

# Surface Free Energies, Interfacial Tensions and Correlation Lengths of the ABF Models

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## Abstract

The surface free energies, interfacial tensions and correlation lengths of the Andrews-Baxter-Forrester models in Regimes III and IV are calculated with fixed boundary conditions. The interfacial tensions are calculated between arbitrary phases and are shown to be additive. The associated critical exponents are given by  $2 - \alpha_s = \mu = \nu$  with  $\nu = (L + 1)/4$  in Regime III and  $4 - 2\alpha_s = \mu = \nu$  with  $\nu = (L + 1)/2$  in Regime IV. Our results are obtained using general commuting transfer matrix and inversion relation methods that may be applied to other solvable lattice models.

## 1 Introduction

There has been a recent convergence of interest in statistical mechanics on systems with a boundary. It is well known that two-dimensional lattice models without a boundary are exactly solvable [1] by commuting transfer matrix methods if the local statistical weights satisfy the celebrated Yang-Baxter equation. It has also been known for some time, from the work of Cherednik [2] and Sklyanin [3], that this integrability extends to vertex models defined on a strip with open boundaries provided the local boundary weights satisfy an additional reflection equation or boundary Yang-Baxter equation. More recently, reflection equations for interaction-round-a-face (IRF) models have been introduced [4, 5, 6], and integrability has been established for lattice spin models defined on a strip with fixed or more general boundary conditions.

Once integrability with a boundary has been established there are various quantities of physical interest, such as the surface free energies and interfacial tensions, that one would like to calculate, and methods to achieve this need to be developed.

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In this direction surface free energies of the Andrews, Baxter and Forrester (ABF) models in Regime III [7, 8], the eight-vertex model [9], and the dilute  $A_L$  models [10] have been obtained by an extension of the inversion relation method [11, 12] used to calculate the bulk free energies.

In this paper we extend the analysis of [7] to obtain the surface free energy of the ABF models in Regime IV. In addition, we extend the generalized inversion relation method to calculate also the interfacial tensions and correlation lengths. In this way, we establish that for solvable lattice models it is possible to obtain the bulk free energies, the surface free energies, the interfacial tensions, the correlation lengths and their associated critical behaviours all by studying the relatively simple inversion relations.

The layout of the paper is as follows. For the rest of this section we follow reference [4] in describing the ABF models with fixed boundary conditions. In Sections 2 and 3, we obtain the bulk and surface free energies. The band structure and ground state degeneracies of the eigenvalue spectra of the transfer matrices are discussed in Section 4. The calculations of the interfacial tensions and correlation lengths are then given in Sections 5 and 6 respectively. We conclude with a brief discussion.

## 1.1 ABF models with fixed boundaries

The ABF models [13] are restricted solid-on-solid models in which heights on the sites of the square lattice take values in the set  $\{1, 2, 3, \dots, L\}$  subject to the condition that the values of heights on adjacent sites of the lattice must differ by  $\pm 1$ . The Boltzmann weights depend on a *crossing parameter*  $\lambda = \pi/(L + 1)$ , and a *spectral parameter*  $u$ . Of interest here are Regimes III and IV, in which we have  $0 < u < \lambda$ . The non-zero face weights are given by

$$W \begin{pmatrix} a \pm 1 & a \\ a & a \mp 1 \end{pmatrix} = \frac{\vartheta_1(\lambda - u)}{\vartheta_1(\lambda)} \quad (1.1)$$

$$W \begin{pmatrix} a & a \pm 1 \\ a \mp 1 & a \end{pmatrix} = \left( \frac{\vartheta_1((a - 1)\lambda)\vartheta_1((a + 1)\lambda)}{\vartheta_1^2(a\lambda)} \right)^{1/2} \frac{\vartheta_1(u)}{\vartheta_1(\lambda)} \quad (1.2)$$

$$W \begin{pmatrix} a & a \pm 1 \\ a \pm 1 & a \end{pmatrix} = \frac{\vartheta_1(a\lambda \pm u)}{\vartheta_1(a\lambda)}. \quad (1.3)$$

The  $\vartheta_1(u) = \vartheta_1(u, p)$  is a standard elliptic theta function with nome  $p$ . The temperature variable  $t = p^2$  measures the deviation from criticality. The critical limit of the ABF models is  $t \rightarrow 0$ , approached from  $t > 0$  in Regime III and  $t < 0$  in Regime IV. We therefore express the nome  $p$  in terms of a real parameter  $\varepsilon > 0$  as

$$p = \begin{cases} e^{-\pi\varepsilon}, & \text{Regime III} \\ ie^{-\pi\varepsilon}, & \text{Regime IV,} \end{cases} \quad (1.4)$$

so that  $t = \pm \exp(-2\pi\varepsilon)$ , and  $t \rightarrow 0$  as  $\varepsilon \rightarrow \infty$ . The product expansions of the functions  $\vartheta_1$  and  $\vartheta_4$  are given by

$$\vartheta_1(u, p) = 2p^{1/4} \sin u \prod_{n=1}^{\infty} (1 - 2p^{2n} \cos 2u + p^{4n})(1 - p^{2n}) \quad (1.5)$$

$$\vartheta_4(u, p) = \prod_{n=1}^{\infty} (1 - 2p^{2n-1} \cos 2u + p^{2(2n-1)})(1 - p^{2n}). \quad (1.6)$$

The  $\vartheta_1$  functions satisfy the quasiperiodicity properties

$$\vartheta_1(u + \pi, p) = -\vartheta_1(u, p) \quad (1.7)$$

$$\vartheta_1(u - i \ln p, p) = -p^{-1} e^{-2iu} \vartheta_1(u, p), \quad (1.8)$$

and the ‘‘conjugate modulus’’ transformations

$$\vartheta_1(u, e^{-\pi\varepsilon}) = \frac{1}{\sqrt{\varepsilon}} e^{-(u-\pi/2)^2/\pi\varepsilon} E(e^{-2u/\varepsilon}, e^{-2\pi/\varepsilon}) \quad (1.9)$$

$$\vartheta_1(u, ie^{-\pi\varepsilon}) = -\frac{1}{\sqrt{2\varepsilon}} e^{i\pi/8} e^{-(u+\pi/4)^2/\pi\varepsilon} E(e^{u/\varepsilon}, -e^{-\pi/2\varepsilon}) \quad (1.10)$$

$$E(x, p) = \prod_{n=1}^{\infty} (1 - p^{n-1}x)(1 - p^n x^{-1})(1 - p^n). \quad (1.11)$$

Following reference [4] we introduce boundary weights. These weights depend on an additional real parameter  $\xi$ , which is independent of  $u$  and may be different for the left and right boundaries. The non-zero boundary weights are

$$K \begin{pmatrix} a \pm 1 & a \\ a & a \end{pmatrix} = -\operatorname{sgn}(\xi) \left( \frac{\vartheta_1((a \pm 1)\lambda)}{\vartheta_1(a\lambda)} \right)^{1/2} \frac{\vartheta_1(u \pm \xi) \vartheta_1(u \mp a\lambda \mp \xi)}{\vartheta_1^2(\lambda)}. \quad (1.12)$$

From the face weights and boundary weights we construct a double-row transfer matrix  $\mathbf{D}(u)$ . For a lattice of width  $N$ , the entry of the transfer matrix corresponding to the rows of heights  $\mathbf{a} = \{a_1, \dots, a_{N+1}\}$  and  $\mathbf{b} = \{b_1, \dots, b_{N+1}\}$  is defined diagrammatically by

$$\langle \mathbf{a} | \mathbf{D}(u) | \mathbf{b} \rangle =$$

The solid heights  $\{c_1, \dots, c_{N+1}\}$  are summed over. As the boundary weights are diagonal, we must have  $a_1 = b_1$  and  $a_{N+1} = b_{N+1}$ . Furthermore, these boundary

heights, which we will call  $a^L$  and  $a^R$ , are fixed to the same values for all entries in the transfer matrix. The parameters  $\xi^L$  and  $\xi^R$  are similarly fixed for all entries. Defined in this way, the double-row transfer matrix exhibits the crossing symmetry

$$\mathbf{D}(\lambda - u) = \mathbf{D}(u). \quad (1.13)$$

More importantly, however, the double-row transfer matrices form a commuting family,

$$\mathbf{D}(u)\mathbf{D}(v) = \mathbf{D}(v)\mathbf{D}(u). \quad (1.14)$$

This implies that the eigenvectors of  $\mathbf{D}(u)$  are independent of  $u$ , so that functional equations satisfied by the transfer matrix are also satisfied by its eigenvalues. In particular, all eigenvalues of the transfer matrix satisfy the crossing symmetry (1.13). It should be emphasized that all the matrices in a commuting family share the same boundary heights  $a^L$  and  $a^R$ , and the same values of  $\xi^L$  and  $\xi^R$ .

To ensure that the largest eigenvalue of the double-row transfer matrix is non-degenerate for all  $0 < u < \lambda$ , we impose the restriction

$$\frac{\lambda}{2} \leq |\xi^L|, |\xi^R| < \lambda, \quad (1.15)$$

and in addition require that  $\xi^{L,R} > 0$  when  $a^{L,R} = 1$  and  $\xi^{L,R} < 0$  when  $a^{L,R} = L$ . Proof of the sufficiency of these restrictions proceeds as follows.

We first show that all the off-diagonal elements of  $\mathbf{D}(u)$  are non-negative. Consider an off-diagonal element  $\langle \mathbf{a} | \mathbf{D}(u) | \mathbf{b} \rangle$ , with  $\mathbf{a}$  and  $\mathbf{b}$  such that  $\langle \mathbf{a} | \mathbf{D}(u) | \mathbf{b} \rangle$  is not identically zero for all  $u$ . Since the element is off-diagonal, we must have  $a_i \neq b_i$  for some  $2 \leq i \leq N$ . We use the boundary crossing relation (equation (3.8) of ref. [4]) to replace the left boundary weight  $K(\lambda - u)$  with  $K(u)$ . This introduces a face with spectral parameter  $2u - \lambda$  and also a (positive) factor of  $\vartheta_1(\lambda)/\vartheta_1(2u)$ . We then use the Yang-Baxter equation (equation (3.4) of ref. [4]) to push this face to the right until it separates the  $(i - 1)$ -th and  $i$ -th pairs of faces. The upper and lower heights  $b_i$  and  $a_i$  differ, so from equation (1.1) the weight of this face is  $\vartheta_1(2\lambda - 2u)/\vartheta_1(\lambda)$ , which is positive for  $0 < u < \lambda$ . Since the other face weights are positive, and since the boundary weights  $K(u)$  are positive for  $0 < u < \min\{|\xi^L|, |\xi^R|\}$ , the entry  $\langle \mathbf{a} | \mathbf{D}(u) | \mathbf{b} \rangle$  is the sum of a sequence of positive numbers. The crossing symmetry (1.13) then dictates that what holds for  $u$  must hold for  $\lambda - u$ , and since  $\lambda/2 \leq \min\{|\xi^L|, |\xi^R|\}$ , the off-diagonal elements of  $\mathbf{D}(u)$  are non-negative for all  $0 < u < \lambda$ .

We now observe that the elements of  $\mathbf{D}(u)$  are bounded, so there exists a real number  $M > 0$  such that all elements of the matrix  $M\mathbf{I} + \mathbf{D}(u)$  are non-negative for  $0 < u < \lambda$ . The largest eigenvalue of this matrix is non-degenerate by the Perron-Frobenius theorem, and it follows immediately that the largest eigenvalue of  $\mathbf{D}(u)$  is also non-degenerate.

When  $u = \lambda/2$ , the symmetry of the face weights is such that the model is isotropic. In this case the values  $\xi = \pm\lambda/2$  deserve special mention since for these choices the isotropic model has all boundary heights fixed. This is easily seen from the definition (1.12) as, for fixed  $a$ , only one of the choices  $a \pm 1$  gives a non-zero boundary weight. The non-zero boundary weights then contribute only a constant factor to each entry of the transfer matrix. Aside from this trivial factor, the lattice exhibits pure fixed boundary conditions, with boundary heights alternating either  $\{a, a + 1, a, a + 1, \dots\}$  or  $\{a, a - 1, a, a - 1, \dots\}$ .

If we divide each of the face weights by  $\vartheta_1(u - \lambda/2)$ , and each of the boundary weights by  $\vartheta_1(u - \lambda/2)^2$ , then the quasiperiodicity (1.8) implies that replacing  $u$  by  $u - i \ln p$  simply introduces a gauge factor to each of the normalized weights. These gauge factors cancel in the entries of  $\mathbf{D}(u)$ , so it follows that the quasiperiodicity of  $\mathbf{D}(u)$  is that of  $\vartheta_1(u - \lambda/2)^{2N+4}$ . If we therefore define a normalized transfer matrix

$$\tilde{\mathbf{D}}(u) = \frac{\mathbf{D}(u)}{\vartheta_1(u - \lambda/2)^{2N+4}}, \quad (1.16)$$

then the entries and eigenvalues of  $\tilde{\mathbf{D}}(u)$  are doubly periodic functions of  $u$  with

$$\text{period rectangle} = \pi \times i\pi\varepsilon^*, \quad \varepsilon^* = \begin{cases} \varepsilon, & \text{Regime III} \\ 2\varepsilon, & \text{Regime IV.} \end{cases} \quad (1.17)$$

In Regime IV there is an additional symmetry within the period rectangle

$$\tilde{\mathbf{D}}(u \pm \pi/2 + i\pi\varepsilon) = \tilde{\mathbf{D}}(u). \quad (1.18)$$

As was shown in [4], the eigenvalues of the ABF models with fixed boundary conditions satisfy the inversion identity

$$s_{-1}s_1D(u)D(u + \lambda) = \varepsilon^L\varepsilon^R s_{-2}s_2f_{-1}f_1 + s_0^2f_0D^{1,2}(u), \quad (1.19)$$

where the functions  $s_k$  and  $f_k$  are given by

$$s_k = \frac{\vartheta_1(2u + k\lambda)}{\vartheta_1(\lambda)}, \quad f_k = (-1)^N \left[ \frac{\vartheta_1(u + k\lambda)}{\vartheta_1(\lambda)} \right]^{2N}. \quad (1.20)$$

The functions  $\varepsilon^L$  and  $\varepsilon^R$  depend on the left and right boundary conditions and, for generic  $a$  and  $\xi$ , take the form

$$\varepsilon^{L,R} = \frac{\vartheta_1(u - \xi)\vartheta_1(u + \xi)\vartheta_1(u - \xi - a\lambda)\vartheta_1(u + \xi + a\lambda)}{\vartheta_1(\lambda)^4}. \quad (1.21)$$

The function  $D^{1,2}(u)$  is an eigenvalue of the double-row transfer matrix at fusion level  $1 \times 2$ . We observe as in the periodic case that for large  $N$ ,  $f_0D^{1,2}/(f_{-1}f_1)$  is

exponentially small in  $-\lambda/2 < \text{Re}(u) < \lambda/2$ . Hence, for  $N$  large, the second term in the inversion identity (1.19) can be neglected. So in calculating bulk and surface properties we just need to solve the simple inversion relation

$$s_{-1}s_1 D(u)D(u+\lambda) = \epsilon^L \epsilon^R s_{-2}s_2 f_{-1}f_1 \quad (1.22)$$

and the crossing relation

$$D(u) = D(\lambda - u) \quad (1.23)$$

subject to the appropriate analyticity and quasiperiodicity in an open strip containing  $0 \leq \text{Re}(u) \leq \lambda$ . These properties determine the bulk and surface quantities uniquely.

The largest eigenvalue  $D_{\max}(u)$  factorizes into contributions from the bulk, the surfaces and the interface

$$D_{\max}(u) \sim \kappa_b(u)^{2N} \kappa_s(u) \Lambda(u) \quad \text{as } N \rightarrow \infty. \quad (1.24)$$

When the left and right boundary conditions coincide, there is no interface and  $\Lambda(u) = 1$ . When the left and right boundaries differ, there is an interface and then the factorization applies to all the eigenvalues in the first band. We say that the boundary conditions coincide when  $n(\alpha, \beta) = 0$  in the notation of Section 4. Clearly, the inversion and crossing relations factorize into bulk and surface terms and can be solved sequentially for the bulk free energies, the surface free energies ( $n(\alpha, \beta) = 0$ ) and the interfacial tensions ( $n(\alpha, \beta) \geq 1$ ). This we do in the following sections.

## 2 Bulk free energies

Equating the bulk terms in the inversion relation (1.22) implies that the bulk partition function per site satisfies the functional equation

$$\kappa_b(u)\kappa_b(u+\lambda) = \frac{\vartheta_1(\lambda-u)\vartheta_1(\lambda+u)}{\vartheta_1(\lambda)^2}. \quad (2.1)$$

This equation is of course independent of the boundary conditions on the lattice and agrees with the equation for periodic boundaries. It has been solved previously, but we include the solution here for completeness. We use the standard techniques developed by Stroganov [11] and Baxter [12].

Since the eigenvectors of  $\mathbf{D}(u)$  are independent of  $u$ , it follows that the eigenvalues  $D(u)$  have the same analyticity and quasiperiodicity as the elements of the transfer matrix. In particular, the eigenvalues must be entire functions of the spectral parameter  $u$ . They are therefore completely characterized by their zeros and growth at infinity. In Regime III the zeros of the largest eigenvalue accumulate on

the lines  $\text{Re}(u) = -\lambda/2$  and  $\text{Re}(u) = 3\lambda/2$ . In addition to these lines, in Regime IV zeros accumulate on  $\text{Re}(u) = (3\lambda - \pi)/2$  and  $\text{Re}(u) = (\pi - \lambda)/2$  in accordance with the periodicity (1.18). The strip  $-\lambda/2 < \text{Re}(u) < 3\lambda/2$  is free of order  $N$  zeros in both regimes. This is confirmed by numerical studies for large finite  $N$ . So inside this strip  $\kappa_b(u)$  is non-zero and  $\ln \kappa_b(u)$  is analytic. The quasiperiodicity

$$\kappa_b(u + i\pi\varepsilon^*) = \begin{cases} e^{i(\lambda-2u)\varepsilon^*} \kappa_b(u), & \text{Regime III} \\ e^{2i(\lambda-2u)\varepsilon^*} \kappa_b(u), & \text{Regime IV} \end{cases} \quad (2.2)$$

implies that the second derivative of  $\ln \kappa_b(u)$  is periodic, with period  $i\pi\varepsilon^*$ . Hence  $\ln \kappa_b(u)$  can be expanded in the form of a generalized Fourier series

$$\ln \kappa_b(u) = Au^2 + Bu + \sum_{k=-\infty}^{\infty} c_k e^{2ku/\varepsilon^*}. \quad (2.3)$$

To evaluate the coefficients  $c_k$ ,  $A$  and  $B$ , we take the logarithm in (2.1), expand the right hand side using the appropriate conjugate modulus transformations (1.9) and (1.10) and equate coefficients.

## 2.1 Regime III

In Regime III we have  $t > 0$ , so we rewrite the right hand side of (2.1) using the conjugate modulus transformation (1.9). With both sides of (2.1) expanded in powers of  $\exp(2u/\varepsilon)$ , we match coefficients and impose the crossing symmetry (1.23) to obtain the solution

$$\ln \kappa_b(u) = \frac{1}{\pi\varepsilon}(\lambda - u)u + 2 \sum_{k=1}^{\infty} \frac{\cosh[(\pi - 2\lambda)k/\varepsilon] \sinh[(\lambda - u)k/\varepsilon] \sinh(uk/\varepsilon)}{k \sinh(\pi k/\varepsilon) \cosh(\lambda k/\varepsilon)}. \quad (2.4)$$

Inside the region  $-\lambda/2 < \text{Re}(u) < 3\lambda/2$  this function gives the bulk behaviour of the largest eigenvalue of the transfer matrix. Applying the Poisson summation formula to the infinite sum gives the behaviour of the free energy in the critical limit  $t \rightarrow 0^+$ . When  $L$  is even  $\ln \kappa_b(u)$  is regular, but when  $L$  is odd the leading-order singularity is [13]

$$\ln \kappa_b \sim t^{\pi/2\lambda} \ln t. \quad (2.5)$$

Since  $\ln \kappa_b \sim t^{2-\alpha}$ , when  $L$  is odd the bulk critical exponent  $\alpha$  is given by

$$2 - \alpha = \frac{\pi}{2\lambda} = \frac{L+1}{2}. \quad (2.6)$$

## 2.2 Regime IV

In Regime IV the temperature variable  $t$  is negative, so we use the conjugate modulus transformation (1.10) to rewrite the right hand side of (2.1) in powers of  $\exp(u/\varepsilon)$ . As before, we match coefficients and use the crossing symmetry (1.23) to obtain

$$\begin{aligned} \ln \kappa_b(u) = & \frac{1}{\pi\varepsilon}(\lambda - u)u + \sum_{k=1}^{\infty} \frac{\cosh[(\pi/2 - 2\lambda)k/\varepsilon] \sinh[(\lambda - u)k/\varepsilon] \sinh(uk/\varepsilon)}{k \sinh(\pi k/2\varepsilon) \cosh(\lambda k/\varepsilon)} \\ & + \sum_{k+\frac{1}{2}=1}^{\infty} \frac{\sinh[(\pi/2 - 2\lambda)k/\varepsilon] \sinh[(\lambda - u)k/\varepsilon] \sinh(uk/\varepsilon)}{k \cosh(\pi k/2\varepsilon) \cosh(\lambda k/\varepsilon)}. \end{aligned} \quad (2.7)$$

Once again we apply the Poisson summation formula to obtain the leading order singularity [13]

$$\ln \kappa_b \sim \begin{cases} (-t)^{\pi/2\lambda} & L \text{ even,} \\ (-t)^{\pi/2\lambda} \ln(-t) & L \text{ odd,} \end{cases} \quad (2.8)$$

so that in Regime IV, the critical exponent  $\alpha$  is given by equation (2.6) for all  $L$ .

## 3 Surface free energies

In order to calculate the surface free energies, we assume that the boundary conditions on the left and the right edges of the lattice coincide, so that  $n(\alpha, \beta) = 0$  in the notation of Section 4. In this case there is a unique largest eigenvalue separated from the other eigenvalues by a gap. We can then divide out the bulk quantities in the inversion relation (1.22) to obtain an inversion relation for the surface partition function per site  $\kappa_s(u)$ . Explicitly,

$$\kappa_s(u)\kappa_s(u + \lambda) = \frac{\vartheta_1(2\lambda - 2u)\vartheta_1(2\lambda + 2u)}{\vartheta_1(\lambda - 2u)\vartheta_1(\lambda + 2u)} \epsilon^L(u)\epsilon^R(u). \quad (3.1)$$

The form of this inversion relation suggests a natural factorization of  $\kappa_s(u)$  into a term independent of the boundary conditions, a term dependent on the left boundary condition, and a term dependent on the right boundary condition. We therefore write

$$\kappa_s(u) = \kappa_s^0(u)\kappa_s^L(u)\kappa_s^R(u). \quad (3.2)$$

The solution of equation (3.1) proceeds in a similar fashion to the solution of the bulk inversion relation, but whereas the analyticity of  $\ln \kappa_b(u)$  depended on the absence of order  $N$  zeros, the analyticity of  $\ln \kappa_s(u)$  depends on the absence of order 1 zeros. Our numerical studies show that when  $n(\alpha, \beta) = 0$  the largest eigenvalue of the

double-row transfer matrix is indeed non-zero for  $0 \leq \text{Re}(u) \leq \lambda$ . We therefore conclude that  $\ln \kappa_s(u)$  is analytic on this strip, and along with the quasiperiodicity

$$\kappa_s(u + i\pi\varepsilon^*) = \begin{cases} e^{4i(\lambda-2u)} e^{4\pi\varepsilon^*} \kappa_s(u), & \text{Regime III} \\ e^{8i(\lambda-2u)} e^{8\pi\varepsilon^*} \kappa_s(u), & \text{Regime IV,} \end{cases} \quad (3.3)$$

this allows us to expand  $\ln \kappa_s(u)$  as a generalized Fourier series.

### 3.1 Regime III

In Regime III we use once again the conjugate modulus transformation (1.9) and the crossing symmetry (1.23) to match Fourier coefficients and obtain

$$\ln \kappa_s^0(u) = \frac{\lambda}{\pi\varepsilon}(\pi - 3\lambda) + 2 \sum_{k=1}^{\infty} \frac{\sinh[(\pi - 3\lambda)k/\varepsilon] \sinh(\lambda k/\varepsilon) \cosh[2(\lambda - 2u)k/\varepsilon]}{k \sinh(\pi k/\varepsilon) \cosh(2\lambda k/\varepsilon)}, \quad (3.4)$$

and for generic  $a$  and  $\xi$ ,

$$\begin{aligned} \ln \kappa_s^{\text{L,R}}(u) &= \frac{1}{\pi\varepsilon} [(a\lambda + \xi)(\pi - 2\xi) + (|\xi| - 2\lambda)\pi + (2 - a^2)\lambda^2 + 2u(\lambda - u)] \\ &- 2 \sum_{k=1}^{\infty} \frac{\cosh[(a\lambda + \xi - |\xi|)k/\varepsilon] \cosh[(\pi - a\lambda - \xi - |\xi|)k/\varepsilon] \cosh[(\lambda - 2u)k/\varepsilon]}{k \sinh(\pi k/\varepsilon) \cosh(\lambda k/\varepsilon)} \\ &+ 2 \sum_{k=1}^{\infty} \frac{\cosh[(\pi - 2\lambda)k/\varepsilon]}{k \sinh(\pi k/\varepsilon)}. \end{aligned} \quad (3.5)$$

We note that, as it should, the height reversal transformation  $a \rightarrow L + 1 - a$  and  $\xi \rightarrow -\xi$  leaves (3.5) unchanged.

Applying the Poisson summation formula to the above expressions for  $\ln \kappa_s^0(u)$  and  $\ln \kappa_s^{\text{L,R}}(u)$ , the dominant behaviour as  $t \rightarrow 0^+$  is seen to come from equation (3.4). If  $L \equiv 2 \pmod{4}$ , however, this term is regular and we must consider the sum in equation (3.5). This latter sum is also regular when  $\xi > 0$  with  $a$  odd and when  $\xi < 0$  with  $a$  even [7], but such exceptions aside, the leading-order singularities of the surface free energy have the form

$$\ln \kappa_s \sim \begin{cases} t^{\pi/4\lambda} & \text{if } L \equiv 0 \text{ or } 1 \pmod{4}, \\ t^{\pi/2\lambda} & \text{if } L \equiv 2 \pmod{4}, \\ t^{\pi/4\lambda} \ln t & \text{if } L \equiv 3 \pmod{4}. \end{cases} \quad (3.6)$$

Since  $\ln \kappa_s \sim t^{2-\alpha_s}$ , the surface critical exponent  $\alpha_s$  [16, 17] in Regime III is given by

$$2 - \alpha_s = \begin{cases} (L + 1)/2 & L \equiv 2 \pmod{4}, \\ (L + 1)/4 & \text{otherwise.} \end{cases} \quad (3.7)$$

### 3.2 Regime IV

In Regime IV we use the conjugate modulus transformation (1.10) to match coefficients and obtain

$$\begin{aligned} \ln \kappa_s^0(u) &= \frac{\lambda}{2\pi\varepsilon}(\pi - 6\lambda) + \sum_{k=1}^{\infty} \frac{\sinh[(\pi/2 - 3\lambda)k/\varepsilon] \sinh(\lambda k/\varepsilon) \cosh[2(\lambda - 2u)k/\varepsilon]}{k \sinh(\pi k/2\varepsilon) \cosh(2\lambda k/\varepsilon)} \\ &\quad + \sum_{k+\frac{1}{2}=1}^{\infty} \frac{\cosh[(\pi/2 - 3\lambda)k/\varepsilon] \sinh(\lambda k/\varepsilon) \cosh[2(\lambda - 2u)k/\varepsilon]}{k \cosh(\pi k/2\varepsilon) \cosh(2\lambda k/\varepsilon)}. \end{aligned} \quad (3.8)$$

The form of  $\ln \kappa_s^{\text{L,R}}(u)$  depends on the values of  $a$  and  $\xi$ . When either  $a \leq (L-1)/2$ , or  $L/2 \leq a \leq (L+2)/2$  with  $\xi < 0$ ,  $\ln \kappa_s^{\text{L,R}}(u)$  is given by

$$\begin{aligned} \ln \kappa_s^{\text{L,R}}(u) &= \frac{1}{2\pi\varepsilon} [(a\lambda + \xi)(\pi - 4\xi) + (|\xi| - 2\lambda)\pi + 2(2 - a^2)\lambda^2 + 4u(\lambda - u)] \\ &\quad + \sum_{k=1}^{\infty} \frac{\cosh[(\pi/2 - 2\lambda)k/\varepsilon]}{k \sinh(\pi/2\varepsilon)} + \sum_{k+\frac{1}{2}=1}^{\infty} \frac{\sinh[(\pi/2 - 2\lambda)k/\varepsilon]}{k \cosh(\pi k/2\varepsilon)} \\ &\quad - \sum_{k=1}^{\infty} \frac{\cosh[(a\lambda + \xi - |\xi|)k/\varepsilon] \cosh[(\pi/2 - a\lambda - \xi - |\xi|)k/\varepsilon] \cosh[(\lambda - 2u)k/\varepsilon]}{k \sinh(\pi k/2\varepsilon) \cosh(\lambda k/\varepsilon)} \\ &\quad - \sum_{k+\frac{1}{2}=1}^{\infty} \frac{\cosh[(a\lambda + \xi - |\xi|)k/\varepsilon] \sinh[(\pi/2 - a\lambda - \xi - |\xi|)k/\varepsilon] \cosh[(\lambda - 2u)k/\varepsilon]}{k \cosh(\pi k/2\varepsilon) \cosh(\lambda k/\varepsilon)}. \end{aligned} \quad (3.9)$$

On the other hand, when either  $a \geq (L+3)/2$ , or  $L/2 \leq a \leq (L+2)/2$  with  $\xi > 0$ ,  $\ln \kappa_s^{\text{L,R}}(u)$  is given by

$$\begin{aligned} \ln \kappa_s^{\text{L,R}}(u) &= \frac{1}{2\pi\varepsilon} [(a\lambda + \xi)(3\pi - 4\xi) + (|\xi| - 2\lambda - \pi)\pi + 2(2 - a^2)\lambda^2 + 4u(\lambda - u)] \\ &\quad + \sum_{k=1}^{\infty} \frac{\cosh[(\pi/2 - 2\lambda)k/\varepsilon]}{k \sinh(\pi k/2\varepsilon)} + \sum_{k+\frac{1}{2}=1}^{\infty} \frac{\sinh[(\pi/2 - 2\lambda)k/\varepsilon]}{k \cosh(\pi k/2\varepsilon)} \\ &\quad - \sum_{k=1}^{\infty} \frac{\cosh[(\pi - a\lambda - \xi - |\xi|)k/\varepsilon] \cosh[(a\lambda - \pi/2 + \xi - |\xi|)k/\varepsilon] \cosh[(\lambda - 2u)k/\varepsilon]}{k \sinh(\pi k/2\varepsilon) \cosh(\lambda k/\varepsilon)} \\ &\quad - \sum_{k+\frac{1}{2}=1}^{\infty} \frac{\cosh[(\pi - a\lambda - \xi - |\xi|)k/\varepsilon] \sinh[(a\lambda - \pi/2 + \xi - |\xi|)k/\varepsilon] \cosh[(\lambda - 2u)k/\varepsilon]}{k \cosh(\pi k/2\varepsilon) \cosh(\lambda k/\varepsilon)}. \end{aligned} \quad (3.10)$$

We note that once again  $\ln \kappa_s(u)$  is unchanged under the height reversal transformation  $a \rightarrow L + 1 - a$  and  $\xi \rightarrow -\xi$ .

The leading-order singular behaviour of the surface free energy is derived from the sums in equation (3.8). We find

$$\ln \kappa_s \sim \begin{cases} (-t)^{\pi/4\lambda} \ln(-t), & \text{if } L \equiv 3 \pmod{4} \\ (-t)^{\pi/4\lambda}, & \text{otherwise.} \end{cases} \quad (3.11)$$

Hence in Regime IV we have for all  $L$

$$2 - \alpha_s = \frac{\pi}{4\lambda} = \frac{L+1}{4}. \quad (3.12)$$

## 4 Band structure and ground state degeneracies

In Regimes III and IV the ABF models admit a number of coexisting phases [13, 14] corresponding to ground state (zero temperature,  $|t| \rightarrow 1$ ) configurations on the lattice given by a pair  $(a, b)$  such that  $a - b = \pm 1$ . Explicitly, the ground states consist of a checkerboard arrangement with height  $a$  on one sublattice and height  $b$  on the other sublattice of the square lattice. At each boundary the boundary height  $a$  together with the choice of  $\xi$  determines the favoured boundary configuration. If  $\xi > 0$  the favoured boundary condition is  $\{a, a + 1, a, a + 1, \dots\}$ , and if  $\xi < 0$ , the favoured boundary condition is  $\{a, a - 1, a, a - 1, \dots\}$ . Clearly, the boundary condition can pick out the phase near the boundaries. When the phase selected on the left and right boundaries is the same there will be a unique phase and a unique ground state. However, when the phase selected by the left boundary differs from that selected by the right boundary, there will be an interface formed between the two coexisting phases. In this case there will be an interfacial tension, the ground state will be highly degenerate corresponding to the many possible locations of the interface, and the double-row transfer matrix will have a continuous band of largest eigenvalues in the thermodynamic limit. More generally, we expect the spectrum of the double-row transfer matrix to consist of many such continuous bands of eigenvalues. These bands can overlap with each other but within each band the states are characterized by a fixed number of domain walls. Furthermore, the number of eigenvalues in each band can be determined by the simple combinatorics of counting the compatible ground state configurations.

### 4.1 Regime III

In Regime III, there are  $2L - 2$  coexisting phases. The corresponding ground state lattice configurations are given by the pairs  $(a, b)$  with  $|a - b| = 1$ , and all values  $1 \leq a, b \leq L$  allowed. Let us denote the left boundary condition with height  $a = a^L$  by  $a^+$  if  $\xi^L > 0$  and  $a^-$  if  $\xi^L < 0$ . Similarly, for the right boundary with height  $b = a^R$  we have  $b^+$  or  $b^-$  depending on the sign of  $\xi^R$ . This is convenient since under

the restriction (1.15) the interfacial tensions and correlation lengths will depend only on the signs of  $\xi^L$  and  $\xi^R$ . The ground state degeneracies for a double-row transfer matrix of  $N$  faces with fixed boundary heights  $a$  and  $b$  are then given by

$$\mathcal{N}(a^+, b^+) = \mathcal{N}(a^-, b^-) = \binom{(N + |a - b|)/2}{|a - b|} \quad (4.1)$$

$$\mathcal{N}(a^+, b^-) = \binom{(N + |a - b + 2|)/2}{|a - b + 1|} \quad (4.2)$$

$$\mathcal{N}(a^-, b^+) = \binom{(N + |a - b - 2|)/2}{|a - b - 1|} \quad (4.3)$$

For later reference, we define  $m(\alpha, \beta)$  and  $n(\alpha, \beta)$  by

$$\mathcal{N}(\alpha, \beta) = \begin{pmatrix} m(\alpha, \beta) \\ n(\alpha, \beta) \end{pmatrix}, \quad (4.4)$$

where  $\alpha = a^\pm$  and  $\beta = b^\pm$ . To see how these formulas arise, consider for simplicity an  $(a^+, b^-)$  boundary height configuration with  $b - a \geq 2$ . Moving from  $a$  to  $b$ , there must be  $b - a - 1$  ground state transitions (domain walls), starting with the state  $\{a, a + 1, a, a + 1, \dots\}$  and ending with the state  $\{\dots, b - 1, b, b - 1, b\}$ . If  $a$  and  $b$  are joined by a path of  $N$  steps, and we let  $m_1, m_2, \dots, m_{b-a-1}$  be the steps at which the transitions occur, then the  $m_i$  can take the values

$$m_1 = 2, 4, \dots, N + a - b + 2 \quad (4.5)$$

$$m_i = m_{i-1} + 1, m_{i-1} + 3, \dots, N + a - b + i + 1 \quad \text{for } 2 \leq i \leq b - a - 1. \quad (4.6)$$

We rewrite and sum over these combinations using the variables  $k_1, k_2, \dots, k_{b-a-1}$  such that  $m_i = 2k_i + i + 1$ . This gives

$$\mathcal{N}(a^+, b^-) = \sum_{k_1=0}^{\frac{N+a-b}{2}} \sum_{k_2=k_1}^{\frac{N+a-b}{2}} \cdots \sum_{k_{b-a-1}=k_{b-a-2}}^{\frac{N+a-b}{2}} 1 = \binom{(N + b - a - 2)/2}{b - a - 1} \quad (4.7)$$

in agreement with equation (4.2). The other expressions may be derived from symmetries under the interchange of  $a$  and  $b$ , and from the equivalence of  $(a^-, \beta)$  with  $N$  steps and  $((a - 1)^+, \beta)$  with  $N + 1$  steps.

## 4.2 Regime IV

First consider  $L$  even. In this case there are  $2L - 4$  phases, and the ground state configurations are as in Regime III, with the exclusion of the  $(L/2, L/2 + 1)$  and  $(L/2 + 1, L/2)$  pairs. In the case when  $L$  is odd, there are again  $2L - 4$  phases and the ground state lattice configurations are as in Regime III, except now the pairs

$((L-1)/2, (L+1)/2)$  and  $((L+1)/2, (L+3)/2)$  are excluded and replaced by the disordered mixture  $((L+1)/2, (L+1)/2 \pm 1)$  where the heights on one sublattice can take the values  $(L-1)/2$  and  $(L+3)/2$  independently at each site.

For both even and odd  $L$ , we find that the number of eigenvalues in the first band is given as follows ( $\lfloor x \rfloor$  denotes the largest integer less than or equal to  $x$ ):

- If  $1 \leq a, b \leq \lfloor (L+1)/2 \rfloor$ , excluding  $\lfloor (L+1)/2 \rfloor^+$ , the degeneracies are the same as in Regime III.
- If  $\lfloor (L+2)/2 \rfloor \leq a, b \leq L$ , excluding  $\lfloor (L+2)/2 \rfloor^-$ , the degeneracies are the same as in Regime III.
- Otherwise, the degeneracies are

$$\mathcal{N}(a^+, b^+) = \mathcal{N}(a^-, b^-) = \binom{(N + |a - b| - 2)/2}{|a - b| - 1} \quad (4.8)$$

$$\mathcal{N}(a^+, b^-) = \binom{(N + |a - b + 2| - 2)/2}{|a - b + 1| - 1} \quad (4.9)$$

$$\mathcal{N}(a^-, b^+) = \binom{(N + |a - b - 2| - 2)/2}{|a - b - 1| - 1}. \quad (4.10)$$

## 5 Interfacial tensions

The interfacial tension  $\sigma^{(\alpha, \beta)}$  between phases  $\alpha = a^\pm$  and  $\beta = b^\pm$  is given by

$$-\sigma^{(\alpha, \beta)} = \lim_{P \rightarrow \infty} \lim_{N \rightarrow \infty} P^{-1} \ln \text{Tr } \mathbf{\Lambda}(u)^{P/2} \quad (5.1)$$

where

$$\mathbf{\Lambda}(u) = \mathbf{\Lambda}^{(\alpha, \beta)}(u) = \frac{\mathbf{D}^{(\alpha, \beta)}(u)}{\left[ D_{\max}^{(\alpha, \alpha)}(u) D_{\max}^{(\beta, \beta)}(u) \right]^{1/2}} \quad (5.2)$$

and a factor of the inverse temperature has been absorbed into the definition of  $\sigma^{(\alpha, \beta)}$ . Here  $D_{\max}^{(\alpha, \alpha)}(u)$  denotes the maximum eigenvalue of the double-row transfer matrix  $\mathbf{D}^{(\alpha, \alpha)}(u)$ . Note that the bulk and surface contributions cancel out in the ratio  $\mathbf{\Lambda}(u)$ . From the inversion relation, crossing, and quasiperiodicity, it follows that the eigenvalues  $\Lambda(u)$  satisfy the simple functional equations

$$\Lambda(u)\Lambda(u + \lambda) = 1, \quad \Lambda(u) = \Lambda(\lambda - u) \quad (5.3)$$

subject to the periodicity

$$\Lambda(u) = \Lambda(u + i\pi\varepsilon^*), \quad \varepsilon^* = \begin{cases} \varepsilon, & \text{Regime III} \\ 2\varepsilon, & \text{Regime IV.} \end{cases} \quad (5.4)$$

In the thermodynamic limit the eigenvalues of  $\mathbf{\Lambda}(u)$  form continuous bands. In this limit the sum in the trace of (5.1) is replaced by integrals over these bands. But in the limit of  $P$  large only the first band of largest eigenvalues is expected to contribute to the interfacial tension. From the counting of degeneracies, this first band contains  $\mathcal{N}(\alpha, \beta)$  eigenvalues. The largest eigenvalue of  $\mathbf{D}^{(\alpha, \beta)}(u)$  has zeros which accumulate on the lines  $\text{Re}(u) = -\lambda/2$  and  $\text{Re}(u) = 3\lambda/2$  in both regimes, and also on the lines  $\text{Re}(u) = (3\lambda - \pi)/2$  and  $\text{Re}(u) = (\pi - \lambda)/2$  in Regime IV. In addition, each eigenvalue in the first band of eigenvalues of  $\mathbf{D}^{(\alpha, \beta)}(u)$  has  $n(\alpha, \beta)$  pairs of zeros in the strip  $-\lambda/2 < \text{Re}(u) < 3\lambda/2$  at

$$u = \lambda/2 \pm i\phi_k, \quad k = 1, 2, \dots, n(\alpha, \beta) \quad (5.5)$$

where each  $\phi_k$  is real and non-zero.

We now seek a solution to the inversion relations (5.3) in the physical strip subject to the given zeros and periodicity. The required solution is

$$\Lambda(u) = \prod_{k=1}^{n(\alpha, \beta)} \Phi(u + i\phi_k)\Phi(u - i\phi_k) \quad (5.6)$$

where

$$\Phi(u) = \frac{\vartheta_1\left(\frac{\pi u}{2\lambda} - \frac{\pi}{4}, |t|^\nu\right)}{\vartheta_1\left(\frac{\pi u}{2\lambda} + \frac{\pi}{4}, |t|^\nu\right)} \quad (5.7)$$

and

$$\nu = \begin{cases} (L+1)/4, & \text{Regime III} \\ (L+1)/2, & \text{Regime IV.} \end{cases} \quad (5.8)$$

Recall that  $|t| = |p^2| = \exp(-2\pi\varepsilon)$  where the nome  $p$  is defined in equation (1.4).

In the thermodynamic limit the distribution of each  $\phi_k$  becomes dense, yielding a density  $\rho(\phi_1, \dots, \phi_{n(\alpha, \beta)})$ . So integrating over this band of eigenvalues for real values of  $u$  in the interval  $0 < u < \lambda$  gives

$$\lim_{N \rightarrow \infty} \text{Tr } \mathbf{\Lambda}(u)^{P/2} \sim \int_0^{\frac{\pi\varepsilon^*}{2}} \dots \int_0^{\frac{\pi\varepsilon^*}{2}} \rho(\phi_1, \dots, \phi_{n(\alpha, \beta)}) \prod_{k=1}^{n(\alpha, \beta)} |\Phi(u + i\phi_k)|^P d\phi_1 \dots d\phi_{n(\alpha, \beta)}. \quad (5.9)$$

For  $P$  large this multiple integral can be evaluated by saddle point methods. The saddle point occurs at

$$u + i\phi_k = \lambda/2 + i\pi\varepsilon^*/2, \quad k = 1, 2, \dots, n(\alpha, \beta). \quad (5.10)$$

Hence

$$\exp[-\sigma^{(\alpha,\beta)}] = \left[ \frac{\vartheta_4(0, |t|^\nu)}{\vartheta_4(\pi/2, |t|^\nu)} \right]^{n(\alpha,\beta)} = k'(|t|^\nu)^{n(\alpha,\beta)/2} \quad (5.11)$$

where  $k'$  is the conjugate elliptic modulus

$$k'(p) = \prod_{n=1}^{\infty} \left( \frac{1 - p^{2n-1}}{1 + p^{2n-1}} \right)^4. \quad (5.12)$$

It follows that the corresponding critical exponent is given by

$$\sigma^{(\alpha,\beta)} \sim |t|^\mu, \quad \mu = \nu = \begin{cases} (L+1)/4, & \text{Regime III} \\ (L+1)/2, & \text{Regime IV.} \end{cases} \quad (5.13)$$

Notice that the phases are linearly ordered and that the only way to get from one phase to another is to pass through any intermediate phases. So let us define  $\sigma^{(\alpha,\beta)} = 0$  for the non-ground state phases. Similarly, for  $L$  odd in Regime IV, let us define  $\sigma^{((L-1)/2^+, (L+1)/2^+)} = 0$  etc. for the disordered phase. Then the interfacial tensions are additive and it follows, for example, that if  $a < b$ ,

$$\sigma^{(a^+, b^+)} = \sum_{k=1}^{b-a} \sigma^{((a+k-1)^+, (a+k)^+)} \quad (5.14)$$

## 6 Correlation lengths

Let  $\varphi(a)$  be a given function of the height  $a$  and let  $a_0$  and  $a_\ell$  be two heights in the same column separated by an even number  $\ell$  of lattice spacings. Then, at large distances, the (truncated) pair correlation function decays exponentially with a correlation length  $\xi$  given by

$$\langle \varphi(a_0)\varphi(a_\ell) \rangle - \langle \varphi(a_0) \rangle \langle \varphi(a_\ell) \rangle \sim \exp(-\ell\xi^{-1}). \quad (6.1)$$

By standard row transfer matrix arguments [1] the correlation length can be calculated from

$$-\xi^{-1} = \lim_{\ell \rightarrow \infty} \lim_{N \rightarrow \infty} \ell^{-1} \ln \sum_{j \neq \max} c_j \Lambda_j(u)^{\ell/2} \quad (6.2)$$

where the  $\Lambda_j(u)$  are eigenvalues of

$$\mathbf{\Lambda}(u) = \frac{\mathbf{D}^{(\alpha,\alpha)}(u)}{D_{\max}^{(\alpha,\alpha)}(u)}, \quad (6.3)$$

$c_j$  denotes  $u$ -independent matrix elements, and the sum excludes the maximal eigenvalue. Again the bulk and surface terms cancel out of the ratio  $\Lambda(u)$ . From the inversion relation and crossing, it follows that the eigenvalues  $\Lambda(u)$  again satisfy the simple functional equations

$$\Lambda(u)\Lambda(u + \lambda) = 1, \quad \Lambda(u) = \Lambda(\lambda - u) \quad (6.4)$$

subject to the previous periodicity (5.4).

In the thermodynamic limit the sum in (6.2) is dominated by the first band of  $(2 - \delta_{\alpha,1^+} - \delta_{\alpha,L^-})\binom{N/2}{2}$  eigenvalues. In this band the eigenvalues  $\Lambda_j(u)$  are analytic and non-zero in an open strip containing  $0 \leq \text{Re}(u) \leq \lambda$  with two pairs of zeros at

$$u = \lambda/2 \pm i\phi_1, \quad u = \lambda/2 \pm i\phi_2. \quad (6.5)$$

The required solution of the inversion relation is therefore

$$\Lambda(u) = \prod_{k=1}^2 \Phi(u + i\phi_k)\Phi(u - i\phi_k) \quad (6.6)$$

where  $\Phi(u)$  is given by (5.7). Passing to the thermodynamic limit for real  $u$  in the interval  $0 < u < \lambda$  we obtain

$$\lim_{N \rightarrow \infty} \sum_{j \neq \max} c_j \Lambda_j(u)^{\ell/2} \sim \int_0^{\frac{\pi \varepsilon^*}{2}} \int_0^{\frac{\pi \varepsilon^*}{2}} c(\phi_1, \phi_2) \prod_{k=1}^2 |\Phi(u + i\phi_k)|^\ell d\phi_1 d\phi_2. \quad (6.7)$$

Carrying out the saddle point analysis as before gives

$$\exp(-\xi^{-1}) = k'(|t|^\nu), \quad (6.8)$$

where  $k'$  is the conjugate elliptic modulus and hence

$$\xi \sim |t|^{-\nu}, \quad \nu = \begin{cases} (L+1)/4, & \text{Regime III} \\ (L+1)/2, & \text{Regime IV.} \end{cases} \quad (6.9)$$

It can be seen that, after allowing for differences in formulation, all of the above results for the interfacial tensions and correlation lengths agree with the known results in the case of the Ising model [1] ( $L = 3$ ) and the interacting hard square model [15] ( $L = 4$ ). Notice also that we have the simple relation

$$\xi^{-1} = 2\sigma^{(1^+, 2^+)}. \quad (6.10)$$

This factor of 2 is easily understood because in the saddle point integrals the contribution to the the inverse correlation length involves two domain walls, whereas the contribution to this interfacial tension derives from just one such domain wall.

Finally, we note that in Regime III the critical exponents  $\alpha$ ,  $\alpha_s$ ,  $\mu$  and  $\nu$  satisfy the scaling relations [1, 16, 17]

$$2 - \alpha = \mu + \nu = 2\nu \quad \alpha_s = \alpha + \nu. \quad (6.11)$$

However these scaling relations break down in Regime IV, since the Regime IV line of exact solution approaches the tricritical point  $t = 0$  at a tangent to the critical line [18].

## 7 Discussion

In this paper we have calculated the surface free energies, interfacial tensions and correlation lengths of the ABF models in Regimes III and IV all by solving the relatively simple inversion relation. The methods are general and can be applied to other solvable lattice models such as the CSOS models and dilute lattice models. The methods employed here could be applied to Regimes I and II with fixed boundary conditions where the heights alternate along the boundary. However, in Regimes I and II, the coexisting phases are selected out by a saw-tooth variation in the ground state configuration [13, 14] and unfortunately these saw-tooth variations on the boundary cannot be handled within our formalism at present. So, strictly speaking, the current methods are best suited to systems that exhibit at most two independent sublattices in the structure of their ground states.

Of course, the correlation length along the strip is a bulk property and not a surface property and so its calculation should not necessitate the introduction of a boundary. Accordingly, the generalized inversion relation method used here to obtain the correlation length can be applied in the case of periodic boundary conditions simply by working with a double-row transfer matrix

$$\mathbf{D}(u) = \mathbf{T}(u)\mathbf{T}(\lambda - u). \quad (7.1)$$

Since the correlation length is independent of the spectral parameter  $u$ , and since  $\mathbf{D}(u)$  and  $\mathbf{T}(u)$  yield the same correlation length at the isotropic point  $u = \lambda/2$ , the two transfer matrices must lead to the same correlation length for all values of  $u$ .

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