PHASE TRANSITIONS, CRITICAL PHENOMENA AND EXACTLY SOLVABLE LATTICE MODELS

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Abstract

An overview is given of the study of phase transitions and critical phenomena in statistical mechanics with an emphasis on the general principle of universality and the role of spatial dimension and symmetry in determining critical behaviour. Although we live in three dimensions, our most complete understanding of critical behaviour is in two dimensions. This circumstance results from the remarkable fact that, in two dimensions, there are many lattice models which not only exhibit critical behaviour but are also exactly solvable by special mathematical techniques related to the Yang-Baxter equation. It turns out that for simple statistical systems in two dimensions there is an exactly solvable lattice model for each possible universality class of critical behaviour.

1 Phase Transitions

We observe the physical effects of phase transitions as a matter of common everyday experience. If we boil a kettle of water we observe a change of phase from a liquid state to a gaseous or vapour phase. Similarly, if the temperature drops below freezing we see water in a pond change phase from a liquid state to solid ice. Lastly, if a horseshoe magnet is heated to a sufficiently high temperature its magnetization will suddenly disapper at a critical temperature called the Curie point. These are all examples of phase transitions. The study of phase transitions has applications in many diverse areas such as liquid crystals, polymers, percolation, and so on. Here I will be concerned with the fundamental mathematical study of phase transitions and critical phenomena [1] and the insights into these problems gained from exactly solvable lattice models [2].

Although phase transitions are common place, they represent a remarkable class of physical phenomena. At a phase transition point the properties of a substance can change discontinuously. Even more dramatically, certain physical quantities can actually diverge at a critical point. This is illustrated in Figures 1 through 5. Figure 1 shows a schematic phase diagram of water. Figure 2 shows the discontinuous jump in density as the temperature passes through the boling (melting) point. Figure 3 shows the phase diagram of a ferromagnetic material such as iron or nickel in an external magnetic field h. Figure 4 shows a jump discontinuity in a typical magnetic isotherm. Finally, Figure 5 shows the divergence in the specific heat at the critical point. The specific heat gives a measure of the response of the substance to a small change in temperature.

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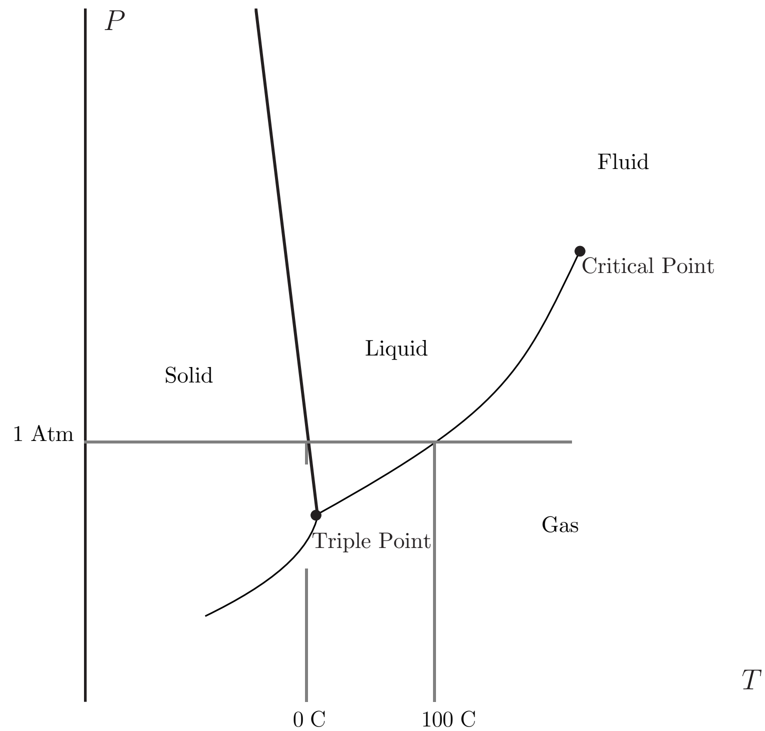


Figure 1. Schematic phase diagram of water showing the solid ice, liquid and gas or vapour phases. Phase transitions between phases occur across the solid lines. There is three phase coexistence at the triple point. The liquid/gas transition line terminates in a critical point. Beyond this point the liquid and gas phases are indistinguishable. The melting ($0°C$) and boiling points ($100°C$) at atmospheric pressure are indicated by dashed lines.

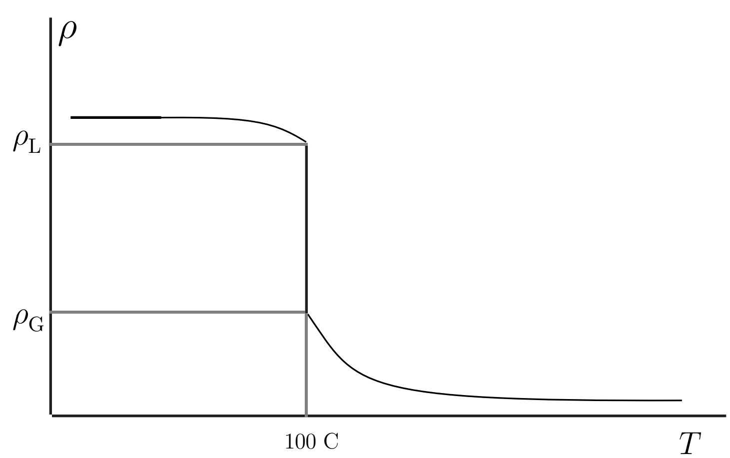


Figure 2. Jump discontinuity in the density ρ of water from the liquid density ρ_L to the gas density ρ_G as the temperature passes through the boiling point at atmospheric pressure.

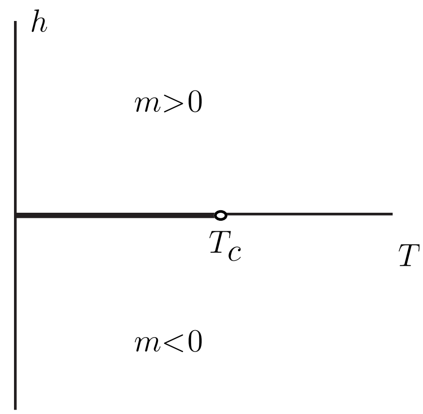


Figure 3. Phase diagram of a ferromagnet such as iron or nickel. Here T is the temperature, h is the external magnetic field and m is the magnetization of the material. The magnetization jumps discontinuously across the phase transition line $h = 0$ which extends from from zero temperature to the critical or Curie point at $T = T_c$.

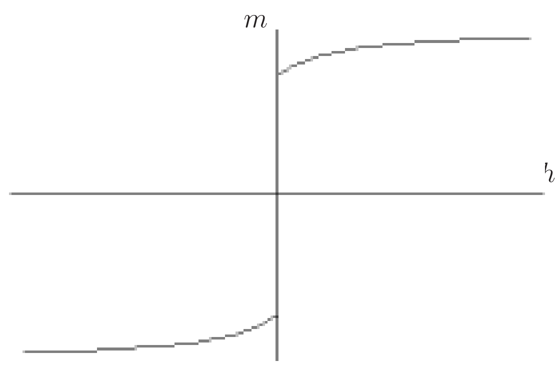


Figure 4. Typical magnetic isotherm for $T < T_c$. At $h = 0$ there is a jump discontinuity in the magnetization.

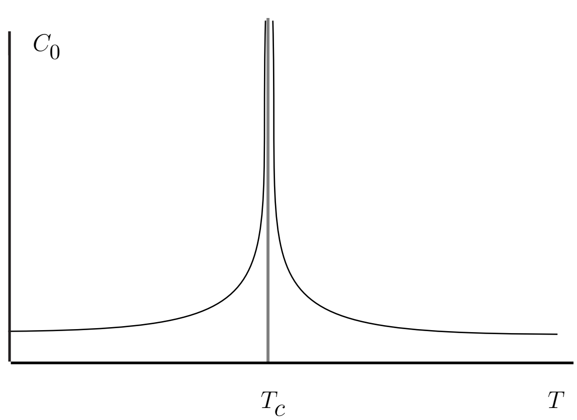


Figure 5. Divergence in the zero-field specific heat C_0 at the critical temperature T_c.

2 Critical Phenomena

2.1 Critical Exponents

The thrust of modern statistical mechanics is to understand the detailed behaviour of systems in the vicinity of critical points. The behaviour of a thermodynamic function $f(x)$ in the vicinity of a critical point at $x = 0$ is characterized by a critical exponent ϵ which describes the power law behaviour close to the critical point

$$f(x) \sim x^{\epsilon} \quad \text{as} \quad x \to 0. \tag{2.1}$$

For example, the magnetization of a ferromagnet in three dimensions vanishes with an exponent $\beta \approx 0.33$ as shown in Figure 6. Some other standard critical exponents for

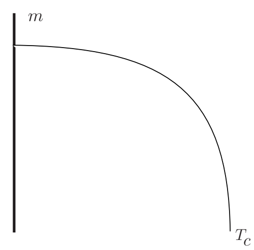


Figure 6. The spontaneous magnetization in zero external field vanishes at the critical temperature T_c with the power law behaviour $m \sim |t|^{\beta}$ where $t = (T - T_c)/T_c$. In three dimensions $\beta \approx 0.33$.

magnetic and fluid systems are defined in Table 1. Typical experimental values for these exponents for both fluid and magnetic systems in three dimensions are

$$\alpha \approx 0.1, \quad \beta \approx 0.33, \quad \gamma \approx 1.2, \quad \delta \approx 4.2 \tag{2.2}$$

Exponent	**Magnet**	**Fluid**
α	$C_0 \sim \|t\|^{-\alpha}, \quad t \to 0$	$C_V \sim \|t\|^{-\alpha}, \quad t \to 0$
β	$m_0 \sim \|t\|^{\beta}, \quad t \to 0-$	$\rho_L - \rho_G \sim \|t\|^{\beta}, \quad t \to 0-$
γ	$\chi_0 \sim \|t\|^{-\gamma}, \quad t \to 0$	$K_T \sim \|t\|^{-\gamma}, \quad t \to 0$
δ	$h \sim \mathrm{sgn}(m)\|m\|^{\delta}, \quad h \to 0, T = T_c$	$P - P_c \sim \mathrm{sgn}(\rho - \rho_c)\|\rho - \rho_c\|^{\delta}$, $\|\rho - \rho_c\| \to 0, T = T_c$

Table 1. The definition of some critical exponents for magnetic and fluid systems. Here $t = (T - T_c)/T_c$, C_0 = zero-field specific heat, χ_0 = zero-field susceptibility, K_T = isothermal compressibility and $\rho_{L,G}$ = liquid, gas density.

2.2 Universality

The critical exponents appear to be insensitive to the microscopic details of the system. This empirical fact is embodied in the following:

Universality Hypothesis

> *All phase transition problems can be divided into a number of different classes depending upon the dimensionality of the system and the symmetries of the ordered state. Within each class, all phase transitions have identical behaviour in the critical region. Only the names of the thermodynamic variables are changed.*

Although it appears to be generally true, there is no mathematical proof of the universality hypothesis which can be summarised in the statement

$$\text{spatial dimension} + \text{symmetries} \quad \Rightarrow \quad \text{universality class} \;\Rightarrow\; \text{critical exponents} \tag{2.3}$$

We have already observed that the critical exponents do not depend on whether the system is a magnet or a fluid. The universality hypothesis, however, tells us much more than this. Universality implies that the critical exponents are not changed if we replace physically complicated interactions with idealized interactions provided the symmetries of the model are preserved. This is a great boon to theoreticians:

> *Simple models suffice to study the universality classes of critical behaviour provided they incorporate the appropriate symmetries of the system to be studied.*

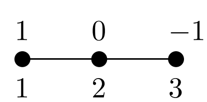
3 Exactly Solvable Lattice Models

In three and more dimensions even the simplest models with appropriate symmetries defy exact mathematical solution. Fortunately, this is not the case in two dimensions! In two dimensions it appears that, at least for simple statistical systems, there is an exactly solvable lattice model for each possible universality class of critical behaviour. Although we of course live in three dimensions, there are many experimental realizations of two-dimensional statistical systems such as gases adsorbed on monlayers and layered ferromagnetic materials with very weak interlayer interactions.

3.1 Ising Model

The first two dimensional model to be solved exactly was the Ising model of a ferromagnet [3]. The Ising model is defined on the square lattice and was solved [4] by Onsager in 1944. With each site of the lattice we associate a spin $a = \pm 1$ representing the local magnetic moment. Essentially we allow the spin to point up or down. For our purposes it is better to allow the spin a to take one of three values which we label either $a = +1, 0, -1$ or $a = 1, 2, 3$. We restrict the values of spins on neighbouring sites of the square lattice to be neighbouring values on the A_3 graph

A_3 *graph* =



= *Ising model*

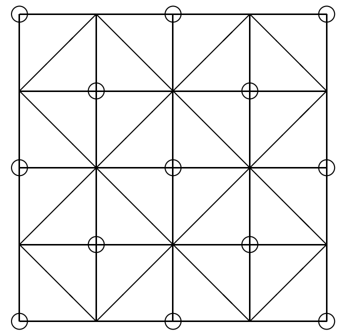
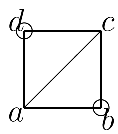
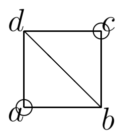


Figure 7. Square lattice showing the two sublattices. The spins on the circled sublattice take the value 0. The $\pm$ spins interact across one of the diagonals of the faces.

Effectively, one of the two sublattices is filled with spins taking the value $a=0$ and on the other sublattice the spins can take either of the values $a=\pm 1$ as shown in Figure 7. The $\pm$ spins interact across the diagonals so these interactions can be associated with the faces of the lattice. Specifically, the energies of interaction are given by the face weights



$$= W\begin{pmatrix} d & c \\ a & b \end{pmatrix} = e^{-Jac/T}, \qquad a, c = \pm 1 \tag{3.4}$$



$$= W\begin{pmatrix} d & c \\ a & b \end{pmatrix} = e^{-Kbd/T}, \qquad b, d = \pm 1 \tag{3.5}$$

where J and K are interaction strengths. The magnetization is given by the thermal average

$$m = \frac{\sum\limits_{\text{spins}} \sigma \prod\limits_{\text{faces}} W\begin{pmatrix} d & c \\ a & b \end{pmatrix}}{\sum\limits_{\text{spins}} \prod\limits_{\text{faces}} W\begin{pmatrix} d & c \\ a & b \end{pmatrix}} \tag{3.6}$$

where $\sigma = \pm 1$ is the spin at a fixed site in the centre of the lattice.

Onsager [4] and Yang [5] showed that in zero magnetic field ($h=0$)

$$\begin{aligned} C_0 &\sim |t|^{-\alpha}, \quad t \to 0, \qquad \alpha = 0 \quad \text{(log divergence)} & (3.7) \\ m_0 &\sim |t|^{\beta}, \quad t \to 0-, \qquad \beta = 1/8. & (3.8) \end{aligned}$$

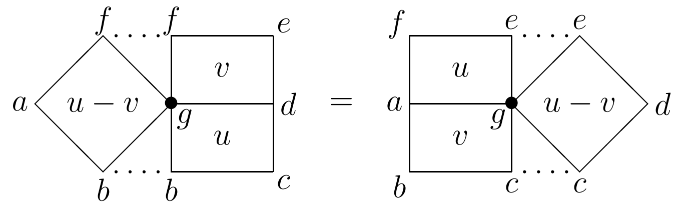
3.2 Yang-Baxter Equations

The Ising model is exactly solvable because, after suitable parametrization, its face weights satisfy the celebrated Yang-Baxter equations [6, 7, 2]

$$\sum_g W\left(\begin{matrix} f & g \\ a & b \end{matrix}\middle| u-v\right) W\left(\begin{matrix} g & d \\ b & c \end{matrix}\middle| u\right) W\left(\begin{matrix} f & e \\ g & d \end{matrix}\middle| v\right) =$$

$$\sum_g W\left(\begin{matrix} a & g \\ b & c \end{matrix}\middle| v\right) W\left(\begin{matrix} f & e \\ a & g \end{matrix}\middle| u\right) W\left(\begin{matrix} e & d \\ g & c \end{matrix}\middle| u-v\right) \tag{3.9}$$

Pictorially, this local relation among face weights can be represented as



Here the parameters u and v are related to interaction strengths J, K and J', K' respectively. More generally, it has emerged that:

> *A lattice model on the square lattice is integrable or exactly solvable whenever its face weights satisfy the Yang-Baxter equations.*

3.3 A–D–E Lattice Models

To construct simple exactly solvable lattice models we need to keep the number of spin configurations as small as possible. This can be achieved by placing restrictions on the allowed values of neighbouring spins as we did for the Ising or A_3 model. For L states these restrictions can be represented by an adjacency graph G_L with L nodes or vertices. Specifically, we restrict neighbouring spins on the square lattice to take neighbouring values on the adjacency graph G_L.

Suppose $L = 2$ and we do not impose any such restrictions, then the number of distinct configurations along a path of N contiguous sites is

$$\#\text{ configurations of an } N\text{-step path} = 2^N. \tag{3.10}$$

We now ask what restrictions will ensure that the growth of the number of configurations is strictly less than 2^N, that is,

$$\#\text{ configurations of an } N\text{-step path} \sim \mu^N, \qquad \mu < 2 \tag{3.11}$$

Remarkably, the only restrictions that do this are given by the *A–D–E* adjacency graphs shown in Figure 8. Even more miraculously, starting with any of the *A–D–E* adjacency conditions, it is possible to construct [8, 9] an exactly solvable lattice model whose face weights satisfy the Yang-Baxter equations! The A_3 model is just the Ising model considered previously.

In the *A–D–E* classification there are two infinite series labelled by A_L and D_L and three exceptional cases labelled $E_{6,7,8}$. In mathematics, these graphs arise as the Dynkin diagrams of classical Lie algebras. Lie algebras are intimately connected with symmetries. For this reason it is perhaps not surprising that there is also an *A–D–E* classification of regular polyhedra as shown in Figure 9.

Each of the *A–D–E* lattice models exhibits a critical point. The critical exponents of the A_L and D_L models have been calculated exactly [8, 9]. The results are given in Table 2.

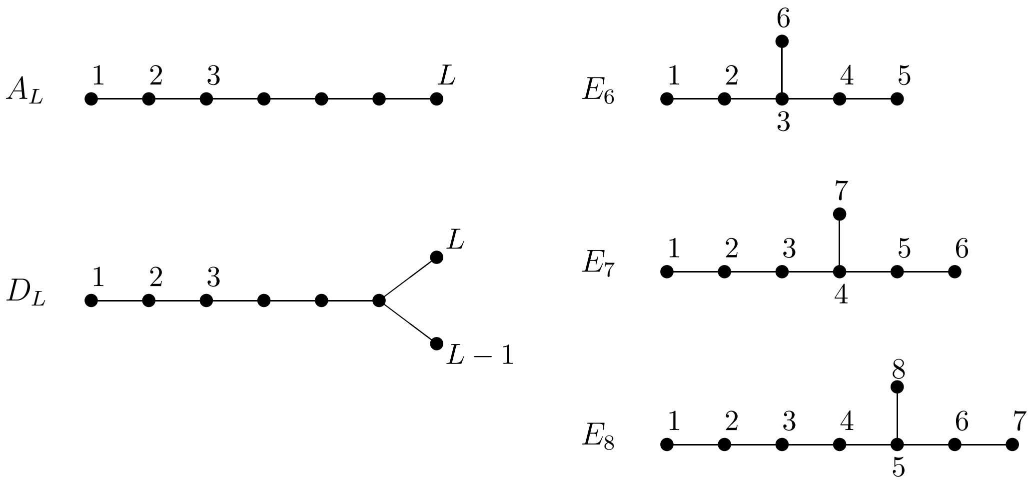


Figure 8. The A–D–E adjacency graphs. There are two infinite series and three exceptional cases.

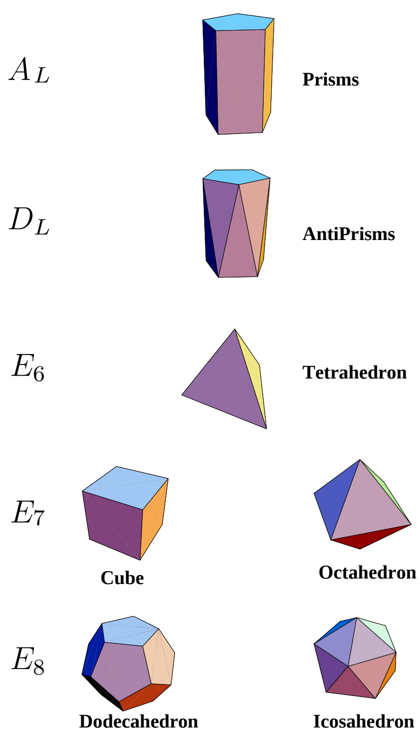


Figure 9. The A–D–E classification of regular polyhedra. The polyhedra are convex and regular in the sense that each vertex is equivalent. A similar classification holds for convex polyhedra with equivalent faces. The two series are related by a face-vertex duality.

Model	α	β_k	
A_L	$\dfrac{3-L}{2}$	$\dfrac{k^2-1}{8L}$,	$k=1,2,\ldots,L-1$
D_L	$3-L$	$\dfrac{k^2-1}{4L(2L-3)}$,	$k=1,3,\ldots,2L-5,L-1$

Table 2. Critical exponents of the A_L and D_L lattice models.

Since, for $L>3$, there is more than one nontrivial order parameter there are many critical exponents β_k. Note also that the exceptional $E_{6,7,8}$ models are not exactly solvable away from the critical point so the exponents α and β are not defined for these models. From the theory of conformal and modular invariance we know that the A–D–E lattice models exhaust [10] the simplest statistical theories, namely, those with central charge $c<1$. There are of course many more exactly solvable lattice models but these have central charges $c\ge 1$ and are beyond the scope of this article.

4 A Mathematical Epilogue

We have seen that the mathematical study of solvable lattice models provides an extremely useful insight into the physics of phase transitions and critical phenomena. However, as is often the case in mathematical physics, the study of physically relevant problems can also lead to new insights into mathematics. To illustrate this point we conclude this review with two examples of beautiful mathematical identities that arise from the rich mathematical structure associated with Yang-Baxter integrable lattice models.

4.1 Rogers Dilogarithm Identities

A complete set of critical exponents for the A_L models can be obtained from the conformal weights

$$\Delta_{r,s}=\frac{[(L+1)r-Ls]^2-1}{4L(L+1)},\qquad 1\le r\le L-1,\quad 1\le s\le L. \tag{4.12}$$

In particular,

$$2-\alpha=\frac{1}{1-\Delta_{1,3}},\qquad \beta_k=\frac{\Delta_{k,k}}{1-\Delta_{1,3}}=\frac{L+1}{2}\Delta_{k,k}. \tag{4.13}$$

In studying the critical A_L models the conformal weights $\Delta_{r,s}$ are determined [11] by the dilogarithm identities [12]

$$S(r,L-2)+1-S(s,L-1)=c-24\Delta_{r,s}+6(r-s)(r-s+1). \tag{4.14}$$

Here

$$S(s,n)=\frac{6}{\pi^2}\sum_{k=1}^{n}L\left(\frac{\sin^2\phi}{\sin^2(k+1)\phi}\right) \tag{4.15}$$

where $L(x)$ is the Rogers dilogarithm, $\phi = s\pi/(n+2)$ and

$$c = 1-\frac{6}{L(L+1)} \tag{4.16}$$

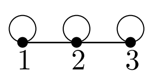
is the central charge. The Rogers dilogarithm $L(x)$ is defined by $L(0)=0$, $L(1)=\pi^2/6$ and

$$L(x)=\begin{cases} -\dfrac{1}{2}\displaystyle\int_0^x dy\left[\frac{\log(1-y)}{y}+\frac{\log y}{1-y}\right], & 0<x<1 \\ \dfrac{\pi^2}{3}-L(x^{-1}), & x>1. \end{cases} \tag{4.17}$$

There is one such dilogarithm identity for each critical exponent of each exactly solvable lattice model. The rational values of the critical exponents derive from the rational values for these special sums of dilogarithms.

4.2 E_8 Rogers-Ramanujan Identity

The two-dimensional Ising model in a magnetic field remains an unsolved problem. However, an insight into this problem can be obtained by studying the dilute A_3 model [13] which lies in the same universality class

Dilute A_3 =  $= \begin{cases}\text{Universality class of Ising} \\ \text{model in a magnetic field.}\end{cases}$ (4.18)

By studying the dilute A_3 model it is possible to prove [14] the identity

$$q^{1/48}\chi_{1,1}^{(3,4)}(q)=\sum_{n_1,\ldots,n_8=0}^{\infty}\frac{q^{\mathsf{n}^T C_{E_8}^{-1}\mathsf{n}}}{(q)_{n_1}\cdots(q)_{n_8}}=\frac{1}{(q)_\infty}\sum_{j=-\infty}^{\infty}\left\{q^{12j^2+j}-q^{12j^2+7j+1}\right\} \tag{4.19}$$

where

$$(q)_m=\prod_{k=1}^{m}(1-q^k), \qquad \boldsymbol{n}=(n_1,n_2,\ldots,n_8)\ \in\ \mathbb{N}^8 \tag{4.20}$$

and the inverse Cartan matrix is

$$C_{E_8}^{-1}=\begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 4 & 2 & 3 \\ 3 & 6 & 8 & 10 & 12 & 8 & 4 & 6 \\ 4 & 8 & 12 & 15 & 18 & 12 & 6 & 9 \\ 5 & 10 & 15 & 20 & 24 & 16 & 8 & 12 \\ 6 & 12 & 18 & 24 & 30 & 20 & 10 & 15 \\ 4 & 8 & 12 & 16 & 20 & 14 & 7 & 10 \\ 2 & 4 & 6 & 8 & 10 & 7 & 4 & 5 \\ 3 & 6 & 9 & 12 & 15 & 10 & 5 & 8 \end{pmatrix}. \tag{4.21}$$

This remarkable identity, which was conjectured by Kedem et al [15], expresses the equivalence of two different representations of the Virasoro character $\chi_{1,1}^{(3,4)}(q)$. It takes the form of a generalized Rogers-Ramanujan [16] identity and establishes a close link between the dilute A_3 model, E_8 and the Ising model in a magnetic field. Indeed, the study [17] of the dilute A_3 model has firmly established the value $\delta = 15$ for the magnetic critical exponent of the Ising model in a magnetic field.

Acknowledgements

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