Boundary Weights for Temperley-Lieb and Dilute Temperley-Lieb Models

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Abstract

We use a direct approach to obtain the general diagonal solutions of the boundary Yang-Baxter equation for the Temperley-Lieb and dilute Temperley-Lieb models and their elliptic extensions.

1. Introduction

A two-dimensional lattice spin model in statistical mechanics can be considered as solvable with periodic boundary conditions if its bulk Boltzmann weights satisfy the Yang-Baxter equation (YBE) [1], and as additionally solvable with certain non-periodic boundary conditions if it admits boundary weights which satisfy the boundary Yang-Baxter equation (BYBE) [2].

Many such models are now known. Restricting our attention to interaction-round-a-face models, these are the eight-vertex solid-on-solid model [3], the cyclic solid-on-solid models [4], the A_L models [5, 6, 7], the dilute A_L models [8], and certain higher-rank models associated with $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ and $A_n^{(2)}$ [9]. Here, we obtain boundary weights for some further, related models.

We begin in Section 2 by stating the YBE and BYBE for interaction-round-a-face models, and defining diagonal and non-diagonal types of BYBE solution. In Sections 3 and 4, we derive the general diagonal solutions of the BYBE for the Temperley-Lieb and dilute Temperley-Lieb models, while in Section 5 we present the general diagonal solutions for the A_L and D_L models, which form the elliptic extensions of the Temperley-Lieb models. We conclude, in Section 6, by stating some further relations satisfied by the previously-presented boundary weights, considering the algebraic properties of some of these weights, presenting general non-diagonal solutions of the BYBE for the A_L models, and discussing alternative techniques for solving the BYBE.

2. The YBE and BYBE

We are considering an *interaction-round-a-face model* whose main features are an *adjacency graph* $\mathcal{G}$ with nodes labelled by the model's spin values, an *adjacency matrix* A with entries defined by

$$A_{ab} = \text{number of bonds of } \mathcal{G} \text{ which connect } a \text{ to } b,$$

a *bulk weight* $W\!\left(\begin{smallmatrix} d & c \\ a & b \end{smallmatrix}\middle| u\right)$ associated with each a, b, c, d satisfying $A_{da}\,A_{ab}\,A_{dc}\,A_{cb}=1$, and a *boundary weight* $B\!\left(b\ \begin{smallmatrix} a \\ c \end{smallmatrix}\middle| u\right)$ associated with each a, b, c satisfying $A_{bc}\,A_{ba}=1$, where u is a *spectral parameter*.

For such a model, the *Yang-Baxter equation* (YBE) is [1]

$$\sum_{\substack{g \\ A_{fg}A_{gb}A_{gd}=1}} W\!\left(\begin{matrix} f & g \\ a & b \end{matrix}\middle| u-v\right) W\!\left(\begin{matrix} g & d \\ b & c \end{matrix}\middle| u\right) W\!\left(\begin{matrix} f & e \\ g & d \end{matrix}\middle| v\right) = \sum_{\substack{g \\ A_{ag}A_{eg}A_{gc}=1}} W\!\left(\begin{matrix} a & g \\ b & c \end{matrix}\middle| v\right) W\!\left(\begin{matrix} f & e \\ a & g \end{matrix}\middle| u\right) W\!\left(\begin{matrix} e & d \\ g & c \end{matrix}\middle| u-v\right), \tag{2.1}$$

which is to hold for all u and v and all a, b, c, d, e, f satisfying $A_{fa}\,A_{ab}\,A_{bc}\,A_{fe}\,A_{ed}\,A_{dc}=1$, and the *boundary Yang-Baxter equation* (BYBE) is [6, 10]

$$\sum_{\substack{fg \\ A_{cf}A_{fe}A_{fg}A_{bg}=1}} W\!\left(\begin{matrix} c & f \\ d & e \end{matrix}\middle| u-v\right) B\!\left(f\ \begin{matrix} g \\ e \end{matrix}\middle| u\right) W\!\left(\begin{matrix} c & b \\ f & g \end{matrix}\middle| u+v\right) B\!\left(b\ \begin{matrix} a \\ g \end{matrix}\middle| v\right) = \sum_{\substack{fg \\ A_{dg}A_{cf}A_{fg}A_{fa}=1}} B\!\left(d\ \begin{matrix} g \\ e \end{matrix}\middle| v\right) W\!\left(\begin{matrix} c & f \\ d & g \end{matrix}\middle| u+v\right) B\!\left(f\ \begin{matrix} a \\ g \end{matrix}\middle| u\right) W\!\left(\begin{matrix} c & b \\ f & a \end{matrix}\middle| u-v\right), \tag{2.2}$$

which is to hold for all u and v and all a, b, c, d, e satisfying $A_{cd}\,A_{de}\,A_{cb}\,A_{ba}=1$.

In practice, one generally begins with specific bulk weights which satisfy the YBE and then attempts to solve the BYBE for corresponding boundary weights. Two classes of solution which are of particular interest, since they may lead to fixed and free boundary conditions respectively, are *diagonal* solutions, for which $B\!\left(b\ \begin{smallmatrix} a \\ c \end{smallmatrix}\middle| u\right)=0$ whenever $a\neq c$, and *non-diagonal* solutions, for which $B\!\left(b\ \begin{smallmatrix} a \\ c \end{smallmatrix}\middle| u\right)\neq 0$ for all a, b, c.

3. Temperley-Lieb Models

We now consider the *Temperley-Lieb models* [11, 12]. The adjacency graph $\mathcal{G}$ of such a model is assumed to contain only bidirectional, single bonds, but may otherwise be chosen arbitrarily. By the Perron-Frobenius theorem, the adjacency matrix of $\mathcal{G}$ has a unique, positive maximum eigenvalue Λ, with an associated eigenvector $(S_{a_1}, S_{a_2}, \ldots)$ which has all positive entries.

The bulk weights, which are known to satisfy the YBE, are

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right) = \frac{s(\lambda-u)}{s(\lambda)}\,\delta_{ac} + \frac{(S_a\,S_c)^{1/2}}{S_b}\,\frac{s(u)}{s(\lambda)}\,\delta_{bd}\,, \tag{3.1}$$

where λ is any solution of

$$\Lambda = 2\,c(\lambda) \tag{3.2}$$

and

$$s(u) = \begin{cases} \sin u\,, & \Lambda<2 \\ u\,, & \Lambda=2 \\ \sinh u\,, & \Lambda>2\,, \end{cases} \qquad c(u) = \begin{cases} \cos u\,, & \Lambda<2 \\ 1\,, & \Lambda=2 \\ \cosh u\,, & \Lambda>2\,. \end{cases} \tag{3.3}$$

Diagonal boundary weights are

$$B\left(b\ \begin{matrix} a \\ a \end{matrix}\middle| u\right) = \tag{3.4}$$

$$\begin{cases} \Big[x_1(a)\,s(u)\,\Big(s(u+\lambda) - \sum_{d\,\in\,\nu(a)} S_d/S_a\,s(u)\Big) + x_2(a)\Big]\,f(a,u)\,, & b\in\nu(a) \\ \Big[-x_1(a)\,s(u)\,\Big(s(u+\lambda) - \sum_{d\,\in\,\nu'(a)} S_d/S_a\,s(u)\Big) + x_2(a)\Big]\,f(a,u)\,, & b\in\nu'(a)\,, \end{cases}$$

where, for each a, $\nu(a)$ and $\nu'(a)$ are any non-empty, non-intersecting sets whose union is the set of neighbours of a, and x_1, x_2 and f are arbitrary.

We now prove that these boundary weights represent the general diagonal solution of the BYBE with bulk weights (3.1).

Setting $B\left(b\ \begin{matrix} a \\ c \end{matrix}\middle| u\right) = K_a(b,u)\,\delta_{ac}$ and using (3.1), we find that the only spin assignments in (2.2) which lead to non-trivial equations are those in which $a=c=e$, and b and d are distinct neighbours of a. These equations are

$$\begin{aligned} 0 = \mathcal{E}_a(b,d) &= s(u+v)\,s(u-v-\lambda)\,(K_a(b,u)\,K_a(d,v) - K_a(d,u)\,K_a(b,v)) \\ &\quad - s(u-v)\,s(u+v-\lambda)\,(K_a(b,u)\,K_a(b,v) - K_a(d,u)\,K_a(d,v)) \\ &\quad + s(u-v)\,s(u+v) \sum_{c\,\in\,\mathcal{N}(a)} S_c\,K_a(c,u)/S_a\,(K_a(b,v) - K_a(d,v))\,, \end{aligned}$$

where $\mathcal{N}(a)$ is the set of neighbours of a.

We shall from now on treat a as fixed. If a has n neighbours, then there are, since $\mathcal{E}_a(b,d) = -\,\mathcal{E}_a(d,b)$, $n(n-1)/2$ distinct equations for the n boundary weights $K_a(b,u)$.

Throughout this proof, we shall also use the eigenvector equation

$$\sum_{b\,\in\,\mathcal{N}(a)} S_b/S_a = 2\,c(\lambda)\,.$$

We now observe that

$$0 \;=\; \mathcal{E}_a(b,c) \;+\; \mathcal{E}_a(c,d) \;+\; \mathcal{E}_a(d,b) \;=\; s(u{+}v)\, s(u{-}v{-}\lambda)\; \det \begin{pmatrix} 1 & 1 & 1 \\ K_a(b,u) & K_a(c,u) & K_a(d,u) \\ K_a(b,v) & K_a(c,v) & K_a(d,v) \end{pmatrix} .$$

The general solution of this system of equations is

$$K_a(b,u) \;=\; y(b)\, g(u) \;+\; h(u) \;,$$

where $y(b)$ are arbitrary constants and g and h are arbitrary functions. Using this solution, we obtain

$$\begin{aligned} 0 \;=\; \mathcal{E}_a(b,d) \;=\; & (y(d) - y(b)) \Big(s(\lambda)\, s(2v)\, g(u)\, h(v) \;-\; s(\lambda)\, s(2u)\, h(u)\, g(v) \\ & +\; s(u{-}v) \Big((y(b) + y(d))\, s(u{+}v{-}\lambda) \;-\; \sum_{c \in \mathcal{N}(a)} y(c)\, S_c/S_a\, s(u{+}v) \Big)\, g(u)\, g(v) \Big) \;. \end{aligned}$$

These equations are satisfied if $y(b)$ are equal for all $b \in \mathcal{N}(a)$. In order to obtain the remaining solutions, we assume $y(\tilde{b}) \neq y(\tilde{d})$ for particular $\tilde{b}$ and $\tilde{d}$. We now transform

$$\begin{aligned} g(u) \;&=\; s(\lambda)\, s(2u)\, \tilde{g}(u) \\ h(u) \;&=\; \Big((y(\tilde{b}) + y(\tilde{d}))\, s(u)\, s(u{-}\lambda) \;-\; \sum_{c \in \mathcal{N}(a)} y(c)\, S_c/S_a\, s(u)^2 \Big)\, \tilde{g}(u) \;+\; \tilde{h}(u) \;, \end{aligned}$$

with $\tilde{g}$ and $\tilde{h}$ arbitrary, which gives

$$\begin{aligned} 0 \;=\; \mathcal{E}_a(b,d) \;=\; & (y(d) - y(b))\, s(\lambda)^2\, s(2u)\, s(2v)\, (\tilde{g}(u)\, \tilde{h}(v) \;-\; \tilde{h}(u)\, \tilde{g}(v) \\ & +\; (y(b) - y(\tilde{b}) + y(d) - y(\tilde{d}))\, s(u{-}v)\, s(u{+}v{-}\lambda)\, \tilde{g}(u)\, \tilde{g}(v)) \;. \end{aligned}$$

The general solution of $\mathcal{E}_a(\tilde{b},\tilde{d}) = 0$ is

$$\tilde{g}(u) = \tilde{x}_1\, f(u) \;, \qquad \tilde{h}(u) = \tilde{x}_2\, f(u) \;,$$

where $\tilde{x}_1$, $\tilde{x}_2$ and f are arbitrary. The remaining cases of $\mathcal{E}_a(b,d) = 0$ now imply that

$$y(b) \;=\; \begin{cases} \tilde{y} \;, & b \in \nu(a) \\ \tilde{y}' \;, & b \in \nu'(a) \;, \end{cases}$$

for some constants $\tilde{y} \neq \tilde{y}'$, and non-intersecting sets $\nu(a)$ and $\nu'(a)$ whose union is $\mathcal{N}(a)$ and which contain $\tilde{b}$ and $\tilde{d}$ respectively. The previous case in which all $y(b)$ are equal is included if we also allow $\tilde{y} = \tilde{y}'$. This now leads to the general solution (3.4), where $x_1(a) = (\tilde{y} - \tilde{y}')\tilde{x}_1$, $x_2(a) = \tilde{x}_2$ and $f(a,u) = f(u)$.

4. Dilute Temperley-Lieb Models

We now consider the *dilute Temperley-Lieb models* [13, 14, 15]. The adjacency graph $\mathcal{G}$ of such a model is assumed to consist of an 'undiluted' adjacency graph $\hat{\mathcal{G}}$ to which are added single bonds which connect each node to itself. The graph $\hat{\mathcal{G}}$ is assumed to be simple and to contain only bidirectional, single bonds, but may otherwise be chosen arbitrarily. The bulk weights, which are known to satisfy the YBE, are

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\,\middle|\, u\right) = \rho_1(u)\,\delta_{abcd} + \rho_2(u)\,\delta_{abc}\,\hat{A}_{ad} + \rho_3(u)\,\delta_{acd}\,\hat{A}_{ab} + \tag{4.1}$$

$$\left(\frac{S_a}{S_b}\right)^{1/2}\rho_4(u)\,\delta_{bcd}\,\hat{A}_{ab} + \left(\frac{S_c}{S_a}\right)^{1/2}\rho_5(u)\,\delta_{abd}\,\hat{A}_{ac} + \rho_6(u)\,\delta_{ab}\,\delta_{cd}\,\hat{A}_{ac} + \rho_7(u)\,\delta_{ad}\,\delta_{bc}\,\hat{A}_{ab} +$$

$$\rho_8(u)\,\delta_{ac}\,\hat{A}_{ab}\,\hat{A}_{ad} + \left(\frac{S_a\,S_c}{S_b\,S_d}\right)^{1/2}\rho_9(u)\,\delta_{bd}\,\hat{A}_{ab}\,\hat{A}_{bc}\,,$$

where $\delta_{a_1\ldots a_m} = \prod_{j=1}^{m-1}\delta_{a_j a_{j+1}}$ and

$$\begin{aligned}
\rho_1(u) &= 1+\frac{\sin u\,\sin(3\lambda-u)}{\sin 2\lambda\,\sin 3\lambda} & \rho_2(u) &= \rho_3(u) = \frac{\sin(3\lambda-u)}{\sin 3\lambda} \\
\rho_4(u) &= \rho_5(u) = \frac{\sin u}{\sin 3\lambda} & \rho_6(u) &= \rho_7(u) = \frac{\sin u\,\sin(3\lambda-u)}{\sin 2\lambda\,\sin 3\lambda} \\
\rho_8(u) &= \frac{\sin(2\lambda-u)\,\sin(3\lambda-u)}{\sin 2\lambda\,\sin 3\lambda} & \rho_9(u) &= -\frac{\sin u\,\sin(\lambda-u)}{\sin 2\lambda\,\sin 3\lambda}\,.
\end{aligned} \tag{4.2}$$

Furthermore, $\hat{A}$ is the adjacency matrix of $\hat{\mathcal{G}}$, Λ is its maximum eigenvalue, $(S_{a_1}, S_{a_2}, \ldots)$ is the associated eigenvector with all positive entries, and λ is any solution of

$$\Lambda = -2\cos 4\lambda\,. \tag{4.3}$$

Diagonal boundary weights are

$$B\left(b\ \begin{matrix} a \\ a \end{matrix}\,\middle|\, u\right) = \begin{cases} \sin(\xi(a)-\frac{\lambda}{2}+u)\ \sin(\xi(a)+\frac{\lambda}{2}-u)\ f(a,u)\,, & b=a \\ \sin(\xi(a)-\frac{\lambda}{2}+u)\ \sin(\xi(a)+\frac{\lambda}{2}+u)\ f(a,u)\,, & b\in\nu(a) \\ \sin(\xi(a)-\frac{\lambda}{2}-u)\ \sin(\xi(a)+\frac{\lambda}{2}-u)\ f(a,u)\,, & b\in\nu'(a)\,, \end{cases} \tag{4.4}$$

where f is arbitrary and, for each a, $\nu(a)$ and $\nu'(a)$ are any non-intersecting sets whose union is the set of neighbours of a on $\hat{\mathcal{G}}$, and $\xi(a)$ is any solution of

$$\tan 2\xi(a) = \frac{\sin 4\lambda}{\cos 4\lambda + \sum_{d\,\in\,\nu(a)} S_d/S_a}\,. \tag{4.5}$$

We now prove that these boundary weights represent the general diagonal solution of the BYBE with bulk weights (4.1).

Setting $B\Big(b\ {a \atop c}\Big|u\Big) = K_a(b,u)\,\delta_{ac}$ and using (4.1), we find that the only classes of spin assignments in (2.2) which lead to non-trivial equations are $a=d=e$ and $b=c$ with $b\in\mathcal{N}(a)$, $a=c=d=e$ with $b\in\mathcal{N}(a)$, and $a=c=e$ with $b,d\in\mathcal{N}(a)$, where $\mathcal{N}(a)$ is the set of neighbours of a on $\hat{\mathcal{G}}$. These give, respectively,

$$\begin{aligned} 0 \;=\; \mathcal{E}^1_a(b) \;=\; &\sin(u{-}v)\,(K_a(a,u)\,K_a(a,v)-K_a(b,u)\,K_a(b,v)) \\ &+\;\sin(u{+}v)\,(K_a(b,u)\,K_a(a,v)-K_a(a,u)\,K_a(b,v))\;, \end{aligned} \tag{4.6a}$$

$$\begin{aligned} 0 \;=\; \mathcal{E}^2_a(b) \;=\; &\rho_4(u{-}v)\,\rho_1(u{+}v)\,K_a(a,u)\,K_a(a,v)\;-\;\rho_1(u{-}v)\,\rho_4(u{+}v)\,K_a(a,u)\,K_a(b,v) \\ &+\rho_8(u{-}v)\,\rho_4(u{+}v)\,K_a(b,u)\,K_a(a,v)\;-\;\rho_4(u{-}v)\,\rho_8(u{+}v)\,K_a(b,u)\,K_a(b,v) \\ &+\sum_{c\in\mathcal{N}(a)} S_c\,K_a(c,u)/S_a\,(\rho_9(u{-}v)\,\rho_4(u{+}v)\,K_a(a,v)-\rho_4(u{-}v)\,\rho_9(u{+}v)\,K_a(b,v))\;, \end{aligned} \tag{4.6b}$$

and

$$\begin{aligned} 0 \;=\; \mathcal{E}^3_a(b,d) \;=\; &\rho_9(u{-}v)\,\rho_8(u{+}v)\,(K_a(b,u)\,K_a(b,v)-K_a(d,u)\,K_a(d,v))\;+ \\ &\rho_8(u{-}v)\,\rho_9(u{+}v)\,(K_a(d,u)\,K_a(b,v)-K_a(b,u)\,K_a(d,v))\;+ \\ &\Big(\rho_4(u{-}v)\,\rho_4(u{+}v)\,K_a(a,u)+\rho_9(u{-}v)\,\rho_9(u{+}v)\sum_{c\in\mathcal{N}(a)}S_c\,K_a(c,u)/S_a\Big)(K_a(b,v)-K_a(d,v)). \end{aligned} \tag{4.6c}$$

We shall from now on treat a as fixed. If a has n neighbours on $\hat{\mathcal{G}}$ then (4.6a) and (4.6b) each provide n equations and (4.6c) provides $n(n-1)/2$ equations for the $n+1$ boundary weights, $K_a(a,u)$ and $K_a(b,u)$ with $b\in\mathcal{N}(a)$.

The general solution of a single case of (4.6a) may be obtained by setting $K_a(a,u) = \cos(u{-}\chi)\,k_1(u){+}\sin(u{-}\chi)\,k_2(u)$ and $K_a(b,u)=\cos(u{+}\chi)\,k_1(u){-}\sin(u{+}\chi)\,k_2(u)$ so that (4.6a) becomes $\sin 2u\,\sin 2v\,(k_1(u)\,k_2(v)-k_2(u)\,k_1(v))=0$. This then gives

$$\begin{aligned} K_a(a,u) \;&=\; (x_1\cos(u{-}\chi)+x_2\sin(u{-}\chi))\,f(u) \\ K_a(b,u) \;&=\; (x_1\cos(u{+}\chi)-x_2\sin(u{+}\chi))\,f(u)\;, \end{aligned}$$

where x_1 and x_2 are arbitrary constants, f is an arbitrary function, and χ may be set to any fixed value, which we shall take here as $\chi=\lambda/2$. Therefore, the general solution of the system of equations (4.6a) is

$$\begin{aligned} K_a(a,u) \;&=\; \prod_{c\in\mathcal{N}(a)}(x_1(c)\cos(u{-}\tfrac{\lambda}{2})+x_2(c)\sin(u{-}\tfrac{\lambda}{2}))\,f(u) \\ K_a(b,u) \;&=\; (x_1(b)\cos(u{+}\tfrac{\lambda}{2})-x_2(b)\sin(u{+}\tfrac{\lambda}{2})) \\ &\qquad\times\prod_{c\in\mathcal{N}(a)-\{b\}}(x_1(c)\cos(u{-}\tfrac{\lambda}{2})+x_2(c)\sin(u{-}\tfrac{\lambda}{2}))f(u), \end{aligned} \tag{4.7}$$

where x_1, x_2 and f are arbitrary.

We also observe that

$$0 \;=\; \sin(u-v-\lambda)\,(\mathcal{E}^2_a(b) \;-\; \mathcal{E}^2_a(d)) \;+\; \sin2\lambda\,\mathcal{E}^3_a(b,d) \;=\; \sin(u+v)\,\sin(u-v-2\lambda)\sin(u-v-3\lambda)/(\sin2\lambda\,\sin^2 3\lambda)\,\mathcal{E}^4_a(b,d)\;,$$

where

$$\begin{aligned}\mathcal{E}^4_a(b,d) \;&=\; \sin(u-v+\lambda)\,K_a(a,u)\,(K_a(b,v)-K_a(d,v))\\ &\quad+\;\sin(u-v-\lambda)\,(K_a(b,u)-K_a(d,u))\,K_a(a,v)\\ &\quad-\;\sin(u+v-\lambda)\,(K_a(b,u)\,K_a(d,v)-K_a(d,u)\,K_a(b,v))\;.\end{aligned}$$

Using (4.7), we now find that

$$\begin{aligned}0 \;=\; \mathcal{E}^4_a(b,d) \;=\; &\prod_{c\in\mathcal{N}(a)-\{b,d\}} (x_1(c)\cos(u-\tfrac{\lambda}{2})+x_2(c)\sin(u-\tfrac{\lambda}{2}))(x_1(c)\cos(v-\tfrac{\lambda}{2})+x_2(c)\sin(v-\tfrac{\lambda}{2}))\\ &\times\,(x_1(b)\,x_2(d)-x_1(d)\,x_2(b))\,(x_1(b)\,x_2(d)+x_1(d)\,x_2(b))\,\sin2u\,\sin2v\,\sin(u-v)\,f(u)\,f(v)\;.\end{aligned}$$

The general solution of this system of equations is

$$x_1(b) \;=\; \tilde{x}_1\,y(b)\;,\qquad x_2(b) \;=\; \begin{cases} \;\;\tilde{x}_2\,y(b)\,, & b\in\nu(a)\\ -\,\tilde{x}_2\,y(b)\,, & b\in\nu'(a)\;,\end{cases}$$

where $\tilde{x}_1$, $\tilde{x}_2$ and y are arbitrary and $\nu(a)$ and $\nu'(a)$ are any non-intersecting sets whose union is $\mathcal{N}(a)$. This gives

$$K_a(a,u) \;=\; (\tilde{x}_1\cos(u-\tfrac{\lambda}{2})+\tilde{x}_2\sin(u-\tfrac{\lambda}{2}))\,(\tilde{x}_1\cos(u-\tfrac{\lambda}{2})-\tilde{x}_2\sin(u-\tfrac{\lambda}{2}))\,\tilde{f}(u)$$

$$K_a(b,u) \;=\; \begin{cases}(\tilde{x}_1\cos(u+\tfrac{\lambda}{2})-\tilde{x}_2\sin(u+\tfrac{\lambda}{2}))\,(\tilde{x}_1\cos(u-\tfrac{\lambda}{2})-\tilde{x}_2\sin(u-\tfrac{\lambda}{2}))\,\tilde{f}(u)\,, & b\in\nu(a)\\ (\tilde{x}_1\cos(u+\tfrac{\lambda}{2})+\tilde{x}_2\sin(u+\tfrac{\lambda}{2}))\,(\tilde{x}_1\cos(u-\tfrac{\lambda}{2})+\tilde{x}_2\sin(u-\tfrac{\lambda}{2}))\,\tilde{f}(u)\,, & b\in\nu'(a)\;,\end{cases}$$

where

$$\tilde{f}(u) \;=\; (\tilde{x}_1\cos(u-\tfrac{\lambda}{2})+\tilde{x}_2\sin(u-\tfrac{\lambda}{2}))^{|\nu(a)|-1}\,(\tilde{x}_1\cos(u-\tfrac{\lambda}{2})-\tilde{x}_2\sin(u-\tfrac{\lambda}{2}))^{|\nu'(a)|-1}\prod_{c\in\mathcal{N}(a)} y(c)\,f(u).$$

We now see that (4.6c) is automatically satisfied for $b,d\in\nu(a)$, or $b,d\in\nu'(a)$, while for $b\in\nu(a)$ and $d\in\nu'(a)$ we have

$$0 \;=\; \mathcal{E}^3_a(b,d) \;=\; 2\sin2u\,\sin2v\,\sin(u-v)\,\sin(u+v)\,\sin(u-v-\lambda)\,\sin(u+v-\lambda)/(\sin2\lambda\,\sin3\lambda)^2 \times\,\tilde{f}(u)\,\tilde{f}(v)\,\mathcal{P}$$

$$0 \;=\; \mathcal{E}^2_a(b) \;=\; (\tilde{x}_1\cos(u-\tfrac{\lambda}{2})-\tilde{x}_2\sin(u-\tfrac{\lambda}{2}))\,(\tilde{x}_1\cos(v-\tfrac{\lambda}{2})-\tilde{x}_2\sin(v-\tfrac{\lambda}{2}))\times\,\sin2u\,\sin2v\,\sin(u-v)\,\sin(u+v)/(\sin2\lambda\,\sin^2 3\lambda)\,\tilde{f}(u)\,\tilde{f}(v)\,\mathcal{P}$$

$$0 \;=\; \mathcal{E}^2_a(d) \;=\; (\tilde{x}_1\cos(u-\tfrac{\lambda}{2})+\tilde{x}_2\sin(u-\tfrac{\lambda}{2}))\,(\tilde{x}_1\cos(v-\tfrac{\lambda}{2})+\tilde{x}_2\sin(v-\tfrac{\lambda}{2}))\times\,\sin2u\,\sin2v\,\sin(u-v)\,\sin(u+v)/(\sin2\lambda\,\sin^2 3\lambda)\,\tilde{f}(u)\,\tilde{f}(v)\,\mathcal{P}\;,$$

where

$$\mathcal{P} \;=\; \sin4\lambda\,(\tilde{x}_1^2 \,-\, \tilde{x}_2^2) \,-\, 2\left(\cos4\lambda + \sum_{d\,\in\,\nu(a)} S_d/S_a\right)\tilde{x}_1\,\tilde{x}_2\;,$$

and we have used the eigenvector equation

$$\sum_{b\,\in\,\mathcal{N}(a)} S_b/S_a \;=\; -2\,\cos4\lambda\;.$$

We must therefore set $\mathcal{P}=0$, the general solution of which is

$$\tilde{x}_1 \;=\; z\,\sin\xi\,, \qquad \tilde{x}_2 \;=\; -\,z\,\cos\xi\;,$$

where z is arbitrary and ξ is any solution of

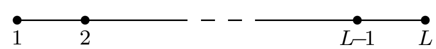
$$\tan 2\xi \;=\; \frac{\sin4\lambda}{\cos4\lambda + \sum_{d\,\in\,\nu(a)} S_d/S_a} \;=\; -\,\frac{\sin4\lambda}{\cos4\lambda + \sum_{d\,\in\,\nu'(a)} S_d/S_a}\;.$$

This now leads to the general solution (4.4), where $\xi(a)=\xi$ and $f(a,u)=z^2\,\tilde{f}(u)$.

5. Elliptic Models

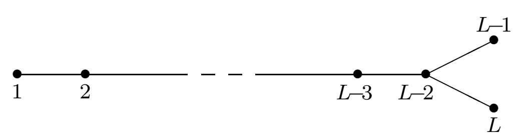
We now consider elliptic extensions of the previous models.

The only Temperley-Lieb models for which such extensions are known are those with adjacency graphs

$A_L \;=$ 

(5.1)

and

$D_L \;=$ 

. (5.2)

It is known that for these cases we may set

$$S_a \;=\; \begin{cases} \sin a\lambda\;; & A_L,\; a=1,\ldots,L\;; \quad D_L,\; a=1,\ldots,L-2 \\ 1/2\;; & D_L,\; a=L-1, L\;, \end{cases} \tag{5.3}$$

with

$$\lambda \;=\; \begin{cases} \dfrac{\pi}{L+1}\;; & A_L \\[2ex] \dfrac{\pi}{2(L-1)}\;; & D_L\;. \end{cases} \tag{5.4}$$

The bulk weights and diagonal boundary weights for the elliptic extensions of these models will be expressed in terms of λ, as given by (5.4), and elliptic theta functions of fixed nome q, with $|q|<1$, as given by

$$
\begin{aligned}
\vartheta_1(u) &= 2\,q^{1/4}\sin u \prod_{n=1}^{\infty}\Big(1-2\,q^{2n}\cos 2u+q^{4n}\Big)\Big(1-q^{2n}\Big)\;, \quad \vartheta_2(u) = \vartheta_1(u+\tfrac{\pi}{2}) \\
\vartheta_3(u) &= \prod_{n=1}^{\infty}\Big(1+2\,q^{2n-1}\cos 2u+q^{4n-2}\Big)\Big(1-q^{2n}\Big)\;, \quad \vartheta_4(u) = \vartheta_3(u+\tfrac{\pi}{2})\;.
\end{aligned}
\tag{5.5}
$$

These bulk weights are [16, 17]

$$
\begin{aligned}
W\!\left(\begin{matrix} a\pm1 & a \\ a & a\mp1 \end{matrix}\,\middle|\, u\right) &= \frac{\vartheta_1(\lambda-u)}{\vartheta_1(\lambda)} \\
W\!\left(\begin{matrix} a & a\pm1 \\ a\mp1 & a \end{matrix}\,\middle|\, u\right) &= \left(\frac{\vartheta_1((a-1)\lambda)\,\vartheta_1((a+1)\lambda)}{\vartheta_1(a\lambda)^2}\right)^{1/2}\frac{\vartheta_1(u)}{\vartheta_1(\lambda)} \\
W\!\left(\begin{matrix} a & a\pm1 \\ a\pm1 & a \end{matrix}\,\middle|\, u\right) &= \frac{\vartheta_1(a\lambda\pm u)}{\vartheta_1(a\lambda)}\;,
\end{aligned}
\tag{5.6a}
$$

which applies to all cases of A_L, and all except the following cases of D_L:

$$
\begin{aligned}
& W\!\left(\begin{matrix} L-1 & L-2 \\ L-2 & L-3 \end{matrix}\,\middle|\, u\right) = W\!\left(\begin{matrix} L-3 & L-2 \\ L-2 & L-1 \end{matrix}\,\middle|\, u\right) \\
&\qquad = W\!\left(\begin{matrix} L & L-2 \\ L-2 & L-3 \end{matrix}\,\middle|\, u\right) = W\!\left(\begin{matrix} L-3 & L-2 \\ L-2 & L \end{matrix}\,\middle|\, u\right) = \frac{\vartheta_1(\lambda-u)}{\vartheta_1(\lambda)} \\
& W\!\left(\begin{matrix} L & L-2 \\ L-2 & L-1 \end{matrix}\,\middle|\, u\right) = W\!\left(\begin{matrix} L-1 & L-2 \\ L-2 & L \end{matrix}\,\middle|\, u\right) = \frac{\vartheta_1(\lambda)\,\vartheta_2(u)-\vartheta_2(\lambda)\,\vartheta_1(u)}{\vartheta_2(0)\,\vartheta_1(\lambda)} \\
& W\!\left(\begin{matrix} L-2 & L-1 \\ L-3 & L-2 \end{matrix}\,\middle|\, u\right) = W\!\left(\begin{matrix} L-2 & L-3 \\ L-1 & L-2 \end{matrix}\,\middle|\, u\right) \\
&\qquad = W\!\left(\begin{matrix} L-2 & L \\ L-3 & L-2 \end{matrix}\,\middle|\, u\right) = W\!\left(\begin{matrix} L-2 & L-3 \\ L & L-2 \end{matrix}\,\middle|\, u\right) = \left(\frac{\vartheta_2(0)\,\vartheta_2(2\lambda)}{2\,\vartheta_2(\lambda)^2}\right)^{1/2}\frac{\vartheta_1(u)}{\vartheta_1(\lambda)} \\
& W\!\left(\begin{matrix} L-2 & L \\ L-1 & L-2 \end{matrix}\,\middle|\, u\right) = W\!\left(\begin{matrix} L-2 & L-1 \\ L & L-2 \end{matrix}\,\middle|\, u\right) = \frac{1}{2}\left(\frac{\vartheta_2(\lambda-u)}{\vartheta_2(\lambda)}-\frac{\vartheta_1(\lambda-u)}{\vartheta_1(\lambda)}\right) \\
& W\!\left(\begin{matrix} L-2 & L-1 \\ L-1 & L-2 \end{matrix}\,\middle|\, u\right) = W\!\left(\begin{matrix} L-2 & L \\ L & L-2 \end{matrix}\,\middle|\, u\right) = \frac{1}{2}\left(\frac{\vartheta_2(\lambda-u)}{\vartheta_2(\lambda)}+\frac{\vartheta_1(\lambda-u)}{\vartheta_1(\lambda)}\right) \\
& W\!\left(\begin{matrix} L-1 & L-2 \\ L-2 & L-1 \end{matrix}\,\middle|\, u\right) = W\!\left(\begin{matrix} L & L-2 \\ L-2 & L \end{matrix}\,\middle|\, u\right) = \frac{\vartheta_1(\lambda)\,\vartheta_2(u)+\vartheta_2(\lambda)\,\vartheta_1(u)}{\vartheta_2(0)\,\vartheta_1(\lambda)}\;.
\end{aligned}
\tag{5.6b}
$$

These bulk weights are known to satisfy the YBE for any fixed q. For $q\to0$, and using (5.3) and (5.4), they reduce to the bulk weights (3.1) of the corresponding Temperley-Lieb models.

Diagonal boundary weights are

$$B\left(a{\pm}1\ \begin{matrix}a\\a\end{matrix}\,\middle|\, u\right) \;=\; \big(x_1(a)\,\vartheta_1(u)\,\vartheta_1(u{\mp}a\lambda)\;+\;x_2(a)\,\vartheta_4(u)\,\vartheta_4(u{\mp}a\lambda)\big)\,f(a,u)\;, \qquad (5.7a)$$

which applies to all cases of A_L, and all except the following cases of D_L:

$$\begin{aligned} B\left(a\ \begin{matrix}L{-}2\\L{-}2\end{matrix}\,\middle|\, u\right) &= K_i(a,u)\,f(L{-}2,u)\;, \qquad a=L{-}3,L{-}1,L \\ B\left(L{-}2\ \begin{matrix}a\\a\end{matrix}\,\middle|\, u\right) &= \vartheta_3(u)\,\vartheta_4(u)\,f(a,u)\;, \qquad a=L{-}1,L\;. \end{aligned} \qquad (5.7b)$$

Here, x_1, x_2 and f are arbitrary, i may be chosen arbitrarily as 1 or 2, and K is given by

$$\begin{aligned} K_1(L{-}3,u) &= x_1(L{-}2)\,\vartheta_1(u)\,\vartheta_2(u{-}\lambda)+x_2(L{-}2)\,\vartheta_4(u)\,\vartheta_3(u{-}\lambda)\\ K_1(L{-}1,u) &= K_1(L,u) \;=\; K_1(L{-}3,-u)\\ K_2(L{-}3,u) &= \big(x_1(L{-}2)\,\vartheta_3(u{-}\lambda)-x_2(L{-}2)\,\vartheta_4(u{-}\lambda)\big)\times\\ &\qquad\qquad \big(x_1(L{-}2)\,\vartheta_3(u{-}\lambda)+x_2(L{-}2)\,\vartheta_4(u{-}\lambda)\big)\,\vartheta_3(u)\,\vartheta_4(u)\\ K_2(L{-}1,u) &= \big(x_1(L{-}2)\,\vartheta_3(u{-}\lambda)-x_2(L{-}2)\,\vartheta_4(u{-}\lambda)\big)\times\\ &\qquad\qquad \big(x_1(L{-}2)\,\vartheta_3(u{+}\lambda)+x_2(L{-}2)\,\vartheta_4(u{+}\lambda)\big)\,\vartheta_3(u)\,\vartheta_4(u)\\ K_2(L,u) &= K_2(L{-}1,-u)\;. \end{aligned}$$

It can be shown, using a method similar to that of Section 3, that these boundary weights represent the general diagonal solution of the BYBE with bulk weights (5.6a)–(5.6b) for any fixed q. For $q \to 0$, and using (5.3), (5.4) and appropriate choices of the arbitrary parameters, they reduce to the boundary weights (3.4) of the corresponding Temperley-Lieb models. In particular, in (5.7a) and in (5.7b) with $i=1$, we should set $x_1(a)=\tilde{x}_1(a)/q^{1/2}$, while for $i=2$ in (5.7b) we should set $x_1(L{-}3)=\vartheta_3(0)\,(\vartheta_4(0)^2\tilde{x}_1+\vartheta_3(0)^2\tilde{x}_2)/q^{1/2}$ and $x_2(L{-}3)=\pm\vartheta_4(0)\,(\vartheta_3(0)^2\tilde{x}_1+\vartheta_4(0)^2\tilde{x}_2)/q^{1/2}$, with $\tilde{x}_1$, $\tilde{x}_2$ and the $+$ or $-$ arbitrary, so that

$$\begin{aligned} K_2(L{-}3,u) &\;\to\; -(\tilde{x}_1{+}\tilde{x}_2)^2\,\sin(u{-}\lambda)^2\;+\;2\,(\tilde{x}_1{+}\tilde{x}_2)\,\tilde{x}_2\\ K_2(L{-}1,u) \;=\; K_2(L,-u) &\;\to\; -(\tilde{x}_1{+}\tilde{x}_2)^2\,\sin(u{\mp}\lambda)^2\;+\;2\,(\tilde{x}_1{+}\tilde{x}_2)\,\tilde{x}_2\;. \end{aligned}$$

Proceeding now to the dilute Temperley-Lieb models, the only cases for which elliptic extensions are known are those with undiluted adjacency graphs A_L. The bulk weights for these extensions are given in [18]. They satisfy the YBE for any fixed value of the elliptic nome q, and for $q \to 0$ they reduce to the bulk weights (4.1) of the corresponding dilute Temperley-Lieb models.

Diagonal boundary weights for these models are given in [8]. It can be shown, using a method similar to that of Section 4, that they represent the general diagonal solution of the BYBE with the bulk weights of [18] for any fixed q, and that for $q \to 0$ they reduce to the boundary weights (4.4) of the corresponding dilute Temperley-Lieb models.

6 . Discussion

We have presented the general diagonal solutions of the BYBE for the Temperley-Lieb and dilute Temperley-Lieb models and for their known elliptic extensions. We now discuss various related issues:

• *Additional relations*. In addition to satisfying the BYBE, the diagonal boundary weights presented here also satisfy the *boundary crossing relation*

$$\eta_a(u)\; B\!\left(b \;\begin{matrix} a \\ c \end{matrix}\;\middle|\; u\right) \;=\; \sum_{\substack{d \\ A_{ad}A_{cd}=1}} \left(\frac{s_d^2}{s_a\, s_c}\right)^{1/2} W\!\left(\begin{matrix} b & a \\ c & d \end{matrix}\;\middle|\; 2u\right)\; B\!\left(d \;\begin{matrix} a \\ c \end{matrix}\;\middle|\; \mu{-}u\right)\;, \tag{6.1}$$

for all u and all a, b, c satisfying $A_{ab}\, A_{bc} = 1$, and the *boundary inversion relation*,

$$\sum_{\substack{d \\ A_{bd}=1}} B\!\left(b \;\begin{matrix} d \\ c \end{matrix}\;\middle|\; u\right)\; B\!\left(b \;\begin{matrix} a \\ d \end{matrix}\;\middle|\; {-u}\right) \;=\; \rho_a(u)\, \delta_{ac}\;, \tag{6.2}$$

for all u and all a, b, c satisfying $A_{ab}\, A_{bc} = 1$, where μ, s, η and ρ are fixed for a particular model. For the models considered here, these parameters and functions are specified in [19, 20].

• *Algebraic properties*. It is known that the bulk weights of the Temperley-Lieb models and dilute Temperley-Lieb models can be expressed in terms of representations of the *Temperley-Lieb algebra* [21] and *dilute Temperley-Lieb algebra* [22] respectively and that they satisfy the YBE through the defining relations of these algebras alone.

We have similarly found that certain cases of the boundary weights (3.4) and (4.4) can also be expressed in terms of such representations and that the BYBE is satisfied through the corresponding algebraic relations alone. Specifically, these cases may be obtained by setting $x_1(a) = 0$, $x_2(a) = 1$ and $f(a,u) = g(u)$, for each a in (3.4), and setting $\nu(a)$ equal to the set of neighbours of a on $\hat{\mathcal{G}}$, $\nu'(a) = \emptyset$, $f(a,u) = g(u)$ and $\xi(a) = -2\lambda$ or $\xi(a) = -2\lambda + \pi/2$, for each a in (4.4), so that

$$B\!\left(b \;\begin{matrix} a \\ c \end{matrix}\;\middle|\; u\right) \;=\; \begin{cases} \delta_{ac}\, g(u)\;, & \text{Temperley-Lieb} \\[2ex] (\hat\rho_1(u)\, \delta_{abc} \;+\; \hat\rho_2(u)\, \delta_{ac}\, \hat{A}_{ab})\, g(u)\;, & \text{dilute Temperley-Lieb}\;, \end{cases} \tag{6.3}$$

where g is arbitrary, $\hat\rho_1(u) \;=\; h(\frac{5\lambda}{2}{-}u)\, h(\frac{3\lambda}{2}{+}u)$, $\hat\rho_2(u) \;=\; h(\frac{5\lambda}{2}{-}u)\, h(\frac{3\lambda}{2}{-}u)$, and h is sin or cos.

• *Non-diagonal solutions*. Obtaining non-diagonal solutions of the BYBE is generally more difficult than obtaining diagonal solutions, and the only interaction-round-a-face models for which non-diagonal solutions are currently known are the A_L models. Specifically, we have found, using a method similar to that of Section 3, that the general non-diagonal solution of the BYBE with A_L bulk weights (5.6a) is

$$B\left(a \begin{array}{c} a\pm1 \\ a\pm1 \end{array} \middle| u\right) = K(a,\pm u)\, f^{\pi_a}(u)$$
$$B\left(a \begin{array}{c} a\mp1 \\ a\pm1 \end{array} \middle| u\right) = \kappa_\pm(a)\, k(a,u)\, f^{\pi_a}(u)\,, \tag{6.4}$$

where π_a is the parity (even or odd) of a, f is arbitrary, K is given by

$$K(a,u) = \Big(x_1^{\pi_a}\,\vartheta_1(u{+}a\lambda)\,\vartheta_2(u)\,\vartheta_3(u)\,\vartheta_4(u) + x_2^{\pi_a}\,\vartheta_1(u)\,\vartheta_2(u{+}a\lambda)\,\vartheta_3(u)\,\vartheta_4(u)$$
$$+ x_3^{\pi_a}\,\vartheta_1(u)\,\vartheta_2(u)\,\vartheta_3(u{+}a\lambda)\,\vartheta_4(u) + x_4^{\pi_a}\,\vartheta_1(u)\,\vartheta_2(u)\,\vartheta_3(u)\,\vartheta_4(u{+}a\lambda)\Big)\Big/\vartheta_1(a\lambda)\,,$$

k is given by

$$k(a,u) = \frac{\vartheta_1(2u)}{\vartheta_1(2\sigma^{\pi_a})}\,,$$

κ_- and κ_+ must satisfy

$$\kappa_-(a)\,\kappa_+(a) = K(a,-\sigma^{\pi_a})\,K(a,\sigma^{\pi_a})\,,$$

and x_1, x_2, x_3, x_4 and σ must satisfy

$$x_1^{\text{even}} + x_2^{\text{even}} + x_3^{\text{even}} + x_4^{\text{even}} = 0$$
$$x_1^{\pi_{L+1}} + x_2^{\pi_{L+1}} - x_3^{\pi_{L+1}} - x_4^{\pi_{L+1}} = 0$$
$$\left(\gamma_{1,-}^{\text{odd}}x_1^{\text{odd}} + \gamma_{2,-}^{\text{odd}}x_2^{\text{odd}} + \gamma_{3,-}^{\text{odd}}x_3^{\text{odd}} + \gamma_{4,-}^{\text{odd}}x_4^{\text{odd}}\right)\times$$
$$\left(\gamma_{1,+}^{\text{odd}}x_1^{\text{odd}} + \gamma_{2,+}^{\text{odd}}x_2^{\text{odd}} + \gamma_{3,+}^{\text{odd}}x_3^{\text{odd}} + \gamma_{4,+}^{\text{odd}}x_4^{\text{odd}}\right) = 0$$
$$\left(\gamma_{1,-}^{\pi_L}x_1^{\pi_L} + \gamma_{2,-}^{\pi_L}x_2^{\pi_L} - \gamma_{3,-}^{\pi_L}x_3^{\pi_L} - \gamma_{4,-}^{\pi_L}x_4^{\pi_L}\right)\left(\gamma_{1,+}^{\pi_L}x_1^{\pi_L} + \gamma_{2,+}^{\pi_L}x_2^{\pi_L} - \gamma_{3,+}^{\pi_L}x_3^{\pi_L} - \gamma_{4,+}^{\pi_L}x_4^{\pi_L}\right) = 0\,,$$

with

$$\gamma_{i,\pm}^{\pi} = \frac{\vartheta_i(\sigma^\pi \pm \lambda)}{\vartheta_i(\sigma^\pi)}\,.$$

Further properties of these solutions are given in [19, 20].

• *Alternative methods.* The BYBE solutions presented here were all derived by direct consideration of the relevant equations, however some of these solutions include cases which can also be obtained by other, indirect means. Specifically, these alternative methods are the construction of new solutions from simpler solutions using fusion [7], which can be shown to lead to certain cases of (5.7a) and (6.4), the generation of solutions for an interaction-round-a-face model using known solutions for a related vertex model and vertex–face intertwiners [3], which can be shown to lead to certain cases of (6.4), and the generation of solutions for an interaction-round-a-face model using known solutions for another such model and face–face intertwiners [19, 20], which can be shown to relate certain cases of (5.7a) for A_{2L-3} with certain cases of (5.7a)–(5.7b) for D_L.

Although it seems that a direct approach is usually needed in order to obtain completely general solutions, it appears that indirect means are nonetheless useful for establishing the existence of solutions, and possibly of arbitrary parameters within solutions, for efficiently deriving particular solutions which are adequate for certain purposes, and for revealing more universal properties of the BYBE.

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