lambda)\sin(3\lambda)}, & \rho_9(u) &= -\frac{\sin u \sin(\lambda - u)}{\sin(2\lambda)\sin(3\lambda)} \end{aligned} \tag{1.4}$$

and the generalized Kronecker delta is

$$\delta_{a,b,c,\ldots} = \begin{cases} 1, & a = b = c = \ldots \\ 0, & \text{otherwise.} \end{cases} \tag{1.5}$$

Here the spectral parameter u lies in the interval $0 < u < 3\lambda$ where there are two choices for the parameter λ

$$\lambda = \frac{(h \pm 1)\pi}{4h}. \tag{1.6}$$

Notice that the effective adjacency condition for the dilute models is given by $I + A$ which is analogous to adding a loop to each node of the A–D–E graph describing the admissible states.

2. Dilute Algebra

Let us consider the Yang-Baxter algebra generated by the face operators $X_j(u)$ of the dilute A–D–E models. If $a = \{a_1, a_2, \ldots\}$ and $a' = \{a'_1, a'_2, \ldots\}$ are consecutive diagonal rows of spins, the local face operators are defined by

$$\langle a|X_j(u)|a'\rangle \;=\; W\left(\begin{matrix} a_{j-1} & a'_j \\ a_j & a_{j+1} \end{matrix}\,\middle|\, u\right) \prod_{k\neq j} \delta(a_k, a'_k)\ . \tag{2.1}$$

These face operators satisfy the Yang-Baxter equation

$$X_{j+1}(u)X_j(v)X_{j+1}(v-u) = X_j(v-u)X_{j+1}(v)X_j(u) \tag{2.2}$$

and the locality condition

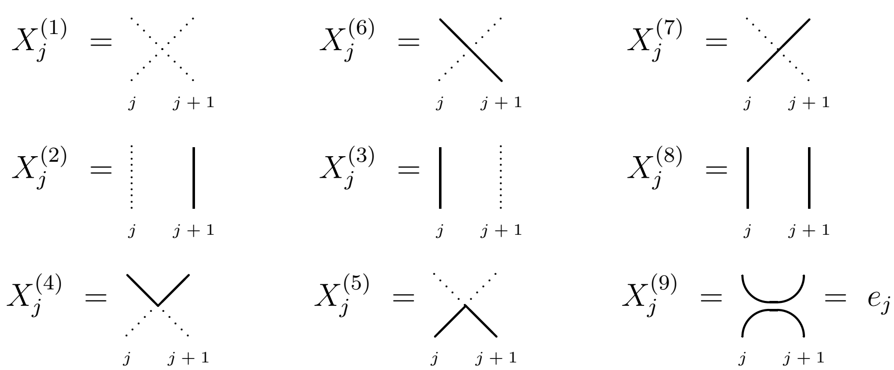
$$X_j(u)X_k(v) = X_k(v)X_j(u), \qquad |j-k| \geq 2. \tag{2.3}$$

These two relations are the defining relations of a Yang-Baxter algebra.

The Yang-Baxter or face operators of the dilute A–D–E models can be decomposed as

$$X_j(u) = \sum_{n=1}^{9} \rho_n(u) X_j^{(n)}. \tag{2.4}$$

Using a convenient graphical notation[5] we can write the u-independent operators $X_j^{(n)}$ as

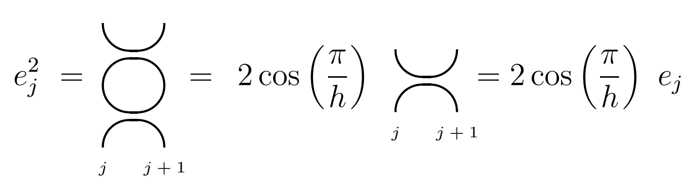


(2.5)

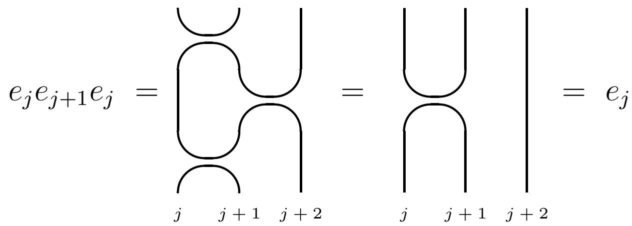
(2.6)

(2.7)

It is easily seen that the operators $X_j^{(9)} = e_j$ satisfy the defining relations of a Temperley-Lieb algebra[6]:

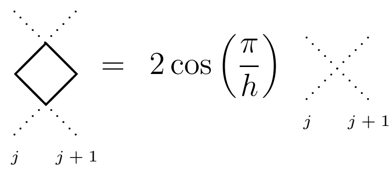


(2.8)

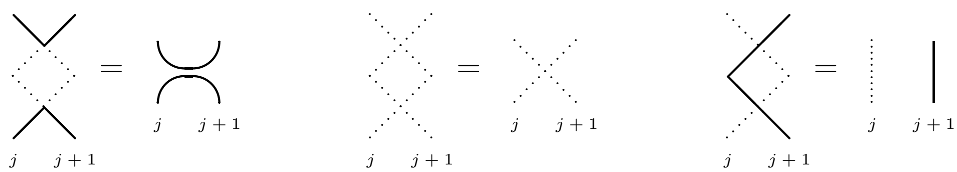


(2.9)

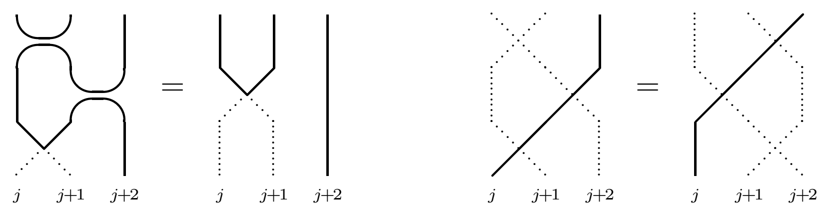
The Temperley-Lieb algebra is therefore a subalgebra of the algebra generated by the operators $X_j^{(n)}$. The operators $X_j^{(n)}$ in fact generate a dilute Temperley-Lieb algebra, which is a particular example of more general multicolour braid-monoid algebras[7], and satisfy many other relations as suggested by their graphical representations such as



(2.10)



(2.11)



(2.12)

These relations have a natural interpretation in terms of isotopy of strings in knots and links. The defining relations of the dilute Temperley-Lieb algebra suffice to ensure that the dilute A–D–E models satisfy the Yang-Baxter equations.

3. Unitary Minimal Series

In appropriate regimes, the critical behaviours of the A–D–E models are described by the unitary minimal series of conformal field theories[8,9] with central charges $c < 1$. More specifically, the modular invariant partition functions (MIPFs) of these models on a torus belong to either of two series classified[10] by a pair (G, G') of A–D–E Lie algebras:

$$
\begin{array}{cc}
\textbf{Critical} & \textbf{Tricritical} \\[6pt]
\boxed{\lambda=\frac{(h+1)\pi}{4h},\quad c=1-\frac{6}{h(h-1)}} & \boxed{\lambda=\frac{(h-1)\pi}{4h},\quad c=1-\frac{6}{h(h+1)}} \\[6pt]
(A,G')=\begin{cases}(A_{h-2},A_{h-1})\\(A_{h-2},D_{(h+2)/2})\\(A_{10},E_6)\\(A_{16},E_7)\\(A_{28},E_8)\end{cases} & (G,A')=\begin{cases}(A_{h-1},A_h)\\(D_{(h+2)/2},A_h)\\(E_6,A_{12})\\(E_7,A_{18})\\(E_8,A_{30})\end{cases}
\end{array}
\tag{3.1}
$$

Here the Coxeter number h belongs to the Lie algebra of A–D–E type.

It has been known for some time that the MIPFs of Pasquier's A–D–E models all belong to the critical series. Recently, however, it has been shown numerically[11] that the MIPFs of the dilute A–D–E models exhaust both the critical and the tricritical series. This result can be paraphrased as follows:

> *The dilute A–D–E lattice models provide an exactly solvable representative for each universality class of the theories characterized by $c < 1$.*

In particular, the dilute D_4 model with $\lambda = 7\pi/24$ lies in the universality class of the critical 3-state Potts model, has $c = 4/5$ and the MIPF is given by (A_4, D_4). By comparison, the dilute D_4 model with $\lambda = 5\pi/24$ lies in the universality class of the tricritical 3-state Potts model, has $c = 6/7$ and the MIPF is given by (D_4, A_6).

4. Off-Critical Solution in Symmetry-Breaking Field

The dilute A_L models admit an elliptic extension[1] and can be solved[3] off-criticality for the free energies, local height probabilities and order parameters. It is therefore possible to obtain the associated critical exponents directly. Unlike the A_L models of Andrews, Baxter and Forrester[12] (ABF) where the elliptic nome plays the role of a temperature-like variable t, the elliptic nome of the dilute A_L models (at least when L is odd) plays the role of a symmetry breaking field h. Specifically, results[12,13] for the order parameters of the ABF models give

$$R^{(k)} \sim t^{\beta_k}, \qquad \beta_k = \frac{(k+1)^2-1}{8L}, \qquad k = 1, 2, \ldots, L-2 \tag{4.1}$$

which for $L = 3$ and $k = 1$ yields the familiar result for the magnetization of the Ising model

$$m \sim t^{\beta}, \qquad \beta = 1/8. \tag{4.2}$$

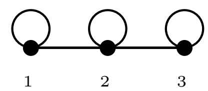
In remarkable contrast, when L is odd, results[3] for the order parameters of the dilute A_L models give

$$\overline{R}^{(k)} \sim h^{1/\delta_k}, \qquad \delta_k = \frac{3L(L+2)}{(k+1)^2-1}, \qquad k = 1,2,\ldots,L-2. \tag{4.3}$$

The dilute A_3 model in fact[14] lies in the universality class of the Ising model in a magnetic field and is therefore of particular interest. In this case the elliptic nome is identified as the leading magnetic field and, accordingly, the direct calculation[3] of the magnetization as a function of field leads for $L = 3$ and $k = 1$ to the celebrated critical exponent

$$m \sim h^{1/\delta}, \qquad \delta = 15. \tag{4.4}$$

Previously, this critical exponent had only been obtained by invoking scaling laws.

Dilute A_3 lattice model $=$ 

$=$ *Universality class of Ising model in a magnetic field*

5. Magnetic Ising Model, Dilute A_3 and E_8

The connections between the dilute A_3 model, the magnetic Ising model and the exceptional Lie algebra E_8 are particularly interesting. In 1989 Zamolodchikov[15] showed that the critical Ising model in a magnetic field is an integrable quantum field theory containing 8 particles with masses related to E_8. Subsequently, in 1994 Bazhanov, Nienhuis and Warnaar[16] showed that, subject to a certain string hypothesis, a Bethe ansatz study of the dilute A_3 model in the scaling limit leads to thermodynamic Bethe ansatz (TBA) equations[17] with an E_8 structure.

In a seemingly unrelated development in 1993, Kedem, Klassen, McCoy and Melzer[18] conjectured fermionic representations for the $c = \frac{1}{2}$ Virasoro characters $\chi^{(3,4)}_{r,s}$ by considering the coset conformal field theory based on E_8. In particular, they conjectured the E_8 Rogers-Ramanujan identity

$$q^{1/48}\chi^{(3,4)}_{1,1}(q) = \sum_{n_1,\ldots,n_8=0}^{\infty} \frac{q^{\boldsymbol{n}^T C^{-1}_{\mathrm{E}_8}\boldsymbol{n}}}{(q)_{n_1}\cdots(q)_{n_8}} = \frac{1}{(q)_\infty}\sum_{j=-\infty}^{\infty}\left\{q^{12j^2+j} - q^{12j^2+7j+1}\right\} \tag{5.1}$$

where $(q)_m = \prod_{k=1}^m (1-q^k)$ for $m \ge 1$, $(q)_0 = 1$ and C_{E_8} is the Cartan matrix

$$C_{\mathrm{E}_8} = 2I - A_{E_8}, \qquad C^{-1}_{\mathrm{E}_8} = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 4 & 2 & 3 \\ 3 & 6 & 8 & 10 & 12 & 8 & 4 & 6 \\ 4 & 8 & 12 & 15 & 18 & 12 & 6 & 9 \\ 5 & 10 & 15 & 20 & 24 & 16 & 8 & 12 \\ 6 & 12 & 18 & 24 & 30 & 20 & 10 & 15 \\ 4 & 8 & 12 & 16 & 20 & 14 & 7 & 10 \\ 2 & 4 & 6 & 8 & 10 & 7 & 4 & 5 \\ 3 & 6 & 9 & 12 & 15 & 10 & 5 & 8 \end{pmatrix}. \tag{5.2}$$

This conjecture has been proved[19] by taking the limit $L \to \infty$ starting with the finite polynomial identity

$$\begin{aligned}
&\sum_{(\boldsymbol{n},\boldsymbol{m})_L} q^{\boldsymbol{n}^T C_{\mathrm{E}_8}^{-1}\boldsymbol{n}} \prod_{i=1}^{8} \begin{bmatrix} n_i+m_i \\ n_i \end{bmatrix}_q \\
&= \sum_{j,k=-\infty}^{\infty} \left\{ q^{12j^2+j+k(k+8j)} \begin{bmatrix} L \\ k,\, k+8j \end{bmatrix}_q - q^{12j^2+7j+1+k(k+8j+2)} \begin{bmatrix} L \\ k,\, k+8j+2 \end{bmatrix}_q \right\}
\end{aligned} \tag{5.3}$$

and using the results

$$\lim_{N\to\infty} \begin{bmatrix} N \\ m \end{bmatrix}_q = \frac{1}{(q)_m}, \qquad \lim_{L\to\infty} \sum_{k=0}^{\infty} q^{k(k+a)} \begin{bmatrix} L \\ k,\, k+a \end{bmatrix}_q = \frac{1}{(q)_\infty}. \tag{5.4}$$

Here the Gaussian polynomials are given by

$$\begin{bmatrix} N \\ m \end{bmatrix}_q = \frac{(q)_N}{(q)_m (q)_{N-m}}, \qquad \begin{bmatrix} N \\ m_1,\, m_2 \end{bmatrix}_q = \frac{(q)_N}{(q)_{m_1}(q)_{m_2}(q)_{N-m_1-m_2}} \tag{5.5}$$

and the constrained sum is over all solutions to the TBA system $(\boldsymbol{n},\boldsymbol{m})_L$:

$$\boldsymbol{n}+\boldsymbol{m} = \frac{1}{2}\left(A_{\mathrm{E}_8}\boldsymbol{m} + L\boldsymbol{e}_1\right) \tag{5.6}$$

$$\boldsymbol{n} = (n_1, n_2, \ldots, n_8),\ \ \boldsymbol{m} = (m_1, m_2, \ldots, m_8),\ \ \boldsymbol{e}_1 = (1, 0, \ldots, 0) \in \mathbb{Z}^8.$$

The polynomials on the right-side of (5.3) are precisely the (bosonic) polynomials that occur in the corner transfer matrix calculation[3] of the magnetization of the dilute A_3 models. The (fermionic) representation[19] of these same polynomials on the left-side of (5.3) makes explicit the hidden E_8 structure in the dilute A_3 lattice model and the critical Ising model in a magnetic field. Similarly, the dilute A_4 and A_6 reveal E_7 and E_6 structures respectively but the situation for the general case of dilute A_L is completely mysterious.

6. Conclusion

Perhaps surprisingly, there are still many new solvable lattice models being discovered. Among these, the dilute lattice models exhibit some very interesting properties. We have summarized some of these in this article. The dilute lattice models in fact consist of many families of solvable lattice models. Indeed, Warnaar and Grimm[21,22] have recently constructed higher rank dilute lattice models which generalize the higher rank models of Jimbo, Miwa and Okado[23]. No doubt when these models are studied further they will also exhibit some interesting properties.

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