

# Analytic Calculation of Conformal Partition Functions: Tricritical Hard Squares with Fixed Boundaries

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## Abstract

An analytic method of calculating conformal partition functions of solvable lattice models is presented. Our technique involves the solution of transfer matrix functional equations in the scaling limit, and unifies the work of Albertini *et al.* on the classification of transfer matrix eigenvalues with that of Klümper and Pearce on finite-size corrections. As an illustration, conformal partition functions of the tricritical hard square model with fixed boundaries are derived. The resulting Virasoro characters arise in their fermionic representation and agree with the Coulomb gas calculations of Saleur and Bauer.

**Keywords:** Conformal partition functions, hard square model, fixed boundaries.

## 1 Introduction

Solvable lattice models have long proven to be valuable tools in the study of critical phenomena. Since the advent of conformal invariance in particular [1, 2], many theoretical predictions have been confirmed by direct analytic calculations on solvable models. Indeed, techniques to calculate such quantities as central charges and conformal weights have now become standard [3–10]. However, confirmation of the predictions of conformal invariance regarding conformal partition functions<sup>1</sup> is a rather more difficult problem, and fully analytic derivations have so far been restricted to the Ising model [11, 12].

There are two ingredients necessary for an analytic calculation of conformal partition functions from solvable lattice models: the complete classification of all eigenvalues of the row-to-row transfer matrix in the leading band; and the exact calculation of finite-size corrections to each of these eigenvalues. Important steps towards such a calculation have been taken in the work of the Stony Brook group [13–

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<sup>1</sup>Following reference [17], we refer to the universal part of the partition function as the “conformal” partition function

17] and the work of Klümper and Pearce [8–10]. The former work, performed in the context of the three-state Potts model, provided a general approach to the classification of all transfer matrix eigenvalues. Although this led to the computation of conformal partition functions, the finite-size corrections to the eigenvalues in the leading band were determined numerically [17]. In contrast, in the latter work an approach was developed for the analytic derivation of such finite-size corrections, but only for the primary scaling dimensions.

In this paper we present an analytic technique for computing conformal partition functions of solvable lattice models. Specifically, we follow the methods of Albertini *et al.* [13] for the classification of transfer matrix eigenvalues. We then extend the methods of Klümper and Pearce [9] to calculate analytically the finite-size corrections to all eigenvalues in the leading band.

To demonstrate our method we derive conformal partition functions for the tricritical hard square model with fixed boundaries, thus confirming the predictions of Saleur and Bauer [18] based on Coulomb gas calculations [19–21]. The model has been chosen as the simplest non-free-fermion model, whilst the fixed boundaries have been chosen to give the simplest possible classification of the leading band of eigenvalues.

This paper is organized as follows. In the next section we introduce Baxter’s tricritical hard square model with fixed boundaries, and give a functional equation satisfied by the row transfer matrix. In section 3 we define the conformal partition function for cylindrical geometries. Sections 4 to 9 are devoted to solving the functional equation to obtain the conformal partition functions. These six sections correspond to the independent choices of boundary, and each yields one of the six  $c = 7/10$  Virasoro characters in its fermionic representation [16]. We conclude with a brief discussion.

## 2 The tricritical hard square model

Baxter’s hard square model is a gas of interacting hard-core particles on the two-dimensional square lattice [22–24]. Each site of the lattice carries a spin  $\mu$  which can take the value 0 or 1. Spins on adjacent sites must satisfy  $\mu\mu' = 0$ , so that allowed spin configurations correspond to configurations of hard-core square particles rotated by  $45^\circ$  with respect to the lattice. Adjacent particles interact, with partition function

$$\Xi(L, M, z) = \sum_{\substack{\text{particle} \\ \text{configurations}}} z^N e^{LN_1 + MN_2} \quad (2.1)$$

where  $N$  is the number of particles,  $N_1$  the number of adjacent particles in the NE-SW direction, and  $N_2$  the number of adjacent particles in the NW-SE direction. Baxter has shown the model to be integrable on the manifold  $z = (1 - e^{-L})(1 - e^{-M})/(e^{L+M} - e^L - e^M)$ . The phase diagram of the hard square model has been discussed by Huse [25], who showed that on a particular line of the solvable manifold the model becomes tricritical.

A convenient reformulation of the model is given by the four-state Andrews-Baxter-Forrester model [26]. In this representation, sites of the square lattice take heights from the set  $\{1, 2, 3, 4\}$  subject to the condition that the values of heights on adjacent sites must differ by  $\pm 1$ . When the heights 1 and 4 are identified with  $\mu = 1$  and the heights 2 and 3 with  $\mu = 0$ , the allowed lattice configurations reproduce the hard square model. The Boltzmann weights of the tricritical hard square model in the ABF formulation are

$$W \begin{pmatrix} d & c \\ a & b \end{pmatrix} = \frac{\sin(\lambda - u)}{\sin \lambda} \delta_{a,c} + \frac{\sin u}{\sin \lambda} \left( \frac{S_a S_c}{S_b S_d} \right)^{\frac{1}{2}} \delta_{b,d} \quad (2.2)$$

where  $\lambda = \pi/5$  and  $S_a = \sin(a\lambda)$ . The parameter  $u$  lies in the range  $0 < u < \lambda$ , with the point  $u = \lambda/2$  corresponding to  $L = M$ .

Since the weights satisfy the Yang-Baxter equation [23], they ensure commuting transfer matrices on a periodic lattice. On a lattice with boundaries it is necessary to introduce left and right boundary weights which satisfy reflection equations [27, 28]. Such weights have been obtained in reference [29], and for our purposes we select

$$K \begin{pmatrix} a & a \pm 1 \\ a & a \pm 1 \end{pmatrix} = \left( \frac{S_{a \pm 1}}{S_a} \right)^{\frac{1}{2}} \frac{\sin(\lambda - u \pm \xi) \sin(\lambda - u \mp a\lambda \mp \xi)}{\sin^2 \lambda} \quad (2.3)$$

$$K \begin{pmatrix} a \pm 1 & a \\ a \pm 1 & a \end{pmatrix} = \left( \frac{S_{a \pm 1}}{S_a} \right)^{\frac{1}{2}}.$$

The parameter  $\xi$  is restricted to the range  $\lambda/2 < \xi < \lambda$ , but is otherwise arbitrary.

We use the face and boundary weights to construct a (double-row) transfer matrix  $\mathbf{T}$  [29]. For a lattice of width  $N$ , the entry of  $\mathbf{T}$  corresponding to rows of heights  $\mathbf{a} = \{a_1, \dots, a_{N-1}\}$  and  $\mathbf{b} = \{b_1, \dots, b_{N-1}\}$  is defined diagrammatically by

$$\mathbf{T}(u)_{\mathbf{a}, \mathbf{b}} = \sum_{c_0, \dots, c_N} \begin{array}{c} \begin{array}{c} r \dots r \\ \vdots \\ r \end{array} \begin{array}{c} b_1 \quad b_2 \quad \dots \quad b_{N-1} \quad s \quad s \\ \vdots \\ s \end{array} \\ \begin{array}{c} \lambda - u \quad \lambda - u \quad \dots \quad \lambda - u \\ \vdots \\ u \quad u \quad \dots \quad u \end{array} \\ \begin{array}{c} c_0 \quad c_1 \quad c_2 \quad \dots \quad c_{N-1} \quad c_N \\ \vdots \\ a_1 \quad a_2 \quad \dots \quad a_{N-1} \end{array} \end{array} \quad (2.4)$$

where the triangles and squares represent the boundary and face weights respectively. The adjacency condition on the heights of the model can be encoded in the  $4 \times 4$  incidence matrix  $A$  with entries  $A_{j,k} = \delta_{|j-k|,1}$ . The dimension of  $\mathbf{T}$  is then simply expressed in terms of  $A^N$  as

$$\dim \mathbf{T} = A_{r,s}^N. \quad (2.5)$$

In reference [29], a functional equation for  $\mathbf{T}(u)$  has been obtained. By defining a normalized transfer matrix

$$\mathbf{t}(u) = (-1)^s \alpha_r(u - \xi) \alpha_{-r}(u + \xi) \beta(u) \gamma_2(u) \mathbf{T}(u) / (\gamma_1(u) \gamma_3(u)) \quad (2.6)$$

this equation takes the simple form

$$\mathbf{t}(u) \mathbf{t}(u + \lambda) = \mathbf{I} + \mathbf{t}(u + 3\lambda). \quad (2.7)$$

In this normalization we have used the definitions

$$\begin{aligned} \gamma_k(u) &= (-1)^N \left( \frac{\sin(u + k\lambda)}{\sin \lambda} \right)^{2N} & \beta(u) &= \frac{\sin^2(2u - \lambda)}{\sin(2u - 3\lambda) \sin(2u + \lambda)} \\ \alpha_a(u) &= \frac{\sin \lambda \sin(u + (3 - a)\lambda) \sin(u + (1 - a)\lambda)}{\sin u \sin(u - \lambda) \sin(u + 2\lambda)}. \end{aligned} \quad (2.8)$$

The arbitrary factor of  $(-1)^s$  in equation (2.6) has been included so that equation (2.7) takes a form independent of  $s$ .

Before we actually solve the functional equation (2.7) in later sections, let us note one property of the eigenvalues of  $\mathbf{t}$  which follows directly from the weights (2.2) and (2.3). Using these weights it follows that

$$\lim_{iu \rightarrow \infty} \sum_g e^{-4iu} \begin{array}{c} a \quad a \quad d \\ \diagdown \quad | \quad \diagup \\ u \quad g \quad c \\ \diagup \quad | \quad \diagdown \\ a \quad a \quad b \end{array} = \lim_{iu \rightarrow \infty} \rho e^{-2iu} \begin{array}{c} d \\ \diagdown \quad \diagup \\ u \quad c \\ \diagup \quad \diagdown \\ b \end{array} \delta_{b,d} \quad (2.9)$$

with  $\rho^{-1} = -4 \exp(i\lambda) \sin^2 \lambda$ , and we conclude that  $\mathbf{t}(u)$  is diagonal in the limit  $iu \rightarrow \infty$ . If we  $N$  times apply (2.9) to the transfer matrix and then use

$$\lim_{iu \rightarrow \infty} \sum_g e^{-2iu} \begin{array}{c} a \quad a \\ \diagdown \quad \diagup \\ u \quad g \\ \diagup \quad \diagdown \\ a \quad a \end{array} = -2\rho \cos(a\lambda) \quad (2.10)$$

we find

$$\lim_{iu \rightarrow \infty} \mathbf{t}(u)_{\mathbf{a}, \mathbf{a}} = \lim_{iu \rightarrow \infty} \frac{(-1)^{s+1}}{(\rho e^{2iu})^{N+1}} \mathbf{T}(u)_{\mathbf{a}, \mathbf{a}} = (-1)^{s+1} 2 \cos(s\lambda) \quad (2.11)$$

independent of  $\mathbf{a}$ . The fact that only the two values  $2 \cos \lambda = (1 + \sqrt{5})/2$  and  $2 \cos(3\lambda) = (1 - \sqrt{5})/2$  occur can be understood by taking the limit  $iu \rightarrow \infty$  in the functional equation (2.7). We will need the result (2.11) in sections 4 to 9.

### 3 Conformal properties of the tricritical hard square model

We here briefly recall some of the properties of the tricritical hard square model with fixed boundaries as predicted by conformal invariance. Consider the finite-size partition function  $Z_{NM}$  of the tricritical hard square model on a cylindrical lattice of width  $N$  and circumference  $M$ . The asymptotic behaviour of  $Z_{NM}$  in the limit of large  $N$  and  $M$  with the ratio  $M/N$  fixed is given by

$$Z_{NM}(u) = \text{Tr}(\mathbf{T}(u)^M) \sim \exp(-NM f_b(u) - M f_s(u)) Z(q). \quad (3.1)$$

Here  $\mathbf{T}$  is the row transfer matrix,  $f_b$  is the bulk free energy,  $f_s$  is the surface free energy, and  $Z(q)$  is the universal conformal partition function with modular parameter

$$q = \exp(-2\pi \sin(5u)M/N). \quad (3.2)$$

To leading orders, each transfer matrix eigenvalue  $T(u)$  reads [30]

$$\ln T(u) = -N f_b(u) - f_s(u) + \frac{2\pi}{N} \left( \frac{c}{24} - \Delta - n \right) \sin(5u) + o\left(\frac{1}{N}\right) \quad (3.3)$$

where  $c = 7/10$  is the central charge,  $\Delta$  is one of the conformal weights in table 1, and  $n$  is a non-negative integer which is zero for the largest eigenvalue. After taking the ratio of each eigenvalue with the largest, we can write

$$Z(q) = \sum_{k=1}^3 \sum_{l=1}^2 q^{-c/24 + \Delta_{k,l}} \mathcal{N}(\Delta_{k,l}) \sum_j q^{n_j} = \sum_{k=1}^3 \sum_{l=1}^2 \mathcal{N}(\Delta_{k,l}) \chi_{k,l}(q) \quad (3.4)$$

with  $\mathcal{N}(\Delta_{k,l}) = 0$  or  $1$  and  $\chi_{k,l}$  one of the  $c = 7/10$  Virasoro characters.

The operator content  $\mathcal{N}(\Delta_{k,l})$  is determined by the particular choice of boundary conditions on the cylinder [18, 31]. For a lattice with the boundary conditions  $(r, s)$  depicted in figure 1, Saleur and Bauer [18] have used Coulomb gas techniques to

	$s$			
4	3/2	7/16	0	
3	3/5	3/80	1/10	
2	1/10	3/80	3/5	
1	0	7/16	3/2	
	1	2	3	$r$

Table 1: Grid of conformal weights  $\Delta_{r,s} = [(5r - 4s)^2 - 1]/80$  for the unitary minimal theory with  $c = 7/10$ .

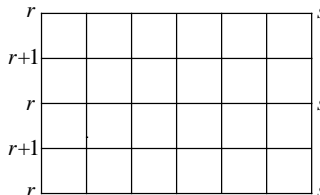


Figure 1: Fixed boundaries on the square lattice. Unlabelled sites are free.

calculate  $\mathcal{N}(\Delta_{k,l}) = \delta_{k,r}\delta_{l,s}$ , so the conformal partition function is given by

$$Z(q) = \chi_{r,s}(q). \quad (3.5)$$

Returning to the transfer matrix defined in equation (2.4), observe that we have chosen our boundary weights (2.3) to correspond to the lattice in figure 1. Every second site on the right boundary is fixed to the same height  $s$ , and the remaining right boundary sites are freely summed over. Furthermore, at the point  $u = \lambda - \xi$  we have

$$K \begin{pmatrix} r & r+1 \\ r & r \end{pmatrix} \neq 0 \quad K \begin{pmatrix} r & r-1 \\ r & r \end{pmatrix} = 0 \quad (3.6)$$

so that near this point the left boundary condition effectively becomes that of figure 1.

## 4 The character $\chi_{1,1}$

We first study the spectrum of the transfer matrix  $\mathbf{T}(u)$  for a lattice of width  $N$  and boundary heights  $(r, s) = (1, 1)$ , as defined by equation (2.4). The adjacency condition implies that  $N$  must be even. We begin by classifying the zeros of the eigenvalues of  $\mathbf{T}(u)$ , and identifying the characteristics of those eigenvalues in the leading band as  $N \rightarrow \infty$ . This allows us to compute the order- $N$ , order-1 and order- $1/N$  behaviour of all eigenvalues in that band, and we will see that the conformal partition function is given by  $Z(q) = \chi_{1,1}(q)$ .

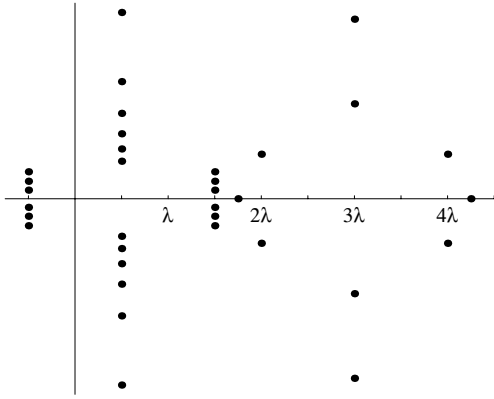


Figure 2: Zeros of a typical eigenvalue of  $\mathbf{T}(u)$  in the complex  $u$ -plane.

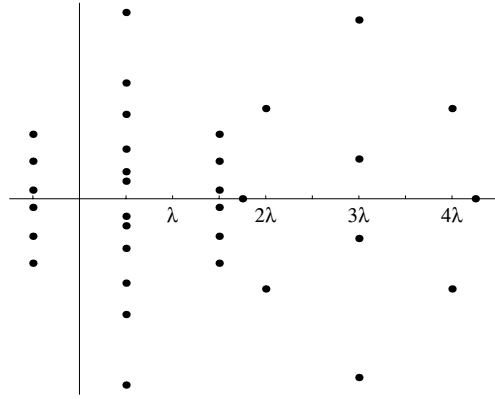


Figure 3: Zeros of another typical eigenvalue of  $\mathbf{T}(u)$  in the complex  $u$ -plane.

#### 4.1 Classification of zeros

From numerical diagonalization of  $\mathbf{T}(u)$  we are able to classify all eigenvalues according to their zeros in the complex plane. For each eigenvalue we find that—apart from a pair of zeros at  $u = \lambda + \xi$  and  $u = 5\lambda - \xi$  induced by the left boundary weight—the zeros of the eigenvalues occur on certain fixed lines. Specifically, two types of 1-strings and two types of 2-strings occur. A 1-string is given by a single zero  $u_j$  such that

$$\operatorname{Re}(u_j) = \begin{cases} \lambda/2 & \text{type 1} \\ 3\lambda & \text{type 2.} \end{cases} \quad (4.1)$$

A 2-string is a pair of zeros  $(u_j, u'_j)$  with equal imaginary part and

$$(\operatorname{Re}(u_j), \operatorname{Re}(u'_j)) = \begin{cases} (-\lambda/2, 3\lambda/2) & \text{type 1} \\ (2\lambda, 4\lambda) & \text{type 2.} \end{cases} \quad (4.2)$$

Distributions of zeros for two typical eigenvalues of  $\mathbf{T}(u)$  for  $N = 16$  are depicted in figures 2 and 3. We note that for finite  $N$  the 2-strings do not fall precisely on the lines given in equation (4.2), but that this deviation vanishes as  $N$  increases.

Due to the crossing symmetry  $\mathbf{T}(u) = \mathbf{T}(\lambda - u)$ , each string occurs with its complex conjugate. We further observe from our numerics that only the exceptional zeros at  $u = \lambda + \xi$  and  $u = 5\lambda - \xi$  occur on the real axis. Hence, given an eigenvalue,

we can denote the number of strings as follows:

$$\begin{aligned}
2m_1 &= \text{number of 1-strings of type 1} \\
2n_1 &= \text{number of 2-strings of type 1} \\
2m_2 &= \text{number of 1-strings of type 2} \\
2n_2 &= \text{number of 2-strings of type 2.}
\end{aligned} \tag{4.3}$$

Since each eigenvalue of  $\mathbf{T}(u)$  has  $2N + 2$  zeros, we have the completeness relation

$$m_1 + m_2 + 2n_1 + 2n_2 = N. \tag{4.4}$$

In addition, we observe the relation

$$m_2 + n_2 = \frac{1}{2}m_1 \tag{4.5}$$

so that both  $m_1$  and  $m_2$  are even (recalling that  $N$  is even). These two equations may be recast in the more familiar form of an  $(\mathbf{m}, \mathbf{n})$ -system [32]

$$\mathbf{m} + \mathbf{n} = \frac{1}{2}(N\mathbf{e}_1 + \mathcal{I}\mathbf{m}) \tag{4.6}$$

where  $\mathbf{m} = (m_1, m_2)$ ,  $\mathbf{n} = (n_1, n_2)$ ,  $\mathbf{e}_1 = (1, 0)$ , and  $\mathcal{I}$  is the  $A_2$  incidence matrix with entries  $\mathcal{I}_{j,k} = \delta_{|j-k|,1}$ .

For each system size, we find that there are many eigenvalues with the same string content  $(\mathbf{m}, \mathbf{n})$ . These eigenvalues are distinguished by the relative orderings of the strings of type 1, and the relative orderings of the strings of type 2. Denoting the imaginary parts of the 1-strings of type  $j$  by  $0 < v_1^{(j)} < \dots < v_{m_j}^{(j)}$ , and the imaginary parts of the 2-strings of type  $j$  by  $0 < w_1^{(j)} < \dots < w_{n_j}^{(j)}$ , we see that in figure 2

$$\begin{aligned}
0 < w_1^{(1)} < w_2^{(1)} < w_3^{(1)} < v_1^{(1)} < v_2^{(1)} < \dots < v_6^{(1)} \\
0 < w_1^{(2)} < v_1^{(2)} < v_2^{(2)}
\end{aligned} \tag{4.7}$$

whereas in figure 3, which has the same string content,

$$\begin{aligned}
0 < w_1^{(1)} < v_1^{(1)} < v_2^{(1)} < w_2^{(1)} < v_3^{(1)} < w_3^{(1)} < v_4^{(1)} < v_5^{(1)} < v_6^{(1)} \\
0 < v_1^{(2)} < w_1^{(2)} < v_2^{(2)}.
\end{aligned} \tag{4.8}$$

Clearly, the total number of possible orderings for a given  $(\mathbf{m}, \mathbf{n})$  is  $\binom{m_1+n_1}{m_1} \binom{m_2+n_2}{m_2}$ . Summing over all allowed string contents  $(\mathbf{m}, \mathbf{n})$  using (4.6) then gives

$$\sum_{\substack{m_1 \text{ even} \\ m_2 \text{ even}}} \binom{m_1+n_1}{m_1} \binom{m_2+n_2}{m_2} = A_{1,1}^N \tag{4.9}$$

which we see from equation (2.5) is indeed the correct number of eigenvalues.

## 4.2 Characterization of leading eigenvalues

Having classified all eigenvalues of  $\mathbf{T}$ , we would like to calculate the order- $1/N$  corrections to the leading band. The eigenvalues in this band are obtained as follows. For an arbitrary system size  $N = N_0$ , consider a pattern of zeros labelled by  $m_1$ ,  $m_2$ ,  $n_1$  and  $n_2$ . Now increase  $N$ , keeping  $m_1$ ,  $m_2$  and  $n_2$  fixed. From the completeness relation (4.4) we see that the number of 2-strings of type 1 must grow linearly with  $N$ . In addition, we keep fixed the relative ordering between  $\{v_k^{(1)}\}_{k=1}^{m_1}$  and  $\{w_k^{(1)}\}_{k=1}^{n_1}$  and the relative ordering between  $\{v_k^{(2)}\}_{k=1}^{m_2}$  and  $\{w_k^{(2)}\}_{k=1}^{n_2}$ . We also specify that the 2-strings of type 1 added as  $N$  is increased all have imaginary parts less than  $\min\{v_1^{(1)}, w_1^{(1)}, v_1^{(2)}, w_1^{(2)}\}$ . To obtain the leading band of eigenvalues in the thermodynamic limit we must finally take  $N_0 \rightarrow \infty$ .

The process of adding 2-strings of type 1 has the effect of pushing the original zeros away from the real  $u$ -axis. We observe from our numerics that this growth is logarithmic, and that the limits

$$\begin{aligned} y_k^{(j)} &= \lim_{N \rightarrow \infty} (5v_k^{(j)} - \ln N) \\ z_k^{(j)} &= \lim_{N \rightarrow \infty} (5w_k^{(j)} - \ln N) \end{aligned} \tag{4.10}$$

are well defined.

## 4.3 Free energies

Before calculating the order- $1/N$  corrections to the leading band of eigenvalues, we need to determine the order- $N$  and order-1 contributions. Because all of the zeros except those on the lines  $\text{Re}(u) = -\lambda/2$  and  $\text{Re}(u) = 3\lambda/2$  are pushed to infinity logarithmically with  $N$ , these contributions are the same for all eigenvalues in the leading band.

Since the number of zeros on the lines  $\text{Re}(u) = -\lambda/2$  and  $\text{Re}(u) = 3\lambda/2$  grows linearly with  $N$  while their extent grows logarithmically, these zeros become dense. Hence in the thermodynamic limit we consider two distinct regions of analyticity,  $-\lambda/2 < \text{Re}(u) < 3\lambda/2$  and  $3\lambda/2 < \text{Re}(u) < 9\lambda/2$ .

Up to this point our discussion has pertained to the transfer matrix  $\mathbf{T}$ . We now want to use the functional equation (2.7) for the normalized transfer matrix  $\mathbf{t}$ , so we have to consider additional zeros and poles introduced by the normalization factors in equation (2.6). The ratio of functions  $\gamma_k(u)$  induces poles of order  $2N$  at  $u = 2\lambda$  and  $u = 4\lambda$  and a zero of order  $2N$  at  $u = 3\lambda$ . Likewise, the function  $\beta(u)$  induces poles of order 1 at  $u = -\lambda/2$ ,  $u = 3\lambda/2$ ,  $u = 2\lambda$  and  $u = 4\lambda$ , and zeros of order 2 at  $u = \lambda/2$  and  $u = 3\lambda$ . Finally, the product  $\alpha_1(u - \xi)\alpha_{-1}(u + \xi)$  implies poles of

order 1 at  $u = \lambda + \xi$  and  $u = 5\lambda - \xi$ . Note that these last poles of order 1 cancel the anomalous zeros on the real axis induced by the boundary weight.

Writing an eigenvalue of  $\mathbf{t}$  as  $t$ , we isolate the order- $N$  and order-1 behaviour,

$$t(u) = f(u)g(u)l(u) \quad (4.11)$$

where the functions  $f$  and  $g$  are defined by

$$[f(u)]^{1/N} = \lim_{N \rightarrow \infty} [t(u)]^{1/N} \quad g(u) = \lim_{N \rightarrow \infty} t(u)/f(u). \quad (4.12)$$

The order- $N$  behaviour of the functional equation (2.7) is then [9, 33]

$$f(u)f(u + \lambda) = 1 \quad (4.13)$$

for  $-\lambda/2 < u < \lambda/2$  or  $3\lambda/2 < u < 7\lambda/2$ . The solution of this equation in accordance with the order- $N$  analyticity is

$$f(u) = \begin{cases} 1 & -\lambda/2 < u < 3\lambda/2 \\ \cot^{2N}(5u/2) & 3\lambda/2 < u < 9\lambda/2. \end{cases} \quad (4.14)$$

This solution in turn implies the order-1 functional equation

$$g(u)g(u + \lambda) = \begin{cases} 1 & -\lambda/2 < u < \lambda/2 \\ 1 + g(u - 2\lambda) & 3\lambda/2 < u < 7\lambda/2. \end{cases} \quad (4.15)$$

In the interval  $-\lambda/2 < u < 3\lambda/2$ , the solution in accordance with the order-1 analyticity is given by

$$g(u) = -\tan^2\left(\frac{5}{2}u - \frac{\pi}{4}\right). \quad (4.16)$$

The solution inside  $3\lambda/2 < u < 9\lambda/2$  can be obtained similarly, but will not concern us as it does not enter the calculation of the order- $1/N$  corrections.

#### 4.4 Computation of the order- $1/N$ corrections

For convenience, we start by selecting a line of constant real part within each of the two strips in the  $u$ -plane. For the functions  $t$ ,  $f$ ,  $g$  and  $l$ , it is then natural to define

$$\begin{aligned} h_1(x) &= h(\lambda/2 + ix/5) & \text{for } |\operatorname{Im}(x)| < \pi \\ h_2(x) &= h(3\lambda + ix/5) & \text{for } |\operatorname{Im}(x)| < 3\pi/2 \end{aligned} \quad (4.17)$$

as well as

$$H_1(x) = 1 + h_1(x) \quad H_2(x) = 1 + h_2(x). \quad (4.18)$$

Substituting (4.11) into the functional equation (2.7), and using equations (4.13) and (4.15), we obtain

$$\begin{aligned} l_1(x - i\frac{\pi}{2})l_1(x + i\frac{\pi}{2}) &= T_2(x) \\ l_2(x - i\frac{\pi}{2})l_2(x + i\frac{\pi}{2}) &= T_1(x)/G_1(x). \end{aligned} \quad (4.19)$$

For a given eigenvalue of string content  $(\mathbf{m}, \mathbf{n})$ , the function  $l_j(x)$  has  $m_j$  pairs of zeros inside the strip  $|\text{Im}(x)| < \pi$ . Hence

$$l_j(x) \prod_{k=1}^{m_j} \tanh[\frac{1}{2}(5v_k^{(j)} - x)] \tanh[\frac{1}{2}(5v_k^{(j)} + x)] \quad \text{for } j = 1, 2 \quad (4.20)$$

is free of poles and zeros inside this strip. Taking the Fourier transform of the logarithmic derivative of equation (4.19) thus gives, for  $|\text{Im}(x)| < \pi$ ,

$$\ln l_1(x) = \sum_{k=1}^{m_1} \ln(\tanh[\frac{1}{2}(5v_k^{(1)} - x)] \tanh[\frac{1}{2}(5v_k^{(1)} + x)]) + k * \ln T_2(x) + D_1 \quad (4.21)$$

$$\ln l_2(x) = \sum_{k=1}^{m_2} \ln(\tanh[\frac{1}{2}(5v_k^{(2)} - x)] \tanh[\frac{1}{2}(5v_k^{(2)} + x)]) + k * \ln(T_1/G_1)(x) + D_2 \quad (4.22)$$

where  $f * g$  denotes a convolution and the kernel  $k$  is

$$k(x) = \frac{1}{2\pi \cosh x}. \quad (4.23)$$

Before we can determine the integration constants  $D_1$  and  $D_2$ , we have to fix the branches of the logarithms of  $l_1$  and  $l_2$ . We first observe from numerical diagonalization of  $\mathbf{T}(u)$  that the complex function  $l_j(x - i\varepsilon)$ , where  $\varepsilon > 0$ , winds  $m_j/2$  times around the origin anticlockwise as  $x$  increases from zero to infinity. An example of this winding is given in figures 4 and 5. In selecting branches of  $\ln l_j$ , we regard  $l_j(x)$  as  $\lim_{\varepsilon \rightarrow 0} l_j(x - i\varepsilon)$ . At each point  $x = \pm 5v_k^{(j)}$  the real function  $l_j(x)$  changes sign. We specify that between  $x = \pm 5v_1^{(j)}$  the logarithm corresponds to the principal branch, so that for  $k = 1, \dots, m_j - 1$

$$\text{Im}(\ln l_j(x)) = \lim_{\varepsilon \rightarrow 0} \text{Im}(\ln l_j(x - i\varepsilon)) = \begin{cases} 0 & 0 \leq x < 5v_1^{(j)} \\ k\pi & 5v_k^{(j)} < x < 5v_{k+1}^{(j)} \\ m_j\pi & 5v_{m_j}^{(j)} < x. \end{cases} \quad (4.24)$$

We note that this definition also implies that  $\text{Im}(\ln l_j(5v_k^{(j)})) = (k - 1/2)\pi$ .

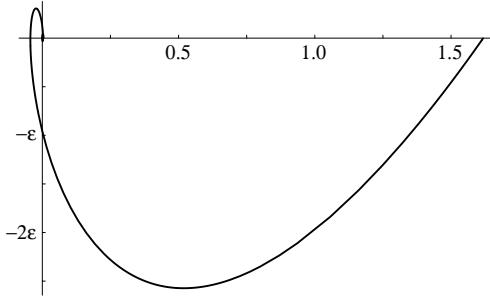


Figure 4: The function  $l_1(x - i\varepsilon)$  in the complex plane between  $x = 0$  and  $x = \infty$ , corresponding to the eigenvalue depicted in figure 2. As  $x$  increases, the function winds anticlockwise.

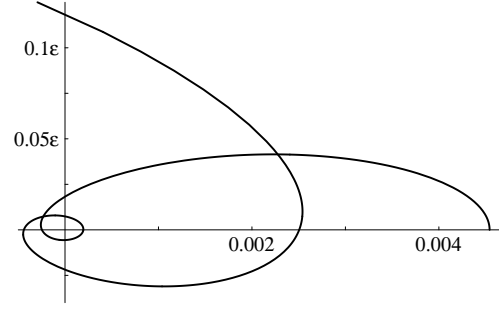


Figure 5: Magnification of figure 4 in the vicinity of the origin. The total of six intersections with the imaginary axis corresponds to  $m_1 = 6$  in figure 2.

We are now able to determine  $D_1$  and  $D_2$ , but it is convenient first to take the scaling limit. Guided by the function  $f_2$ , which behaves as

$$\lim_{N \rightarrow \infty} f_2(x + \ln N) = \exp(-4e^{-x}) \quad (4.25)$$

we assume the general scaling forms

$$\hat{h}(x) = \lim_{N \rightarrow \infty} h(x + \ln N). \quad (4.26)$$

It will be important later to note that

$$\begin{aligned} \lim_{x \rightarrow \infty} \hat{h}(x) &= \lim_{x \rightarrow \infty} h(x) \\ \lim_{x \rightarrow -\infty} \hat{h}(x) &= \lim_{x \rightarrow -\infty} \lim_{N \rightarrow \infty} h(x) \end{aligned} \quad (4.27)$$

as may be explicitly verified for the function  $f_2$ .

Now taking the scaling limit of (4.21) and (4.22) gives

$$\ln \hat{t}_1(x) = \sum_{k=1}^{m_1} \ln(\tanh[\frac{1}{2}(y_k^{(1)} - x)]) + k * \ln \hat{T}_2(x) \quad (4.28)$$

$$\ln \hat{t}_2(x) = -4e^{-x} + \sum_{k=1}^{m_2} \ln(\tanh[\frac{1}{2}(y_k^{(2)} - x)]) + k * \ln \hat{T}_1(x). \quad (4.29)$$

To obtain these expressions we have used the factorization (4.11), equations (4.10) and (4.25), the property

$$(k * \ln H)(\infty) = \ln H(\infty) \int_{-\infty}^{\infty} k(y) dy = \frac{1}{2} \ln H(\infty) \quad (4.30)$$

and the limits  $x \rightarrow \infty$  of equations (2.7), (4.13) and (4.15). From these and equation (4.24) we have  $D_1 = D_2 = 0$ .

We now return to the calculation of the finite-size corrections and express the order- $1/N$  contributions to  $\ln l_1$  in terms of scaling functions. Taking the first term on the right-hand side of equation (4.21), we use equation (4.10) to write

$$\sum_{k=1}^{m_1} \ln(\tanh[\frac{1}{2}(5v_k^{(1)} - x)] \tanh[\frac{1}{2}(5v_k^{(1)} + x)]) = -\frac{4}{N} \cosh x \sum_{k=1}^{m_1} e^{-y_k^{(1)}} + o\left(\frac{1}{N}\right). \quad (4.31)$$

Next we consider

$$\begin{aligned} k * \ln T_2(x) &= \int_0^\infty [k(x-y) + k(x+y)] \ln T_2(y) dy \\ &= \int_{-\ln N}^\infty [k(x-y-\ln N) + k(x+y+\ln N)] \ln T_2(y+\ln N) dy \end{aligned} \quad (4.32)$$

and assume that this can be replaced by

$$k * \ln T_2(x) = \frac{2}{\pi N} \cosh x \int_{-\infty}^\infty e^{-y} \ln \hat{T}_2(y) dy + o\left(\frac{1}{N}\right). \quad (4.33)$$

This assumption can be verified explicitly when  $T_2$  is replaced by  $1 + f_2$ . Hence, combining equations (4.21), (4.31) and (4.33), we have to leading order

$$\ln l_1(x) = -\frac{4}{N} \cosh x \left[ \sum_{k=1}^{m_1} e^{-y_k^{(1)}} - \frac{1}{2\pi} \int_{-\infty}^\infty e^{-y} \ln \hat{T}_2(y) dy \right]. \quad (4.34)$$

We now rewrite this expression by manipulating the functional equations (4.28) and (4.29). Evaluating (4.28) at  $x = y_k^{(2)} - i\frac{\pi}{2}$  and (4.29) at  $x = y_k^{(1)} - i\frac{\pi}{2}$  gives

$$\begin{aligned} 0 &= \frac{1}{2\pi} \int_{-\infty}^\infty \frac{\ln \hat{T}_2(y)}{\sinh(y_k^{(2)} - y)} dy - i \sum_{l=1}^{m_1} \ln(\tanh[\frac{1}{2}(y_l^{(1)} - y_k^{(2)} + i\frac{\pi}{2})]) \\ &\quad + i \ln \hat{t}_1(y_k^{(2)} - i\frac{\pi}{2}) \end{aligned} \quad (4.35)$$

and

$$\begin{aligned} 4e^{-y_k^{(1)}} &= \frac{1}{2\pi} \int_{-\infty}^\infty \frac{\ln \hat{T}_1(y)}{\sinh(y_k^{(1)} - y)} dy - i \sum_{l=1}^{m_2} \ln(\tanh[\frac{1}{2}(y_l^{(2)} - y_k^{(1)} + i\frac{\pi}{2})]) \\ &\quad + i \ln \hat{t}_2(y_k^{(1)} - i\frac{\pi}{2}). \end{aligned} \quad (4.36)$$

The scaling limit of the functional equation (2.7) implies that

$$\begin{aligned} \hat{t}_1(y_k^{(2)} - i\frac{\pi}{2}) &= \hat{t}_1(z_k^{(2)} - i\frac{\pi}{2}) = -1 \\ \hat{t}_2(y_k^{(1)} - i\frac{\pi}{2}) &= \hat{t}_2(z_k^{(1)} - i\frac{\pi}{2}) = -1. \end{aligned} \quad (4.37)$$

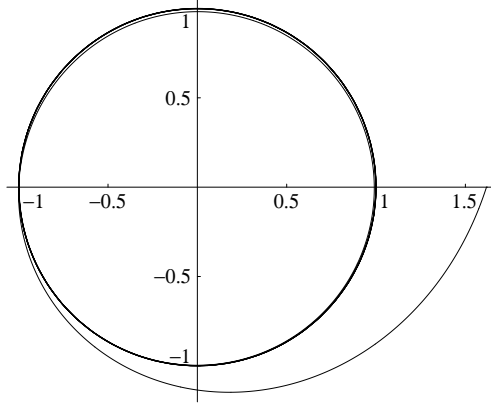


Figure 6: The function  $t_2(x - i\frac{\pi}{2})$  in the complex plane between  $x = 0$  and  $x = \infty$ , corresponding to the eigenvalue depicted in figure 2. As  $x$  decreases from infinity to  $w_1^{(1)} > 0$ , the argument of the function decreases by a total of  $(2m_1 + 2n_1 - 1)\pi = 17\pi$ . The function  $t_1(x - i\frac{\pi}{2})$  behaves similarly, but with a total decrease in argument of  $(2m_2 + 2n_2 - 1)\pi = 5\pi$ .

We now determine the branch of the logarithm at the points  $\hat{t}_j(y_k^{(j')} - i\frac{\pi}{2})$ , where  $j' = 3 - j$ . Decreasing  $x$  from infinity in the function  $\hat{t}_j(x - i\frac{\pi}{2})$ , we encounter the  $m_{j'} + n_{j'}$  points at which  $\hat{t}_j = -1$ . Between successive encounters  $\hat{t}_j$  decreases its argument by  $2\pi$ , a property which can be observed numerically for  $t_j$  as in figure 6. Since equation (4.24) implies  $\text{Im}(\ln \hat{t}_j(\infty)) = m_j\pi$ , we see that

$$\begin{aligned} \ln \hat{t}_1(y_k^{(2)} - i\frac{\pi}{2}) &= (2k + m_1 - 2m_2 - 1 - 2I_k^{(2)})i\pi \\ \ln \hat{t}_2(y_k^{(1)} - i\frac{\pi}{2}) &= (2k + m_2 - 2m_1 - 1 - 2I_k^{(1)})i\pi \end{aligned} \quad (4.38)$$

where  $I_k^{(j)}$  is the number of  $z_l^{(j)}$  greater than  $y_k^{(j)}$ . Since  $y_{k+1}^{(j)} > y_k^{(j)}$ , the integers  $I_k^{(j)}$  must satisfy

$$n_j \geq I_1^{(j)} \geq I_2^{(j)} \geq \dots \geq I_{m_j}^{(j)} \geq 0. \quad (4.39)$$

If we now sum equation (4.35) from  $k = 1$  to  $m_2$ , equation (4.36) from  $k = 1$  to  $m_1$  and use the relation

$$\ln(\tanh(i\frac{\pi}{4} + x)) + \ln(\tanh(i\frac{\pi}{4} - x)) = i\pi \quad (4.40)$$

we arrive at

$$\frac{4}{\pi} \sum_{k=1}^{m_1} e^{-y_k^{(1)}} = \frac{1}{2} \mathbf{m} C \mathbf{m} + \sum_{j=1}^2 \sum_{k=1}^{m_j} \left[ 2I_k^{(j)} + \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \frac{\ln \hat{T}_j(y)}{\sinh(y_k^{(j)} - y)} dy \right] \quad (4.41)$$

where  $C$  is the  $A_2$  Cartan matrix with entries given by  $C_{j,k} = 2\delta_{j,k} - \delta_{|j-k|,1}$ . Substituting (4.41) into equation (4.34) gives

$$\ln l_1(x) = -\frac{\pi}{N} \cosh x \left( \frac{1}{2} \mathbf{m} C \mathbf{m} + \sum_{j=1}^2 \sum_{k=1}^{m_j} \left[ 2I_k^{(j)} + \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \frac{\ln \hat{T}_j(y)}{\sinh(y_k^{(j)} - y)} dy \right] - \frac{2}{\pi^2} \int_{-\infty}^{\infty} e^{-y} \ln \hat{T}_2(y) dy \right). \quad (4.42)$$

It remains to evaluate the integrals in equation (4.42). Following reference [9], we consider the quantities

$$S_j = \int_{-\infty}^{\infty} \left( [\ln \hat{t}_j(x)]' \ln \hat{T}_j(x) - \ln \hat{t}_j(x) [\ln \hat{T}_j(x)]' \right) dx \quad \text{for } j = 1, 2. \quad (4.43)$$

On the one hand we can use the explicit expressions for  $\ln \hat{t}_j$  given in equations (4.28) and (4.29) to write

$$S_1 + S_2 = 8 \int_{-\infty}^{\infty} e^{-x} \ln \hat{T}_2(x) dx - \sum_{j=1}^2 \left[ i\pi m_j \hat{T}_j(\infty) + 2 \sum_{k=1}^{m_j} \int_{-\infty}^{\infty} \frac{\ln \hat{T}_j(x)}{\sinh(y_k^{(j)} - x)} dx \right] \quad (4.44)$$

where due to the symmetry  $k(x) = k(-x)$  all terms involving the kernel have cancelled.

On the other hand, we can change integration variables in equation (4.43) to evaluate  $S_j$  directly. Dividing the line of integration into the  $m_j + 1$  intervals between  $-\infty < y_1^{(j)} < \dots < y_{m_j}^{(j)} < \infty$ , we note that for all  $k = 1, \dots, m_j - 1$ ,

$$\begin{aligned} \int_{y_k^{(j)}}^{y_{k+1}^{(j)}} \left( [\ln \hat{t}_j(x)]' \ln \hat{T}_j(x) - \ln \hat{t}_j(x) [\ln \hat{T}_j(x)]' \right) dx \\ = \int_{\hat{t}_j(y_k^{(j)})}^{\hat{t}_j(y_{k+1}^{(j)})} \left( \frac{\ln(1+x)}{x} - \frac{\ln x + i\pi k}{1+x} \right) dx = 0. \end{aligned} \quad (4.45)$$

Hence we are left with

$$\begin{aligned} S_j &= \int_{\hat{t}_j(-\infty)}^0 \left( \frac{\ln(1+x)}{x} - \frac{\ln x}{1+x} \right) dx + \int_0^{\hat{t}_j(\infty)} \left( \frac{\ln(1+x)}{x} - \frac{\ln x + i\pi m_j}{1+x} \right) dx \\ &= \int_{\hat{t}_j(-\infty)}^{\hat{t}_j(\infty)} \left( \frac{\ln(1+x)}{x} - \frac{\ln x}{1+x} \right) dx - i\pi m_j \hat{T}_j(\infty) \end{aligned} \quad (4.46)$$

From equations (2.11), (4.14), (4.16) and (4.27), we obtain the limits

$$\hat{t}_1(\infty) = \hat{t}_2(\infty) = 2 \cos \lambda \quad \hat{t}_1(-\infty) = g_1(-\infty) = 1 \quad \hat{t}_2(-\infty) = f_2(-\infty) = 0 \quad (4.47)$$

which give, using the values of the Rogers dilogarithm in the appendix,

$$S_1 + S_2 + i\pi \sum_{j=1}^2 m_j \hat{T}_j(\infty) = 4L\left(\frac{1}{2\cos\lambda}\right) - 2L\left(\frac{1}{2}\right) = \frac{7\pi^2}{30}. \quad (4.48)$$

Equating (4.44) and (4.48), we finally arrive at

$$\frac{2}{\pi^2} \int_{-\infty}^{\infty} e^{-x} \ln \hat{T}_2(x) dx + \frac{1}{2\pi^2} \sum_{j=1}^2 \sum_{k=1}^{m_j} \frac{\ln \hat{T}_j(x)}{\sinh(y_k^{(j)} - x)} dx = \frac{7}{120}. \quad (4.49)$$

Substituting equation (4.49) into equation (4.42), then recalling equation (4.17), we have

$$\ln l(u) = \frac{2\pi}{N} \sin(5u) \left[ \frac{7}{240} - \frac{1}{4} \mathbf{m} C \mathbf{m} - \sum_{j=1}^2 \sum_{k=1}^{m_j} I_k^{(j)} \right]. \quad (4.50)$$

Since  $m_1$  and  $m_2$  are even, the “minimal” solution of equation (4.6) is  $m_1 = m_2 = 0$ . Thus comparison of (4.50) with equation (3.3) shows that  $c = 7/10$  and  $\Delta = 0$ , and hence that the integer  $n$  is to be identified with

$$n = \frac{1}{4} \mathbf{m} C \mathbf{m} + \sum_{j=1}^2 \sum_{k=1}^{m_j} I_k^{(j)}. \quad (4.51)$$

From equation (3.4) we see that to calculate the conformal partition function  $Z(q)$  we must sum over  $q^n$ . From equation (4.51), this amounts to summing over all allowed values  $\mathbf{m}$  and  $I_k^{(j)}$ . We first perform the sum over  $I_k^{(j)}$  using the restrictions (4.39)

$$\sum_{I_1^{(j)}=0}^{n_j} \sum_{I_2^{(j)}=0}^{I_1^{(j)}} \dots \sum_{I_{m_j}^{(j)}=0}^{I_{m_j-1}^{(j)}} q^{I_1^{(j)} + \dots + I_{m_j}^{(j)}} = \begin{bmatrix} m_j + n_j \\ m_j \end{bmatrix}. \quad (4.52)$$

Here the  $q$ -binomial coefficient is defined by

$$\begin{bmatrix} m+n \\ m \end{bmatrix} = \begin{cases} \frac{(q)_{m+n}}{(q)_m (q)_n} & m, n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.53)$$

with  $(q)_m = (1-q) \cdots (1-q^m)$  for  $m \geq 1$  and  $(q)_0 = 1$ . Now summing over  $\mathbf{m}$  gives

$$Z_{N_0}(q) = q^{-c/24} \sum_{\substack{m_1 \text{ even} \\ m_2 \text{ even}}} q^{\frac{1}{4} \mathbf{m} C \mathbf{m}} \begin{bmatrix} m_1 + n_1 \\ m_1 \end{bmatrix} \begin{bmatrix} m_2 + n_2 \\ m_2 \end{bmatrix} \quad (4.54)$$

with  $\mathbf{m} + \mathbf{n} = \frac{1}{2}(N_0 \mathbf{e}_1 + \mathcal{I}\mathbf{m})$ . Following our description of the leading band of eigenvalues in section 4.2, we take  $N_0 \rightarrow \infty$  to obtain

$$Z(q) = q^{-c/24} \sum_{\substack{m_1 \text{ even} \\ m_2 \text{ even}}} \frac{q^{\frac{1}{4}\mathbf{mCm}}}{(q)_{m_1}} \left[ \begin{matrix} \frac{1}{2}m_1 \\ m_2 \end{matrix} \right] = \chi_{1,1}(q). \quad (4.55)$$

Thus we see that the conformal partition function for the  $(r, s) = (1, 1)$  boundary condition is the  $\chi_{1,1}(q)$  Virasoro character, and that it arises naturally in its fermionic representation [34].

## 5 The character $\chi_{1,2}$

We next consider a lattice with boundary heights  $(r, s) = (1, 2)$ . We will see that the main difference in comparison to the  $(1, 1)$  boundary occurs in the classification of the transfer matrix eigenvalues, and we find the conformal partition function  $Z(q) = \chi_{1,2}(q)$ .

### 5.1 Classification of zeros

Our numerics show that the zero patterns of eigenvalues of  $\mathbf{T}$  with the  $(1, 2)$  boundary condition are qualitatively the same as those with the  $(1, 1)$  boundary condition. That is, the same four string types occur, and the same zeros on the real  $u$ -axis are induced by the left boundary weight. Therefore we again define  $(\mathbf{m}, \mathbf{n})$  by equation (4.3). The difference between the two boundary conditions lies in the orderings of the strings of type 1. Previously, given a string content  $(\mathbf{m}, \mathbf{n})$ , eigenvalues corresponding to all  $\binom{m_1+n_1}{m_1}$  orderings of strings of type 1 were seen to occur. In this case, however, we observe that for certain string contents, the 1-string of type 1 furthest from the real  $u$ -axis is “frozen”, so that no eigenvalue occurs with  $v_{m_1}^{(1)} < w_{n_1}^{(1)}$ . Hence for these contents, only  $\binom{m_1+n_1-1}{m_1-1}$  orderings of the strings of type 1 are permitted.

To treat both cases simultaneously, we introduce a variable  $\sigma = \pm 1$ , with  $\sigma = 1$  for contents with a frozen 1-string, and  $\sigma = -1$  for the other contents. We then find that the strings satisfy the  $(\mathbf{m}, \mathbf{n})$ -system

$$\begin{aligned} m_1 + n_1 &= \frac{1}{2}(N + m_2 + \sigma) \\ m_2 + n_2 &= \frac{1}{2}(m_1 - \sigma). \end{aligned} \quad (5.1)$$

Since  $N$  is odd, this implies that  $m_1$  is odd and  $m_2$  is even. Using (5.1), the total

number of eigenvalues is correctly computed to be

$$\sum_{\sigma} \sum_{\substack{m_1 \text{ odd} \\ m_2 \text{ even}}} \binom{m_1 + n_1 - \delta_{\sigma,1}}{m_1 - \delta_{\sigma,1}} \binom{m_2 + n_2}{m_2} = A_{1,2}^N. \quad (5.2)$$

## 5.2 Computation of the order-1/ $N$ corrections

As the order- $N$  analyticity is independent of the choice of boundary conditions, the solution of the functional equation (4.13) for  $f$  is once again given by (4.14). The order-1 analyticity, and hence the solution of equation (4.15) for  $g$ , does depend on the boundary conditions, but in this case it is unchanged from the description of section 4.3. Thus the function  $g$  for the (1,2) boundary is again given by equation (4.16).

Recalling the definition of  $l_j$  from equations (4.11) and (4.17), and proceeding with the Fourier transformation as performed in section 4.4, we again obtain equations (4.21) and (4.22). Examination of the winding of the function  $l_j(x - i\varepsilon)$  shows the same behaviour as before, and we choose branches of the logarithms in accordance with equation (4.24). Now taking the scaling limit of (4.21) and (4.22), and again using the limits  $x \rightarrow \infty$  of equations (2.7), (4.13) and (4.15) in conjunction with (4.10) and (4.24), we obtain

$$\begin{aligned} \ln \hat{t}_1(x) &= \sum_{k=1}^{m_1} \ln(\tanh[\frac{1}{2}(y_k^{(1)} - x)]) + k * \ln \hat{T}_2(x) \\ \ln[-\hat{t}_2(x)] &= -4e^{-x} + \sum_{k=1}^{m_2} \ln(\tanh[\frac{1}{2}(y_k^{(2)} - x)]) + k * \ln \hat{T}_1(x). \end{aligned} \quad (5.3)$$

We note that since the limit  $\lim_{iu \rightarrow \infty} \mathbf{t}(u)_{\mathbf{a}, \mathbf{a}}$  in equation (2.11) is negative when  $s = 2$ , our signs inside the logarithms of  $\hat{t}_1$  and  $\hat{t}_2$  are such that

$$\text{Im}(\ln \hat{t}_1(\infty)) = m_1 \pi \quad \text{Im}(\ln[-\hat{t}_2(\infty)]) = m_2 \pi \quad (5.4)$$

are consistent with the parities of  $m_1$  and  $m_2$ .

Following the same procedure as in section 4.4, we will use (5.3) to rewrite the order-1/ $N$  term (4.34). The crucial step, and the one at which differences between this and the (1,1) boundary case are evident, is the determination of  $\ln \hat{t}_1(y_k^{(2)} - i\frac{\pi}{2})$  and  $\ln[-\hat{t}_2(y_k^{(1)} - i\frac{\pi}{2})]$ . When  $\sigma = -1$ , the behaviours of  $t_1(x - i\frac{\pi}{2})$  and  $t_2(x - i\frac{\pi}{2})$  are given respectively in figures 7 and 8. When  $\sigma = 1$ , the qualitative features of the two plots are interchanged. In each case the arguments are decreasing as  $x$  decreases from infinity, but we see that in figure 7 the function first passes through

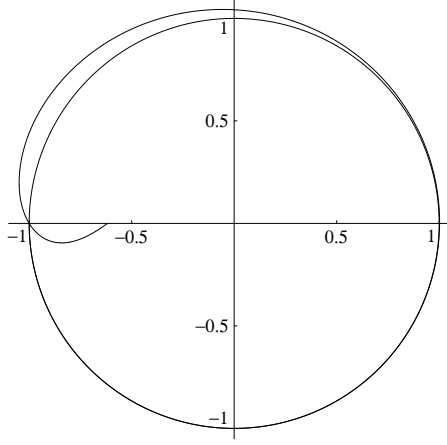


Figure 7: The function  $t_1(x - i\frac{\pi}{2})$  in the complex plane between  $x = 0$  and  $x = \infty$ . As  $x$  decreases the winding is clockwise.

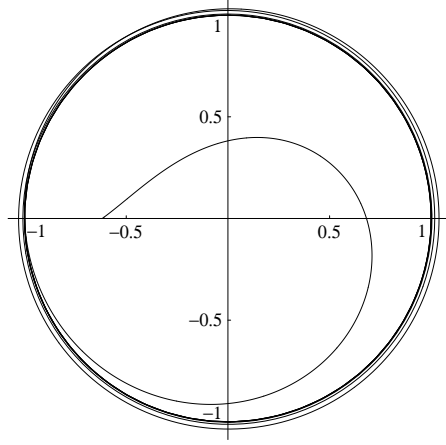


Figure 8: The function  $t_2(x - i\frac{\pi}{2})$  in the complex plane between  $x = 0$  and  $x = \infty$ . As  $x$  decreases the winding is clockwise.

the point  $-1$  with the same argument as at  $x = \infty$ , while in figure 8 the function's argument has decreased by  $2\pi$  at the corresponding point. Recalling equation (5.4), we therefore see that

$$\begin{aligned} \ln \hat{t}_1(y_k^{(2)} - i\frac{\pi}{2}) &= (2k + m_1 - 2m_2 - \sigma - 1 - 2I_k^{(2)})i\pi \\ \ln[-\hat{t}_2(y_k^{(1)} - i\frac{\pi}{2})] &= (2k + m_2 - 2m_1 + \sigma - 1 - 2I_k^{(1)})i\pi \end{aligned} \quad (5.5)$$

where  $I_k^{(j)}$  is the number of the  $z_i^{(j)}$  greater than  $y_k^{(j)}$ . The integers  $I_k^{(j)}$  again satisfy the restrictions (4.39), but with the additional restriction  $I_{m_1}^{(1)} = 0$  when  $\sigma = 1$ .

With the branches of the logarithms in (5.5), the manipulation of (4.21) leads to the expression

$$\begin{aligned} \ln l_1(x) &= -\frac{\pi}{N} \cosh x \left( \frac{1}{2} \mathbf{m} \mathbf{C} \mathbf{m} - (m_1 - m_2) \sigma \right. \\ &\quad \left. + \sum_{j=1}^2 \sum_{k=1}^{m_j} \left[ 2I_k^{(j)} + \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \frac{\ln \hat{T}_j(y)}{\sinh(y_k^{(j)} - y)} dy \right] - \frac{2}{\pi^2} \int_{-\infty}^{\infty} e^{-y} \ln \hat{T}_2(y) dy \right). \end{aligned} \quad (5.6)$$

This corresponds to equation (4.42) of section 4.4. As in that section, we evaluate the integrals in equation (5.6) by considering

$$\sum_{j=1}^2 \int_{-\infty}^{\infty} \left( [\ln \hat{t}_j(x)]' \ln \hat{T}_j(x) - \ln[\pm \hat{t}_j(x)] [\ln \hat{T}_j(x)]' \right) dx. \quad (5.7)$$

Substituting the expressions (5.3) for  $\ln \hat{t}_1$  and  $\ln[-\hat{t}_2]$  leads once again to equation (4.44). Equating this to the expression obtained by changing variables of inte-

gration and using the limits

$$\hat{t}_1(\infty) = \hat{t}_2(\infty) = 2 \cos(3\lambda) \quad \hat{t}_1(-\infty) = 1 \quad \hat{t}_2(-\infty) = 0 \quad (5.8)$$

we arrive at

$$\begin{aligned} \frac{2}{\pi^2} \int_{-\infty}^{\infty} e^{-x} \ln \hat{T}_2(x) dx + \frac{1}{2\pi^2} \sum_{j=1}^2 \sum_{k=1}^{m_j} \frac{\ln \hat{T}_j(x)}{\sinh(y_k^{(j)} - x)} dx \\ = -\frac{1}{4\pi^2} \left[ 4L(-2 \cos(3\lambda)) + 2L\left(\frac{1}{2}\right) \right] = -\frac{17}{120}. \end{aligned} \quad (5.9)$$

Now substituting (5.9) into equation (5.6), we obtain

$$\ln l(u) = -\frac{2\pi}{N} \sin(5u) \left[ \frac{17}{240} + \frac{1}{4} \mathbf{mCm} - \frac{1}{2} (m_1 - m_2) \sigma + \sum_{j=1}^2 \sum_{k=1}^{m_j} I_k^{(j)} \right]. \quad (5.10)$$

The minimal solution of equation (5.1) is  $m_1 = 1$  and  $m_2 = 0$  with  $\sigma = 1$ , so in this case  $\Delta = 1/10$  and the integer  $n$  of equation (3.3) is given by

$$n = \frac{1}{4} \mathbf{mCm} - \frac{1}{2} (m_1 - m_2) \sigma + \sum_{j=1}^2 \sum_{k=1}^{m_j} I_k^{(j)}. \quad (5.11)$$

Summing over all eigenvalues in the leading band gives, recalling the restrictions on the integers  $I_k^{(j)}$ ,

$$Z_{N_0}(q) = q^{-\frac{c}{24} + \Delta_{1,2}} \sum_{\sigma} \sum_{\substack{m_1 \text{ odd} \\ m_2 \text{ even}}} q^{\frac{1}{4} \mathbf{mCm} - \frac{1}{2} (m_1 - m_2) \sigma} \begin{bmatrix} m_1 + n_1 - \delta_{\sigma,1} \\ m_1 - \delta_{\sigma,1} \end{bmatrix} \begin{bmatrix} m_2 + n_2 \\ m_2 \end{bmatrix}. \quad (5.12)$$

Eliminating  $n_1$  and  $n_2$  using equation (5.1), we employ the standard  $q$ -binomial recurrence

$$\begin{bmatrix} N \\ m \end{bmatrix} = \begin{bmatrix} N-1 \\ m-1 \end{bmatrix} + q^m \begin{bmatrix} N-1 \\ m \end{bmatrix} \quad (5.13)$$

to perform the sum over  $\sigma$ . We thus obtain

$$Z_{N_0}(q) = q^{-\frac{c}{24} + \Delta_{1,2}} \sum_{\substack{m_1 \text{ even} \\ m_2 \text{ even}}} q^{\frac{1}{4} \mathbf{mCm} - \frac{1}{2} m_1} \begin{bmatrix} \frac{1}{2}(N_0 + m_2 + 1) \\ m_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} m_1 \\ m_2 \end{bmatrix}. \quad (5.14)$$

Finally, taking  $N_0 \rightarrow \infty$  gives

$$Z(q) = q^{-\frac{c}{24} + \Delta_{1,2}} \sum_{\substack{m_1 \text{ even} \\ m_2 \text{ even}}} \frac{q^{\frac{1}{4} \mathbf{mCm} - \frac{1}{2} m_1}}{(q)_{m_1}} \begin{bmatrix} \frac{1}{2} m_1 \\ m_2 \end{bmatrix} \quad (5.15)$$

which is the  $\chi_{1,2}(q)$  Virasoro character [34].

## 6 The character $\chi_{2,1}$

### 6.1 Classification of zeros

Under the boundary condition (2, 1) the zeros of eigenvalues of  $\mathbf{T}$  again form the same four string types defined in section 4.1. In addition to the strings, however, we find that some eigenvalues have zeros on the real  $u$ -axis. Such eigenvalues have either a single pair of zeros at  $u = 3\lambda \pm \alpha$ , or a single quadruplet of zeros at  $u = 2\lambda \pm \alpha$  and  $u = 4\lambda \pm \alpha$ . The numbers  $\alpha$  satisfy  $0 < \alpha < \lambda/2$ , and are in general different for each eigenvalue. We make the crucial observation that if we fix a set of eigenvalues for an arbitrary system size  $N_0$ , and increase the system size by adding 2-strings of type 1 in accordance with the description of the leading band, then for each eigenvalue there exists a  $K$  such that  $\alpha = 0$  for all  $N > K$ . Indeed, we see that each pair of real zeros becomes a 1-string of type 2, and that each quadruplet of real zeros becomes a 2-string of type 2.

It is therefore clear that any zeros on the real  $u$ -axis should be interpreted as forming strings of type 2. With this interpretation, and with the definitions of equation (4.3), we observe that the strings satisfy the  $(\mathbf{m}, \mathbf{n})$ -system

$$\begin{aligned} m_1 + n_1 &= \frac{1}{2}(N + m_2) \\ m_2 + n_2 &= \frac{1}{2}(m_1 + 1) \end{aligned} \tag{6.1}$$

so that  $m_1$  and  $m_2$  are odd. Using (6.1) we correctly compute

$$\sum_{\substack{m_1 \text{ odd} \\ m_2 \text{ odd}}} \binom{m_1 + n_1}{m_1} \binom{m_2 + n_2}{m_2} = A_{2,1}^N. \tag{6.2}$$

### 6.2 Free energies

The order- $N$  behaviour is, of course, the same as for the previous two boundary conditions, but the boundary spin  $r = 2$  induces different order-1 analyticity on the real  $u$ -axis. We see from section 6.1 that in the limit  $N \rightarrow \infty$  the eigenvalues of  $\mathbf{T}$  in the leading band are free of zeros on the real  $u$ -axis. However, the product  $\alpha_2(u - \xi)\alpha_{-2}(u + \xi)$  in the normalization (2.6) of  $\mathbf{t}$  now introduces zeros of order 1 at  $u = \xi - \lambda$  and  $u = 2\lambda - \xi$ , and poles of order 1 at  $u = \lambda - \xi$ ,  $u = \xi$  and  $u = 3\lambda \pm \xi$ . Of these points,  $u = \xi - \lambda$ ,  $u = \lambda - \xi$ ,  $u = \xi$  and  $u = 2\lambda - \xi$  fall into the strip  $-\lambda/2 < u < 3\lambda/2$ . The zeros and poles of  $\beta$  are unchanged, so the solution of equation (4.15) inside this strip is now

$$g(u) = \tan^2\left(\frac{5}{2}u - \frac{\pi}{4}\right) \tan\left[\frac{5}{2}(u + \xi)\right] \cot\left[\frac{5}{2}(u - \xi)\right]. \tag{6.3}$$

It immediately follows that  $\lim_{x \rightarrow \pm\infty} g_1(x) = -1$ .

### 6.3 Computation of the order-1/ $N$ corrections

The computation of the order-1/ $N$  corrections is very similar to that presented in section 4.4. We need simply to replace  $\hat{t}_1$  and  $\hat{t}_2$  by their negatives on the left-hand sides of equations (4.28) and (4.29), and hence in equations (4.35), (4.36) and (4.38). This is necessary for our choice of the branches of the logarithms of  $\hat{t}_1$  and  $\hat{t}_2$  to be consistent with the parities of  $m_1$  and  $m_2$ . Once this change is made, the whole of section 4.4 from equation (4.17) to equation (4.42) holds for this boundary condition.

Instead of the expression (4.43), we now consider

$$\sum_{j=1}^2 \int_{-\infty}^{\infty} \left( [\ln \hat{t}_j(x)]' \ln \hat{T}_j(x) - \ln[-\hat{t}_j(x)] [\ln \hat{T}_j(x)]' \right) dx. \quad (6.4)$$

The left-hand side of equation (4.49) is unchanged, but using the limits

$$\hat{t}_1(\infty) = \hat{t}_2(\infty) = 2 \cos \lambda \quad \hat{t}_1(-\infty) = -1 \quad \hat{t}_2(-\infty) = 0 \quad (6.5)$$

the right-hand side is replaced by

$$\frac{1}{4\pi^2} \left[ 4L\left(\frac{1}{2 \cos \lambda}\right) + 2L(1) \right] = \frac{11}{60}. \quad (6.6)$$

Hence in place of (4.50) we obtain

$$\ln l(u) = -\frac{2\pi}{N} \sin(5u) \left[ -\frac{11}{120} + \frac{1}{4} \mathbf{mCm} + \sum_{j=1}^2 \sum_{k=1}^{m_j} I_k^{(j)} \right]. \quad (6.7)$$

The minimal solution  $m_1 = m_2 = 1$  of equation (6.1) gives  $\mathbf{mCm} = 2$ , so we have  $\Delta = 7/16$  and the identification

$$n + \frac{1}{2} = \frac{1}{4} \mathbf{mCm} + \sum_{j=1}^2 \sum_{k=1}^{m_j} I_k^{(j)}. \quad (6.8)$$

Summing over the leading band of eigenvalues thus gives

$$Z_{N_0}(q) = q^{-\frac{c}{24} + \Delta_{2,1} - \frac{1}{2}} \sum_{\substack{m_1 \text{ odd} \\ m_2 \text{ odd}}} q^{\frac{1}{4} \mathbf{mCm}} \begin{bmatrix} m_1 + n_1 \\ m_1 \end{bmatrix} \begin{bmatrix} m_2 + n_2 \\ m_2 \end{bmatrix}. \quad (6.9)$$

Eliminating  $n_1$  and  $n_2$  using equation (6.1) and taking  $N_0 \rightarrow \infty$  finally yields

$$Z(q) = q^{-\frac{c}{24} + \Delta_{2,1} - \frac{1}{2}} \sum_{\substack{m_1 \text{ odd} \\ m_2 \text{ odd}}} \frac{q^{\frac{1}{4} \mathbf{mCm}}}{(q)_{m_1}} \begin{bmatrix} \frac{1}{2}(m_1 + 1) \\ m_2 \end{bmatrix} = \chi_{2,1}(q). \quad (6.10)$$

## 7 The character $\chi_{2,2}$

### 7.1 Classification of zeros

Under the (2, 2) boundary condition we see, in addition to the usual four string types, both the “frozen” zeros of section 5.1 and the zeros on the real  $u$ -axis of section 6.1. Once again we see that these latter zeros become strings of type 2 for  $N$  sufficiently large.

As in section 5.1, we let  $\sigma = 1$  for contents with the frozen 1-string and  $\sigma = -1$  otherwise. With the definitions of equation (4.3), we then observe the  $(\mathbf{m}, \mathbf{n})$ -system

$$\begin{aligned} m_1 + n_1 &= \frac{1}{2}(N + m_2 + \sigma) \\ m_2 + n_2 &= \frac{1}{2}(m_1 - \sigma + 1) \end{aligned} \quad (7.1)$$

so that  $m_1$  is even and  $m_2$  is odd. Using equation (7.1), the total number of eigenvalues is correctly computed to be

$$\sum_{\sigma} \sum_{\substack{m_1 \text{ even} \\ m_2 \text{ odd}}} \binom{m_1 + n_1 - \delta_{\sigma,1}}{m_1 - \delta_{\sigma,1}} \binom{m_2 + n_2}{m_2} = A_{2,2}^N. \quad (7.2)$$

### 7.2 Computation of the order-1/ $N$ corrections

With the same height  $r = 2$ , the free energies under this boundary condition are the same as those given in section 6.2. We now proceed precisely in the manner of section 5.2, with the simple replacement of  $\hat{t}_1$  by  $-\hat{t}_1$  and  $-\hat{t}_2$  by  $\hat{t}_2$  in accordance with the parities of  $m_1$  and  $m_2$ . The calculation of section 5.2 then applies from equation (5.3) to equation (5.7). The limits

$$\hat{t}_1(\infty) = \hat{t}_2(\infty) = 2 \cos(3\lambda) \quad \hat{t}_1(-\infty) = -1 \quad \hat{t}_2(-\infty) = 0 \quad (7.3)$$

change the right-hand side of (5.9) to

$$-\frac{1}{4\pi^2} [4L(-2 \cos(3\lambda)) - 2L(1)] = -\frac{1}{60}. \quad (7.4)$$

Hence we obtain

$$\ln l(u) = -\frac{2\pi}{N} \sin(5u) \left[ \frac{1}{120} + \frac{1}{4} \mathbf{m} C \mathbf{m} - \frac{1}{2} (m_1 - m_2) \sigma + \sum_{j=1}^2 \sum_{k=1}^{m_j} I_k^{(j)} \right]. \quad (7.5)$$

The minimal solution  $m_1 = 0$  and  $m_2 = 1$  with  $\sigma = -1$  of equation (7.1) leads to the identifications  $\Delta = 3/80$  and

$$n = \frac{1}{4} \mathbf{m} C \mathbf{m} - \frac{1}{2} (m_1 - m_2) \sigma + \sum_{j=1}^2 \sum_{k=1}^{m_j} I_k^{(j)} \quad (7.6)$$

with  $I_{m_1}^{(1)} = 0$  if  $\sigma = 1$ . Summing over the leading band of eigenvalues, we have

$$Z_{N_0}(q) = q^{-\frac{c}{24} + \Delta_{2,2}} \sum_{\sigma} \sum_{\substack{m_1 \text{ even} \\ m_2 \text{ odd}}} q^{\frac{1}{4} \mathbf{m} \mathbf{C} \mathbf{m} - \frac{1}{2} (m_1 - m_2) \sigma} \begin{bmatrix} m_1 + n_1 - \delta_{\sigma,1} \\ m_1 - \delta_{\sigma,1} \end{bmatrix} \begin{bmatrix} m_2 + n_2 \\ m_2 \end{bmatrix}. \quad (7.7)$$

Eliminating  $n_1$  and  $n_2$  using equation (7.1), and summing over  $\sigma$  using the recurrence (5.13), we obtain

$$Z_{N_0}(q) = q^{-\frac{c}{24} + \Delta_{2,2}} \sum_{\substack{m_1 \text{ odd} \\ m_2 \text{ odd}}} q^{\frac{1}{4} \mathbf{m} \mathbf{C} \mathbf{m} - \frac{1}{2} m_1} \begin{bmatrix} \frac{1}{2}(N_0 + m_2 + 1) \\ m_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(m_1 + 1) \\ m_2 \end{bmatrix}. \quad (7.8)$$

Finally, taking  $N_0 \rightarrow \infty$  gives

$$Z(q) = q^{-\frac{c}{24} + \Delta_{2,2}} \sum_{\substack{m_1 \text{ odd} \\ m_2 \text{ odd}}} \frac{q^{\frac{1}{4} \mathbf{m} \mathbf{C} \mathbf{m} - \frac{1}{2} m_1}}{(q)_{m_1}} \begin{bmatrix} \frac{1}{2}(m_1 + 1) \\ m_2 \end{bmatrix} = \chi_{2,2}(q). \quad (7.9)$$

## 8 The character $\chi_{3,1}$

### 8.1 Classification of zeros

Under the (3, 1) boundary condition, the zero patterns of eigenvalues of  $\mathbf{T}$  are very simple. We see the usual string types, no zeros on the real  $u$ -axis, and no “frozen” zeros. Defining  $(\mathbf{m}, \mathbf{n})$  by equation (4.3) we then observe

$$\begin{aligned} m_1 + n_1 &= \frac{1}{2}(N + m_2 + 1) \\ m_2 + n_2 &= \frac{1}{2}m_1 \end{aligned} \quad (8.1)$$

so that  $m_1$  is even and  $m_2$  is odd. Using equation (8.1), the total number of eigenvalues is correctly computed to be

$$\sum_{\substack{m_1 \text{ even} \\ m_2 \text{ odd}}} \binom{m_1 + n_1}{m_1} \binom{m_2 + n_2}{m_2} = A_{3,1}^N. \quad (8.2)$$

### 8.2 Computation of the order-1/ $N$ corrections

As there are no zeros of  $\mathbf{T}$  on the real  $u$ -axis, the order-1 analyticity of  $\mathbf{t}$  is determined solely by the zeros and poles of the normalizing functions  $\beta$  and  $\alpha$ . Since the function  $\beta$  is the same for all boundary conditions, the order-1 analyticity is as given in section 4.3, but with the product  $\alpha_3(u - \xi)\alpha_{-3}(u + \xi)$  now inducing zeros at

$u = 2\lambda + \xi$  and  $u = 4\lambda - \xi$ , and poles at  $u = \lambda + \xi$ ,  $u = 3\lambda \pm \xi$  and  $u = 5\lambda - \xi$ . These additional zeros and poles all fall inside  $3\lambda/2 < u < 9\lambda/2$ , so the solution (4.16) for  $g$  inside the interval  $-\lambda/2 < u < 3\lambda/2$  is unchanged for the (3, 1) boundary condition.

Indeed, the differences between the computation of the order-1/ $N$  corrections for the (3, 1) and the (1, 1) boundary conditions are also very small. All of the calculation from equation (4.17) to equation (4.50) proceeds in an identical fashion, with the simple replacement of  $\hat{t}_2$  by its negative in accordance with the change in parity of  $m_2$ .

The minimal solution of equation (8.1) is  $m_1 = 2$  and  $m_2 = 1$ , so  $\Delta = 3/2$  and equation (4.51) is replaced by

$$n + \frac{3}{2} = \frac{1}{4} \mathbf{m} C \mathbf{m} + \sum_{j=1}^2 \sum_{k=1}^{m_j} I_k^{(j)}. \quad (8.3)$$

Summing over the leading band of eigenvalues, we therefore have

$$Z_{N_0}(q) = q^{-\frac{c}{24}} \sum_{\substack{m_1 \text{ even} \\ m_2 \text{ odd}}} q^{\frac{1}{4} \mathbf{m} C \mathbf{m}} \begin{bmatrix} m_1 + n_1 \\ m_1 \end{bmatrix} \begin{bmatrix} m_2 + n_2 \\ m_2 \end{bmatrix}. \quad (8.4)$$

Eliminating  $n_1$  and  $n_2$  using equation (8.1), and taking  $N_0 \rightarrow \infty$ , we finally obtain the partition function

$$Z(q) = q^{-\frac{c}{24}} \sum_{\substack{m_1 \text{ even} \\ m_2 \text{ odd}}} \frac{q^{\frac{1}{4} \mathbf{m} C \mathbf{m}}}{(q)_{m_1}} \begin{bmatrix} \frac{1}{2} m_1 \\ m_2 \end{bmatrix} = \chi_{3,1}(q). \quad (8.5)$$

## 9 The character $\chi_{3,2}$

### 9.1 Classification of zeros

For the (3, 2) boundary condition we see the usual four strings, no zeros on the real  $u$ -axis, and the “frozen” 1-string of type 1 for certain string contents. As in sections 5.1 and 7.1, we let  $\sigma = 1$  for contents with the frozen 1-string and  $\sigma = -1$  otherwise. With the definitions of equation (4.3), we then observe the  $(\mathbf{m}, \mathbf{n})$ -system

$$\begin{aligned} m_1 + n_1 &= \frac{1}{2}(N + m_2 + \sigma + 1) \\ m_2 + n_2 &= \frac{1}{2}(m_1 - \sigma) \end{aligned} \quad (9.1)$$

so that  $m_1$  and  $m_2$  are odd. Using equation (9.1), the total number of eigenvalues is correctly computed to be

$$\sum_{\sigma} \sum_{\substack{m_1 \text{ odd} \\ m_2 \text{ odd}}} \binom{m_1 + n_1 - \delta_{\sigma,1}}{m_1 - \delta_{\sigma,1}} \binom{m_2 + n_2}{m_2} = A_{3,2}^N. \quad (9.2)$$

## 9.2 Computation of the order-1/ $N$ corrections

Under the (3, 2) boundary condition, the order-1 analyticity is the same as under the (3, 1) boundary. Hence the function  $g$  is again given by equation (4.16). The order-1/ $N$  calculation now proceeds exactly as in section 5.2, with the replacement of  $-\hat{t}_2$  by  $\hat{t}_2$  in accordance with the change in parity of  $m_2$ .

With this single alteration, the computation of section 5.2 holds from equation (5.3) to equation (5.10). However, the change in parity of  $m_2$  means that the minimal solution to the  $(\mathbf{m}, \mathbf{n})$ -system (9.1) is  $m_1 = m_2 = 1$  with  $\sigma = -1$ . Hence  $\Delta = 3/5$  and the integer  $n$  of equation (3.3) is given by

$$n + \frac{1}{2} = \frac{1}{4} \mathbf{m} C \mathbf{m} - \frac{1}{2} (m_1 - m_2) \sigma + \sum_{j=1}^2 \sum_{k=1}^{m_j} I_k^{(j)}. \quad (9.3)$$

Summing over all eigenvalues in the leading band then gives

$$Z_{N_0}(q) = q^{-\frac{c}{24} + \Delta_{3,2} - \frac{1}{2}} \sum_{\sigma} \sum_{\substack{m_1 \text{ odd} \\ m_2 \text{ odd}}} q^{\frac{1}{4} \mathbf{m} C \mathbf{m} - \frac{1}{2} (m_1 - m_2) \sigma} \begin{bmatrix} m_1 + n_1 - \delta_{\sigma,1} \\ m_1 - \delta_{\sigma,1} \end{bmatrix} \begin{bmatrix} m_2 + n_2 \\ m_2 \end{bmatrix}. \quad (9.4)$$

Eliminating  $n_1$  and  $n_2$  using equation (9.1), and summing over  $\sigma$  using the recurrence (5.13), we obtain

$$Z_{N_0}(q) = q^{-\frac{c}{24} + \Delta_{3,2} - \frac{1}{2}} \sum_{\substack{m_1 \text{ even} \\ m_2 \text{ odd}}} q^{\frac{1}{4} \mathbf{m} C \mathbf{m} - \frac{1}{2} m_1} \begin{bmatrix} \frac{1}{2} (N_0 + m_2 + 2) \\ m_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} m_1 \\ m_2 \end{bmatrix}. \quad (9.5)$$

Finally, taking  $N_0 \rightarrow \infty$  gives

$$Z(q) = q^{-\frac{c}{24} + \Delta_{3,2} - \frac{1}{2}} \sum_{\substack{m_1 \text{ even} \\ m_2 \text{ odd}}} \frac{q^{\frac{1}{4} \mathbf{m} C \mathbf{m} - \frac{1}{2} m_1}}{(q)_{m_1}} \begin{bmatrix} \frac{1}{2} m_1 \\ m_2 \end{bmatrix} = \chi_{3,2}(q). \quad (9.6)$$

## 10 Discussion

In this paper we have computed conformal partition functions of the tricritical hard square model with fixed boundaries by calculating analytically the order-1/ $N$  corrections to all leading eigenvalues of the transfer matrix. Each partition function is found to be a single  $c = 7/10$  Virasoro character  $\chi_{r,s}$ , with  $r$  and  $s$  depending on the choice of boundary condition. This is in agreement with the results of Saleur and Bauer [18].

It is worthwhile noting that, as a bonus, the calculations of this paper also yield partition functions of the critical hard hexagon model [23]. This model has the

same Boltzmann weights as tricritical hard squares, but with  $u < 0$  in the ABF formulation. Once again, the heights 1 and 4 are identified with  $\mu = 1$  and the heights 2 and 3 with  $\mu = 0$ . There are three degenerate ground states, which correspond to the complete filling ( $\mu = 1$ ) of one of the sublattices A, B or C shown below.

$$\begin{array}{cccccc} C & A & B & C & A & B \\ A & B & C & A & B & C \\ C & A & B & C & A & B \\ A & B & C & A & B & C \end{array}$$

We denote by  $Z_{P,Q}$  the conformal partition function of the model on a lattice with left boundary labelled by P and right boundary by Q. Generally, P and Q may correspond to sums of ground states.

By making the transformation  $q \rightarrow 1/q$  in the finite partition functions  $Z_{N_0}$  and then taking  $N_0 \rightarrow \infty$ , we obtain  $Z_{A,A} = \chi_{1,1} + \chi_{4,1} = b_0^0$ ,  $Z_{A,B} = Z_{A,C} = \chi_{1,3} = b_2^0$ ,  $Z_{A,B+C} = \chi_{2,1} + \chi_{3,1} = b_0^2$ , and  $Z_{A,A+B} = Z_{A,A+C} = \chi_{2,3} = b_1^1$ . Here  $\chi_{r,s}$  are now the  $c = 4/5$  Virasoro characters and  $b_m^l$  are the branching functions corresponding to the  $\mathbb{Z}_3$  parafermion models. We note that for the (2, 1) and (2, 2) boundaries the classification of eigenvalues depends on the interpretation of zeros on the real  $u$ -axis. Hence in these cases the transformation is meaningless, since we have no description of the leading band of eigenvalues in the hard hexagon regime. The above results are in accordance with those of references [18, 31] for the three-state Potts model, which lies in the same universality class as the hard hexagon model.

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## A The Rogers dilogarithm

In this appendix we briefly collect some results for the Rogers dilogarithm which enable us to evaluate the integrals that occur in our calculation. For details we refer the reader to, for example, reference [35]. The Rogers dilogarithm is defined for  $0 < x < 1$  by

$$L(x) = -\frac{1}{2} \int_0^x \left( \frac{\ln(1-x)}{x} + \frac{\ln x}{1-x} \right) dx. \quad (\text{A.1})$$

When  $a > 0$ , we can make the identification

$$\int_0^a \left( \frac{\ln(1+x)}{x} - \frac{\ln x}{1+x} \right) dx = 2L\left(\frac{a}{1+a}\right). \quad (\text{A.2})$$

For our calculation we need the particular values

$$L(1) = \frac{\pi^2}{6} \quad L\left(\frac{1}{2}\right) = \frac{\pi^2}{12} \quad L\left(\frac{1}{2\cos\lambda}\right) = L(-2\cos(3\lambda)) = \frac{\pi^2}{10} \quad (\text{A.3})$$

where  $\lambda = \pi/5$ .

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