

# Random matrix ensembles with an effective extensive external charge

T.H. Baker, P.J. Forrester and P.A. Pearce

*Department of Mathematics, University of Melbourne, Parkville, Victoria 3052, Australia*

Recent theoretical studies of chaotic scattering have encountered ensembles of random matrices in which the eigenvalue probability density function contains a one-body factor with an exponent proportional to the number of eigenvalues. Two such ensembles have been encountered: an ensemble of unitary matrices specified by the so-called Poisson kernel, and the Laguerre ensemble of positive definite matrices. Here we consider various properties of these ensembles. Jack polynomial theory is used to prove a reproducing property of the Poisson kernel, and a certain unimodular mapping is used to demonstrate that the variance of a linear statistic is the same as in the Dyson circular ensemble. For the Laguerre ensemble, the scaled global density is calculated exactly for all even values of the parameter  $\beta$ , while for  $\beta = 2$  (random matrices with unitary symmetry), the neighbourhood of the smallest eigenvalue is shown to be in the soft edge universality class.

## 1 Introduction

In the theory of random matrices, a primary task is to compute the probability density function (p.d.f.) for the eigenvalues from knowledge of the p.d.f. for the ensemble of matrices. Two examples of random matrix ensembles of interest in this paper are Dyson's [Dys62] circular ensembles of symmetric, unitary and self dual quaternion unitary random matrices (labelled by  $\beta = 1, 2$  and  $4$  respectively), and the Laguerre ensemble of random Wishart matrices  $A = X^\dagger X$ , where  $X$  is a random  $M \times N$  ( $M \geq N$ ) matrix which has either real ( $\beta = 1$ ), complex ( $\beta = 2$ ) or quaternion real ( $\beta = 4$ ) Gaussian random elements. In the circular ensemble, the p.d.f. for the matrices is uniquely specified by requiring that it be uniform and unchanged by mappings of the form  $U \mapsto VUV'$ , where  $V$  is an arbitrary  $N \times N$  unitary matrix and  $V' = V^T$  for  $\beta = 1$ ,  $V'$  is arbitrary for  $\beta = 2$  and  $V' = V^D$  for  $\beta = 4$  ( $D$  denotes the quaternion dual). In the Laguerre ensemble, the distribution of the elements of  $X$  are taken to be the Gaussian  $Ae^{-\beta\text{Tr}(X^\dagger X)/2}$  which is equivalent to choosing each element independently with a Gaussian distribution  $A'e^{-\beta|x_{jk}|^2/2}$ .

The corresponding p.d.f. for the eigenvalues  $e^{i\theta_j}$ ,  $j = 1, \dots, N$  in the circular ensemble is

$$\frac{1}{C_{\beta N}} \prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^\beta, \quad (1.1)$$

while for the Laguerre ensemble the p.d.f. is given by

$$\frac{1}{C'_{a\beta N}} \prod_{j=1}^N e^{-\beta\lambda_j/2} \lambda_j^{a\beta/2} \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^\beta, \quad \lambda_j > 0, \quad (1.2)$$

where  $a := M - N + 1 - 2/\beta$ . In both cases the eigenvalue p.d.f.'s can be written in the form of a Boltzmann factor for a classical gas, with potential energy consisting of one and two body terms only and interacting at inverse temperature  $\beta$ ,

$$\exp\left(-\beta\left(\sum_{j=1}^N V_1(x_j) + \sum_{1 \leq j < k \leq N} V_2(x_j, x_k)\right)\right). \quad (1.3)$$

Thus for the circular ensemble

$$V_1(x) = 0, \quad V_2(x, y) = -\log |e^{ix} - e^{iy}|, \quad (1.4)$$

while for the Laguerre ensemble

$$V_1(x) = \frac{x}{2} - \frac{a}{2} \log x, \quad V_2(x, y) = -\log |x - y|, \quad (1.5)$$

which displays the well known fact that the analogous classical gas has a two body logarithmic potential. The logarithmic potential is special in that it is the Coulomb potential between like charges in two dimensions. In (1.4) the two dimensional charges are confined to a unit circle, while in (1.5) the two dimensional charges are free to move on the half line  $x > 0$ , but are confined to the neighbourhood of the origin by the one body potential.

In this work we will focus attention on a subclass of random matrix ensembles with eigenvalue p.d.f.'s of the form (1.3) in which the one body potential contains a term proportional to  $N \log |1 - \mu^* e^{ix}|$  (unitary matrices) or  $-N \log |x|$  (Laguerre ensemble). Again the two body term is logarithmic. In the classical gas these one body potentials can be interpreted as being due to an external fixed charge at the point  $1/\mu^*$  in the complex plane with strength proportional to  $-N$  (unitary matrices), and as an external charge fixed at the origin of strength proportional to  $N$  (Laguerre ensemble). We see from (1.2) that an example of a random matrix of this type in the Laguerre case is a Wishart matrix with  $X$  a rectangular matrix in which the number of columns is some fixed fraction of the number of rows. In the theory of random unitary matrices, this type of eigenvalue p.d.f. results as a special case of the the ensemble of random matrices defined by the Poisson kernel

$$\frac{1}{C} \frac{\det(1 - \bar{S}S^\dagger)^{\beta(N-1)/2+1}}{|\det(1 - \bar{S}^\dagger S)|^{\beta(N-1)+2}}, \quad (1.6)$$

where the notation  $\bar{S}$  denotes the average of  $S$ . Random unitary matrices with this p.d.f. occur in the study of scattering problems in nuclear physics [MPS85] and mesoscopic systems [Bro95]. In the case  $\bar{S} = 0$ , this reduces to the p.d.f. specifying Dyson's circular ensemble (all members equally probable). In the case that  $\bar{S} = \mu 1_N$ ,  $|\mu| < 1$  ( $1_N$  denotes the  $N \times N$  unit matrix), the corresponding eigenvalue p.d.f. is given by

$$\frac{1}{C_{\beta N}} \prod_{j=1}^N \frac{(1 - |\mu|^2)^{\beta a'/2}}{|1 - \mu^* e^{i\theta}|^{\beta a'}} \prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^\beta \quad (1.7)$$

where  $a' = (N - 1 + 2/\beta)$  and  $C_{\beta N}$  is as in (1.1), and so in the classical gas picture there is a fixed charge of opposite sign at the point  $1/\mu^*$ , and the magnitude of this charge is indeed proportional to  $N$ .

In Section 2 we will consider the eigenvalue p.d.f. (1.7) for the Poisson kernel. First the special reproducing property of the p.d.f.'s (1.6) and (1.7) for  $\beta = 1, 2$  and 4 will be revised, and extended to all  $\beta > 0$  in the case of (1.7). Our tool here is Jack polynomial theory [Mac95]. Then we will consider the effect of the  $N$ -dependent exponent in (1.7) on the one and two body correlation functions, as well as on the fluctuation formula for the variance of a linear statistic.

In Section 3 a physical problem giving rise to the Laguerre ensemble in which  $a$  in (1.2) is equal to  $N$  will be revised. Then we will revise known theorems in the cases  $\beta = 1$  and 2 for the global density and the distribution of the largest and smallest eigenvalues. Next known integral formulas for the density [BF97a] at general even  $\beta$ , deduced from the theory of generalized Selberg integrals [Kan93] and their relationship to Jack polynomial theory, will be analyzed in the appropriate limit to deduce that the formula for the global density holds independent of  $\beta$ . For the special coupling  $\beta = 2$  the local distribution functions in the neighbourhood of the smallest eigenvalue are analyzed and shown to belong to the universality class of the soft edge, giving rise to the Airy kernel [For93, TW94a, KF97b]. We conclude the section with an analysis of some nonlinear equations [TW94a, TW94b], which explicitly demonstrates the universality of the distribution of the smallest eigenvalue.

## 2 The Poisson kernel

### 2.1 Physical origin of the Poisson kernel

The scattering matrix for  $N$  channels entering and leaving a chaotic cavity via a non-ideal lead containing a tunnel barrier has as its p.d.f. the Poisson kernel (1.6) [Bro95]. Also, in scattering problems in nuclear physics, (1.6) has been used [MPS85] to describe situations in which the average of the scattering matrix is non-zero. It was in the latter problem that the Poisson kernel first appeared in an application of random matrix theory. In [MPS85] a requirement for the p.d.f. of the ensemble of scattering matrices was the special reproducing property

$$f(\bar{S}) := f(\langle S \rangle) = \langle f(S) \rangle \quad (2.1)$$

for  $f$  analytic in  $S$ . It was noted that a result of Hua [Hua63] gives that

$$f(\bar{S}) = \frac{1}{C} \int f(S) \frac{\det(1_N - \bar{S} \bar{S}^\dagger)^{\beta(N-1)/2+1}}{|\det(1_N - \bar{S}^\dagger S)|^{\beta(N-1)+2}} \mu(dS) \quad (2.2)$$

where  $\mu(dS)$  is the invariant measure associated with the Dyson circular ensemble, and thus that the Poisson kernel exhibits the reproducing property (2.1).

## 2.2 The reproducing property for general $\beta$

To make any sense out of (2.2) it is necessary that  $\beta = 1, 2$  or  $4$ , so that the measure  $\mu(dS)$  has meaning. However, in the case  $\bar{S} = \mu 1_N$ , the corresponding eigenvalue p.d.f. (1.7) can be interpreted as a Boltzmann factor and it makes sense to consider all  $\beta > 0$ . In this case, for  $\beta = 1, 2$  and  $4$ , (1.7) used in (2.2) gives that the reproducing property restricted to analytic functions of the eigenvalues  $e^{i\theta_j}$  reads

$$f(\mu, \dots, \mu) = \frac{1}{C_{\beta N}} \prod_{l=1}^N \int_0^{2\pi} d\theta_l \frac{(1 - |\mu|^2)^{\beta a'/2}}{|1 - \mu^* e^{i\theta_l}|^{\beta a'}} f(e^{i\theta_1}, \dots, e^{i\theta_N}) \prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^\beta. \quad (2.3)$$

This equation is well defined for all  $\beta > 0$ , but its validity has only been established for  $\beta = 1, 2$  and  $4$ . Here we will establish its validity for general  $\beta > 0$ , using properties of the orthogonal polynomials associated with the p.d.f. (1.1). Note that in the case  $N = 1$  (2.3) reads

$$f(\mu) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - |\mu|^2)}{|1 - \mu^* e^{i\theta}|^2} f(e^{i\theta}) d\theta, \quad (2.4)$$

which is the celebrated Poisson formula on a circle, giving the value of an analytic function  $f$  for  $|\mu| < 1$  in terms of its value on the unit circle. Thus (2.3) can be regarded as an  $N$ -dimensional generalization of this result.

The multivariable orthogonal polynomials corresponding to (1.1) are the symmetric Jack polynomials  $P_\kappa^{(2/\beta)}(z)$  [Sta89, Mac95], as they possess the orthogonality property

$$\prod_{l=1}^N \int_0^{2\pi} d\theta_l P_\kappa^{(2/\beta)}(z) P_\sigma^{(2/\beta)}(z^*) \prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^\beta = \mathcal{N}_\kappa \delta_{\kappa, \sigma}, \quad (2.5)$$

and form a complete set for the space of analytic functions. Here the labels  $\kappa$  and  $\sigma$  are partitions consisting of  $N$  parts or less, and  $z := (e^{i\theta_1}, \dots, e^{i\theta_N})$ . The normalization  $\mathcal{N}_\kappa$  is given by

$$\frac{\mathcal{N}_\kappa}{\mathcal{N}_0} = \frac{P_\kappa^{(2/\beta)}(1^N) d'_\kappa}{[1 + \beta(N - 1)/2]_\kappa^{(2/\beta)}} \quad (2.6)$$

where

$$d'_\kappa := \prod_{s \in \kappa} \left( \frac{2}{\beta} a(s) + l(s) + 1 \right), \quad [u]_\kappa^{(\alpha)} := \prod_{j=1}^N \frac{\Gamma(u - \frac{\beta}{2}(j-1) + \kappa_j)}{\Gamma(u - \frac{\beta}{2}(j-1))}.$$

The notation  $s \in \kappa$  refers to the diagram of the partition  $\kappa$ , and  $a(s) = \kappa_i - j$  is the arm length while  $l(s) = \kappa'_j - i$  is the leg length ( $\kappa'$  refers to the conjugate partition of  $\kappa$ ); see e.g. [Mac95]. The normalization  $\mathcal{N}_0$  is the same quantity as the normalization  $C_{\beta N}$  in (1.1), and has the explicit value

$$\mathcal{N}_0 = C_{\beta N} = (2\pi)^N \frac{(N\beta/2)!}{(\beta/2)!^N}. \quad (2.7)$$

The quantity  $P_\kappa^{(2/\beta)}(1^N)$  in (2.6) also has an explicit evaluation [Sta89, Mac95, BF97b], but it suits our purposes to leave it unevaluated.

With these preliminaries, we now pose the problem of specifying the kernel  $K(w, z^*)$ ,  $w = (e^{i\phi_1}, \dots, e^{i\phi_N})$ ,  $\text{Im}(\phi_j) > 0$ , which is an analytic function of  $z^*$ , and has the reproducing property

$$f(w) = \prod_{l=1}^N \int_0^{2\pi} d\theta_l K(w, z^*) f(z) \prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^\beta, \quad (2.8)$$

for  $f$  analytic and symmetric in  $w_1, \dots, w_N$ . The proof of (2.3) will then consist first of evaluating  $K(w, z^*)$  at  $w = (\mu, \dots, \mu)$ , then transforming the equation (2.8) so that  $K(w, z^*)$  becomes real. Using the orthogonality property (2.5), as well as the completeness of  $\{P_\kappa^{(2/\beta)}(z)\}$  for analytic functions, it is a simple exercise to check that the required kernel  $K$  is uniquely given by

$$\begin{aligned} K(w, z^*) &= \sum_\kappa \frac{P_\kappa^{(2/\beta)}(w) P_\kappa^{(2/\beta)}(z^*)}{\mathcal{N}_\kappa} \\ &= \frac{1}{\mathcal{N}_0} \sum_\kappa [1 + \beta(N-1)/2]_\kappa^{(2/\beta)} \frac{P_\kappa^{(2/\beta)}(w) P_\kappa^{(2/\beta)}(z^*)}{d'_\kappa P_\kappa^{(2/\beta)}(1^N)}. \end{aligned} \quad (2.9)$$

where the second equality follows from (2.6). But in general the generalized hypergeometric function  ${}_1\mathcal{F}_0^{(2/\beta)}(a; w, z^*)$  is defined by

$${}_1\mathcal{F}_0^{(2/\beta)}(a; w, z^*) = \sum_\kappa [a]_\kappa^{(2/\beta)} \frac{P_\kappa^{(2/\beta)}(w) P_\kappa^{(2/\beta)}(z^*)}{d'_\kappa P_\kappa^{(2/\beta)}(1^N)} \quad (2.10)$$

so we have

$$K(w, z^*) = \frac{1}{\mathcal{N}_0} {}_1\mathcal{F}_0^{(2/\beta)}(1 + \beta(N-1)/2; w; z^*). \quad (2.11)$$

In general there is no known expression for  ${}_1\mathcal{F}_0^{(2/\beta)}(a; w, z^*)$  in terms of elementary functions. However, the case  $w = (\mu, \dots, \mu)$  is an exception, for then we have

$${}_1\mathcal{F}_0^{(2/\beta)}(a; w, z^*) \Big|_{w=(\mu, \dots, \mu)} = {}_1\mathcal{F}_0^{(2/\beta)}(a; \mu z^*) := \sum_\kappa \frac{\mu^{|\kappa|} [a]_\kappa^{(2/\beta)} P_\kappa^{(2/\beta)}(z^*)}{d'_\kappa}. \quad (2.12)$$

The significance of this is that  ${}_1\mathcal{F}_0^{(2/\beta)}(a; \mu z^*)$  can be summed according to the generalized binomial formula [Kan93]

$${}_1\mathcal{F}_0^{(2/\beta)}(a; \mu z^*) = \prod_{j=1}^N \frac{1}{(1 - \mu z_j^*)^a}, \quad |\mu z_j| < 1. \quad (2.13)$$

Comparing (2.11) – (2.13) we therefore have

$$K(w, z^*) \Big|_{w=(\mu, \dots, \mu)} = \frac{1}{\mathcal{N}_0} \prod_{j=1}^N \frac{1}{(1 - \mu z_j^*)^{1+\beta(N-1)/2}}. \quad (2.14)$$

Although this is an explicit solution to the problem of determining the kernel in (2.8) in the case  $w = (\mu, \dots, \mu)$ , it does not immediately establish (2.3) as the kernel (2.14) is not real. Note that in the case  $N = 1$ , (2.14) is the Cauchy kernel from elementary complex analysis. For general  $N$ , to obtain a real (Poisson) kernel from the Cauchy kernel, we proceed as in the one dimensional case and simply make the replacement

$$f \mapsto \prod_{j=1}^N \frac{1}{(1 - \mu^* z_j)^{1+\beta(N-1)/2}} f$$

in (2.8) with  $w = (\mu, \dots, \mu)$ . The formula (2.3) results, thus establishing its validity for general  $\beta > 0$ .

## 2.3 Fluctuation formulas

In the application of random matrix theory, an important class of observables are the linear statistics  $A = \sum_{j=1}^N a(\lambda_j)$ . The first two moments of these statistics are given by

$$\langle A \rangle = \int_0^{2\pi} \rho_{(1)}(\theta) a(\theta) d\theta \quad (2.15)$$

$$\text{Var}(A) := \int_0^{2\pi} d\theta_1 a(\theta_1) \int_0^{2\pi} d\theta_2 a(\theta_2) S(\theta_1, \theta_2) \quad (2.16)$$

where  $S(\lambda_1, \lambda_2)$  denotes the structure function

$$S(\lambda_1, \lambda_2) := \rho_{(2)}^T(\lambda_1, \lambda_2) + \rho_{(1)}(\lambda_1) \delta(\lambda_1 - \lambda_2),$$

with  $\rho_{(1)}$  denoting the density and  $\rho_{(2)}^T$  denoting the truncated two particle distribution function. In this subsection these quantities will be considered for the eigenvalue p.d.f. (1.7).

It is instructive to first consider the case  $\mu = 0$ , when (1.7) reduces to the eigenvalue p.d.f. (1.1) for the circular ensemble. In this case [For95]

$$\rho_{(1)}(\theta^{(0)}) = \frac{N}{2\pi} \quad (2.17)$$

$$S(\theta_1^{(0)}, \theta_2^{(0)}) \underset{N \rightarrow \infty}{\sim} -\frac{1}{\beta\pi^2} \frac{\partial^2}{\partial\theta_1^{(0)} \partial\theta_2^{(0)}} \log |\sin(\theta_1^{(0)} - \theta_2^{(0)})/2| \quad (2.18)$$

(the use of the superscript (0) indicates that  $\mu = 0$  in (1.7)), where in the asymptotic expression (2.18) all oscillatory terms, each of which have period some integer multiple of  $2\pi/N$ , are ignored. For example, we have the exact result (see e.g. [Meh91])

$$\begin{aligned} \rho_{(2)}^T(\theta_1^{(0)}, \theta_2^{(0)}) &= -\left(\frac{1}{2\pi}\right)^2 \frac{\sin^2 N(\theta_1^{(0)} - \theta_2^{(0)})/2}{\sin^2(\theta_1^{(0)} - \theta_2^{(0)})/2} \\ &= -\frac{1}{2} \left(\frac{1}{2\pi}\right)^2 \left( \frac{1}{\sin^2(\theta_1^{(0)} - \theta_2^{(0)})/2} - \frac{\cos N(\theta_1^{(0)} - \theta_2^{(0)})}{\sin^2(\theta_1^{(0)} - \theta_2^{(0)})/2} \right). \end{aligned}$$

Ignoring the oscillatory term with the factor of  $\cos N(\theta_1^{(0)} - \theta_2^{(0)})$  gives

$$\rho_{(2)}^T(\theta_1^{(0)}, \theta_2^{(0)}) \underset{N \rightarrow \infty}{\sim} -\frac{1}{2(2\pi)^2} \frac{1}{\sin^2(\theta_1^{(0)} - \theta_2^{(0)})/2} = -\frac{1}{2\pi^2} \frac{\partial^2}{\partial\theta_1^{(0)} \partial\theta_2^{(0)}} \log |\sin(\theta_1^{(0)} - \theta_2^{(0)})/2|$$

valid for  $\theta_1^{(0)} \neq \theta_2^{(0)}$ . The validity of (2.18) for  $\theta_1^{(0)} = \theta_2^{(0)}$  is then deduced by noting that if we write (define)

$$\int_0^{2\pi} \frac{\partial^2}{\partial\theta_1^{(0)} \partial\theta_2^{(0)}} \log |\sin(\theta_1^{(0)} - \theta_2^{(0)})/2| d\theta_1^{(0)} = \frac{\partial}{\partial\theta_2^{(0)}} \int_0^{2\pi} \frac{\partial}{\partial\theta_1^{(0)}} \log |\sin(\theta_1^{(0)} - \theta_2^{(0)})/2| d\theta_1^{(0)} \quad (2.19)$$

then this integral vanishes. This is the perfect screening sum rule for the underlying log-gas system, and is a fundamental requirement of  $S(\theta_1^{(0)}, \theta_2^{(0)})$  [Mar88].

Substituting (2.18) in (2.16), interchanging the order of differentiation and integration according to (2.19), and using the Fourier expansion

$$\log |\sin(\theta_1^{(0)} - \theta_2^{(0)})/2| = \sum_{p=-\infty}^{\infty} \alpha_p e^{ip(\theta_1^{(0)} - \theta_2^{(0)})}, \quad \alpha_p = -\frac{1}{2|p|} \quad (p \neq 0), \quad \alpha_0 = -2\pi \log 2,$$

allows (2.16) to be evaluated as

$$\text{Var}(A)^{(0)} = \frac{4}{\beta} \sum_{n=1}^{\infty} n a_n a_{-n}, \quad a(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}. \quad (2.20)$$

This has the well known feature of being independent of  $N$  – fluctuations are therefore strongly suppressed. Furthermore, it has been rigorously proved by Johansson [Joh88, Joh95] that the corresponding full distribution of the linear statistic  $A$  is given by the central limit-type theorem

$$\Pr(u = A) \underset{N \rightarrow \infty}{\sim} \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-(u - \langle A \rangle)^2 / 2\sigma^2} \quad (2.21)$$

where  $\langle A \rangle$  is given by (2.15) with the substitution (2.17), and  $\sigma^2 = \text{Var}(A)$  as given by (2.20).

Let us now consider the situation for general  $\mu$ ,  $|\mu| < 1$ . It is well known [Hua63], and is simple to verify, that under the transformation

$$e^{i\theta_j} = \frac{e^{i\theta_j^{(0)}} - \mu}{1 - \mu^* e^{i\theta_j^{(0)}}} \quad (2.22)$$

(note that the RHS has unit modulus and this mapping is one-to-one), the p.d.f. for general  $\mu$  is transformed into the p.d.f. with  $\mu = 0$  according to

$$\begin{aligned} & \prod_{j=1}^N \left( \frac{1 - |\mu|^2}{|1 - \mu^* e^{i\theta_j}|^2} \right)^{\beta(N-1)/2+1} \prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^{\beta} d\theta_1 \cdots d\theta_N \\ &= \prod_{1 \leq j < k \leq N} |e^{i\theta_k^{(0)}} - e^{i\theta_j^{(0)}}|^{\beta} d\theta_1^{(0)} \cdots d\theta_N^{(0)}. \end{aligned} \quad (2.23)$$

This means that the correlation functions for general  $\mu$  can be obtained from the correlation functions for  $\mu = 0$  by applying the inverse of the transformation (2.22),

$$e^{i\theta_j^{(0)}} = \frac{\mu + e^{i\theta_j}}{1 + \mu^* e^{i\theta_j}}, \quad (2.24)$$

and noting that

$$d\theta^{(0)} = \frac{(1 - |\mu|^2)}{|1 - \mu^* e^{i\theta}|^2} d\theta. \quad (2.25)$$

Hence, from (2.17), we have

$$\rho_{(1)}(\theta) = \frac{N}{2\pi} \frac{(1 - |\mu|^2)}{|1 - \mu^* e^{i\theta}|^2} \quad (2.26)$$

independent of  $\beta$ , and so from (2.15),

$$\langle A \rangle = \frac{N}{2\pi} \int_0^{2\pi} \frac{(1 - |\mu|^2)}{|1 - \mu^* e^{i\theta}|^2} \rho(\theta) d\theta. \quad (2.27)$$

For the structure function, substituting (2.24) and (2.25) in (2.18) gives

$$\begin{aligned} S(\theta_1^{(0)}, \theta_2^{(0)}) d\theta_1^{(0)} d\theta_2^{(0)} &:= -\frac{1}{\beta\pi^2} \left( \frac{\partial^2}{\partial\theta_1^{(0)} \partial\theta_2^{(0)}} \log |e^{i\theta_1^{(0)}} - e^{i\theta_2^{(0)}}| \right) d\theta_1^{(0)} d\theta_2^{(0)} \\ &= -\frac{1}{\beta\pi^2} \left( \frac{\partial^2}{\partial\theta_1 \partial\theta_2} \log \left| \frac{e^{i\theta_1} - e^{i\theta_2}}{|1 + \mu^* e^{i\theta_1}| |1 + \mu^* e^{i\theta_2}|} \right| \right) d\theta_1 d\theta_2 \\ &= -\frac{1}{\beta\pi^2} \left( \frac{\partial^2}{\partial\theta_1 \partial\theta_2} \log |e^{i\theta_1} - e^{i\theta_2}| \right) d\theta_1 d\theta_2. \end{aligned} \quad (2.28)$$

Thus  $S(\theta_1, \theta_2)$  for general  $\mu$  is identical to  $S(\theta_1^{(0)}, \theta_2^{(0)})$  for  $\mu = 0$ , and consequently

$$\text{Var}(A)^{(0)} = \text{Var}(A). \quad (2.29)$$

This fact illustrates a universality feature of the underlying log-gas:  $\text{Var}(A)$  is invariant with respect to the particular one body potential modifying (1.1), provided the corresponding one body density is a well behaved function.

Finally, we note that (2.29) can be demonstrated via a numerical experiment. The experiment is performed by first generating random unitary matrices with uniform distribution (matrices from the CUE). This can be done by diagonalizing matrices from the GUE (random Hermitian matrices): the matrix of eigenvectors, when multiplied by a diagonal matrix with entries  $e^{i\theta_j}$ , ( $j = 1, \dots, n$ ) where  $\theta_j$  is a random angle between 0 and  $2\pi$  with uniform distribution, gives a matrix belonging to the CUE. Next we calculate the eigenvalues of each matrix (which will have distribution (1.1) with  $\beta = 2$ ), and transform them according to (2.22) with a specific value of  $\mu$ . The resulting eigenvalues will have distribution as on the l.h.s. of (3.5). For each set  $k$  of eigenvalues  $\{e^{i\theta_j}\}_{j=1, \dots, N}$  we then calculate  $A_k := \sum_{j=1}^N a(\theta_j)$  for some particular choices of  $a$ . From the resulting list of values  $\{A_k\}$ , the empirical mean and standard derivation are calculated according to the usual formulas. In table 1 we present the result of performing this numerical experiment with  $N$ , the dimension of the unitary matrix, equal to 15, and  $a(\theta) = \cos j\theta$  ( $j = 1, \dots, 5$ ). These empirical values are compared with the theoretical prediction for the variance in the limit  $N \rightarrow \infty$  as given by (2.20) (note that with  $a(\theta) = \cos j\theta$ ,  $a_j = a_{-j} = 1/2$ ,  $a_n = 0$  otherwise; thus (2.20) gives  $\text{Var}(A) = j/2$ ).

$a(\theta)$	Empirical $\text{Var}(A)$	Theoretical $\text{Var}(A)$
$\cos \theta$	0.509	0.5
$\cos 2\theta$	0.972	1
$\cos 3\theta$	1.6	1.5
$\cos 4\theta$	1.8	2
$\cos 5\theta$	2.6	2.5

**Table 1.** The second column contains the empirical variance of the quantity  $\sum_{j=1}^N a(\theta_j)$ , with  $a(\theta)$  as specified. This was calculated for 500  $15 \times 15$  matrices with eigenvalue distribution given by the l.h.s. of (3.5) with  $\beta = 2$  and  $\mu = .5$ . The final column contains the theoretical variance for the same quantity in the  $N \rightarrow \infty$  limit.

### 3 Laguerre ensemble with an $N$ -dependent exponent

#### 3.1 Motivation

Recently Brouwer et al. [BFB97] have considered the problem of the distribution of the eigenvalues of the Wigner-Smith matrix  $Q = -i\hbar S^{-1}\partial S/\partial E$ . Here  $S$  refers to the scattering matrix coupled to a perfect lead which supports  $N$  channels of the same energy  $E$ . It was found that for each of the three possible symmetries of  $S$ , orthogonal ( $\beta = 1$ ), unitary ( $\beta = 2$ ) and symplectic ( $\beta = 4$ ), the p.d.f. for the reciprocal of the eigenvalues of  $Q$  is given by (1.2) with  $a = N$ . This motivates a study of some of the properties of the distribution functions and fluctuation formulas associated with (1.2) for general  $a = YN$ ,  $Y > 0$ .

#### 3.2 $\beta = 1$ and 2

As remarked in the introduction, the p.d.f. (1.2) for  $\beta = 1, 2$  and 4 is realized as the eigenvalue p.d.f. of random Wishart matrices  $A = X^\dagger X$ , where  $X$  has dimension  $M \times N$ . For  $\beta = 1$  and 2, and with  $a$  proportional to  $N$ , the limiting form of the global density and the statistical properties of the largest and smallest eigenvalues have been extensively studied (see [Ede88] and references therein). In particular, with

$$a = YN \tag{3.1}$$

it is known that the global eigenvalue density is given by

$$\lim_{N \rightarrow \infty} \rho(4Nx) = \begin{cases} \frac{1}{\pi x} \sqrt{(x - t_1(Y))(t_2(Y) - x)}, & t_1(Y) < x < t_2(Y) \\ 0, & \text{otherwise} \end{cases} \tag{3.2}$$

where

$$t_1(Y) = \frac{1}{4}(\sqrt{1+Y} - 1)^2, \quad t_2(Y) = \frac{1}{4}(\sqrt{1+Y} + 1)^2. \tag{3.3}$$

In qualitative terms, the result (3.2) says that the support of the density, which is  $(0, 1)$  when  $Y = 0$ , is repelled from the origin and elongated as  $Y$  increases. This is consistent with the log-gas interpretation of (1.2) with  $a$  given by (3.1), as then there is an external charge of strength  $YN$  placed at the origin. This charge repels the  $N$  mobile charges of unit strength away from the origin. The fact that (3.2) is independent of  $\beta$  is also consistent with the log-gas interpretation. In fact macroscopic electrostatics says that the one-body potential in (1.5) results from a neutralizing background charge density  $\rho_b(y)$  according to

$$\frac{x}{2} - \frac{YN}{2} \log x + C = \int_I \rho_b(y) \log |x - y| dy, \quad x \in I \tag{3.4}$$

where  $I$  is an interval in  $\mathbf{R}^+$ . The quantity  $\beta$  does not appear in this equation, so  $\rho_b(y)$  is independent of  $\beta$ . But to leading order the particle density will equal the background density, as in general Coulomb systems strongly suppress charge fluctuations [Mar88]. The expected independence of (3.2) on  $\beta$  follows.

### 3.3 $\beta$ even

For  $\beta$  even, an exact  $\beta$ -dimensional integral representation of the density in the finite system is available [For94, BF97a], which allows the global density limit to be taken explicitly.

Now, in a system of  $N + 1$  particles, the one-body density  $\rho(x)$  in the Laguerre ensemble is given by

$$\rho_{N+1}(x) := \frac{N+1}{Z_{N+1}(a, \beta)} e^{-\beta x/2} x^{\beta a/2} I_N(a, \beta; x) \quad (3.5)$$

where

$$Z_{N+1}(a, \beta) := \prod_{l=1}^{N+1} \int_0^\infty dx_l e^{-\beta x_l/2} x_l^{\beta a/2} \prod_{1 \leq j < k \leq N+1} |x_k - x_j|^\beta \quad (3.6)$$

$$I_N(a, \beta; x) := \prod_{l=1}^N \int_0^\infty dx_l |x - x_l|^\beta e^{-\beta x_l/2} x_l^{\beta a/2} \prod_{1 \leq j < k \leq N} |x_k - x_j|^\beta \quad (3.7)$$

The normalization (3.6) is a well known limiting case of the Selberg integral, and has the exact evaluation (see e.g. [Ask80])

$$Z_N(a, \beta) = \left(\frac{2}{\beta}\right)^{N(1+\beta a/2)+\beta N(N-1)/2} \prod_{j=0}^{N-1} \frac{\Gamma(1+\beta(j+1)/2)\Gamma(1+\beta(j+a)/2)}{\Gamma(1+\beta/2)}. \quad (3.8)$$

Notice that for  $\beta$  even the integral (3.7) is a polynomial in  $x$ . In this case (3.7) has been shown to be expressible in terms of a certain generalized Laguerre polynomial based on Jack polynomials [For94]. Furthermore, this generalized Laguerre polynomial has a different integral representation, which allows the  $N$ -dimensional integral (3.7) to be expressed as a  $\beta$ -dimensional integral. This reads [BF97a]

$$\frac{1}{Z_N(a+2, \beta)} I_N(a, \beta; x) = \frac{1}{Q(a, \beta)} |f(a, \beta; x)| \quad (3.9)$$

where

$$f(a, \beta; x) := \int_{\mathcal{C}_1} dt_1 \cdots \int_{\mathcal{C}_\beta} dt_\beta \prod_{j=1}^{\beta} e^{x t_j} t_j^{-N-3+2/\beta} (1-t_j)^{a+N+2/\beta-1} \prod_{1 \leq j < k \leq \beta} (t_k - t_j)^{4/\beta}. \quad (3.10)$$

The contours of integration in (3.10) must be simple loops which start at  $x = 1$  and enclose the origin, and the quantity  $Q(a, \beta)$  in (3.9) is chosen so that at  $x = 0$ , the r.h.s. equals 1 (the l.h.s. has this property). Also, the modulus sign in (3.9) has been included for convenience to eliminate terms of unit modulus which otherwise occur in  $Q(a, \beta)$ ; this is valid for  $x \in \mathbf{R}$  since then the l.h.s. is positive. Choosing each  $\mathcal{C}_j$  in (3.10) to be the unit circle gives

$$Q(a, \beta) = (2\pi)^\beta M_\beta(a+2/\beta-1, N, 2/\beta) \quad (3.11)$$

where

$$\begin{aligned} M_N(a', b', c') &:= \prod_{l=1}^N \int_{-1/2}^{1/2} e^{\pi i \theta_l (a' - b')} |1 + e^{2\pi i \theta_l}|^{a' + b'} \prod_{1 \leq j < k \leq N} |e^{2\pi i \theta_k} - e^{2\pi i \theta_j}|^{2c'} \\ &= \prod_{j=1}^N \frac{\Gamma(a' + b' + 1 + (j-1)c') \Gamma(1 + c'j)}{\Gamma(a' + 1 + (j-1)c') \Gamma(b' + 1 + (j-1)c') \Gamma(1 + c')} \end{aligned} \quad (3.12)$$

(the integral  $M_N$  is due to Morris [Mor82]).

Our interest is in the large- $N$  asymptotic form of  $\rho_{N+1}(4Nx)\Big|_{a=YN}$ , which from (3.5) and (3.9) requires the large- $N$  form of  $f(YN, \beta; x)$ . The necessary technique is a generalized saddle point analysis as introduced in [For94] and presented in detail in [BF97a]. The saddle points occur at the stationary points of

$$N\left(4xt_j - \log t_j + (Y+1)\log(1-t_j)\right), \quad (3.13)$$

which is the  $N$ -dependent term of the integrand of  $f(YN, \beta; x)$  when expressed as an exponential. A simple calculation gives that there are two stationary points,  $t_+$  and  $t_-$  say, given by

$$t_{\pm} = \frac{x - Y/4 \pm ((x - Y/4)^2 - x)^{1/2}}{2x}. \quad (3.14)$$

Note that  $t_+ = t_-^*$  for

$$(x - Y/4)^2 - x < 0. \quad (3.15)$$

Assuming (3.15), following the strategy of [BF97a], the leading large- $N$  asymptotic behaviour is obtained by deforming  $\beta/2$  of the contours through  $t_+$ , and the remaining  $\beta/2$  of the contours through  $t_-$  (this introduces a factor of  $\binom{\beta}{\beta/2}$  to account for the number of ways of dividing the  $\beta$  contours into these two classes). The calculation now proceeds in a conventional way, with the exponent (3.13) being expanded about  $t_{\pm}$  to second order, and the  $N$ -independent terms in the integrand replaced by their value at  $t_+$  or  $t_-$  as appropriate. After this step we have

$$\begin{aligned} |f(YN, \beta; x)| &\sim \binom{\beta}{\beta/2} |g_2(t_+, Y)|^{\beta} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_{\beta/2} e^{-\frac{N}{2}g_1(t_+, Y)(t_1^2 + \cdots + t_{\beta/2}^2)} \\ &\quad \times \prod_{1 \leq j < k \leq \beta/2} |t_k - t_j|^{4/\beta} \\ &= \binom{\beta}{\beta/2} |g_2(t_+, Y)|^{\beta} |Ng_1(t_+, Y)|^{-\beta+1} \left(V_{\beta/2}(2/\beta)\right)^2 \end{aligned} \quad (3.16)$$

where

$$g_1(t_+, Y) := \frac{1}{t_+^2} - \frac{1+Y}{(1-t_+)^2} \quad (3.17)$$

$$g_2(t_+, Y) := e^{N(4xt_+ - \log t_+ + (Y+1)\log(1-t_+))} t_+^{-3+2/\beta} (1-t_+)^{2/\beta-1} (t_+ - t_-) \quad (3.18)$$

$$\begin{aligned} V_N(c) &:= \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_N e^{-\frac{1}{2}(t_1^2 + \cdots + t_N^2)} \prod_{1 \leq j < k \leq N} |t_k - t_j|^{2c} \\ &= (2\pi)^{N/2} \prod_{j=0}^{N-1} \frac{\Gamma(1+c(j+1))}{\Gamma(1+c)}. \end{aligned} \quad (3.19)$$

The equality in (3.16) follow from a simple change of variables, while (3.19) is known as Mehta's integral, and can be evaluated as a limiting case of the Selberg integral [Ask80] (for a direct evaluation see [Eva94]).

From the definition (3.14) we have that

$$|x_+|^2 = \frac{1}{4x}, \quad |1-x_+|^2 = \frac{1+Y}{4x}, \quad |t_+ - t_-|^2 = \frac{1}{x^2} \left(x - (x - Y/4)^2\right),$$

and use of these results in the formulas (3.17) and (3.18) defining  $g_1$  and  $g_2$  shows

$$\frac{|g_2(t_+, Y)|^\beta}{|g_1(t_+, Y)|^{\beta-1}} = e^{2\beta N(x-Y/4)}(1+Y)^{(1+Y)N\beta/2}(4x)^{-YN\beta/2}(1+Y)^{1/2}\frac{1}{x}\left(x-(x-Y/4)^2\right)^{1/2}. \quad (3.20)$$

Furthermore, from (3.8)

$$\frac{Z_N(a+2, \beta)}{Z_{N+1}(a, \beta)} = \left(\frac{2}{\beta}\right)^{N\beta-\frac{\beta}{2}} \frac{\Gamma(1+\frac{\beta}{2})}{\Gamma(1+\frac{\beta}{2}(N+1))} \frac{\Gamma(\frac{\beta}{2}(a+N+1)+1)}{\Gamma(\frac{\beta}{2}a+1)\Gamma(\frac{\beta}{2}(a+1)+1)}, \quad (3.21)$$

while use of (3.12) in (3.11) and comparison with (3.19) shows that

$$\frac{1}{Q(a, \beta)} = \frac{1}{(2\pi)^{\beta/2}V_\beta(2/\beta)} \prod_{j=1}^{\beta} \frac{\Gamma(a+\frac{2}{\beta}j)\Gamma(N+1+\frac{2}{\beta}(j-1))}{\Gamma(a+N+\frac{2}{\beta}j)}, \quad (3.22)$$

and use of (3.19) gives

$$\frac{(V_{\beta/2}(2/\beta))^2}{V_\beta(2/\beta)} = (\beta/2)^{\beta/2} \frac{\Gamma(1+\frac{\beta}{2})}{\Gamma(1+\beta)}. \quad (3.23)$$

With (3.9) substituted in (3.5), the remaining task is to use Stirling's formula to compute the leading large- $N$  asymptotic behaviour of (3.21) and (3.22) with  $a = YN$ . Performing this task, and substituting the resulting expression together with (3.20) and (3.23) in (3.9), shows that for  $x$  such that (3.15) is true

$$\lim_{N \rightarrow \infty} \rho_{N+1}(4Nx) \Big|_{a=YN} = \frac{1}{2\pi x} \left(x - (x - Y/4)^2\right)^{1/2}. \quad (3.24)$$

Outside the interval (3.15), i.e. outside  $x \in [\frac{1}{2}(1 + \frac{Y}{2} - \sqrt{1+Y}), \frac{1}{2}(1 + \frac{Y}{2} + \sqrt{1+Y})]$ , this limit must vanish. This is seen from the fact that the density is positive, and must satisfy the normalization

$$\int_0^\infty \rho_{N+1}(4Nx) dx \sim \frac{1}{4},$$

which is satisfied by the r.h.s. of (3.24). The result (3.24) for the scaled density, established for even  $\beta$ , is identical to the result (3.2) known for  $\beta = 1$  and 2, as expected.

## 4 The distribution functions in the neighbourhood of the smallest eigenvalue

### 4.1 The $n$ -point distributions

For fixed  $N$  and  $\beta = 1, 2$  or 4, the exact expressions for the general  $n$ -point distribution function in the Laguerre ensemble are known [Bro65, NW91, NF95]. With  $a = YN$  and  $N$  large, it is natural to move the origin to the (mean) location of the smallest eigenvalue, and to scale the eigenvalues so that the mean spacing near the spectrum edge is  $O(1)$ . One anticipates that the limiting  $n$ -point distribution function will correspond to the  $n$ -point distribution function for the so-called soft edge, which is the edge of the spectrum for the Gaussian random matrix ensemble, with the eigenvalues appropriately scaled.

In quantitative terms, we expect that for appropriate  $\nu(N)$  independent of  $\beta$ ,

$$\lim_{N \rightarrow \infty} \left( \nu(N) \right)^n \rho_{(n)} \left( N(1 - \sqrt{1+Y})^2 - \nu(N)x_1, \dots, N(1 - \sqrt{1+Y})^2 - \nu(N)x_n \right) = \rho_{(n)}^{\text{soft}}(x_1, \dots, x_n). \quad (4.1)$$

On the l.h.s.,  $\rho_{(n)}$  refers to the  $n$ -point distribution function for the Laguerre ensemble with  $a = YN$ . Note from (3.2) and (3.24) that  $N(1 - \sqrt{1+Y})^2$  is the location of the smallest eigenvalue (to leading order in  $N$ ). On the r.h.s.

$$\rho_{(n)}^{\text{soft}}(x_1, \dots, x_n) := \lim_{N \rightarrow \infty} \left( \frac{1}{2^{1/2} N^{1/6}} \right)^n \rho_{(n)} \left( (2N)^{1/2} + \frac{x_1}{2^{1/2} N^{1/6}}, \dots, (2N)^{1/2} + \frac{x_n}{2^{1/2} N^{1/6}} \right), \quad (4.2)$$

where here  $\rho_{(n)}$  refers to the  $n$ -point distribution function for the Gaussian ensemble defined by the eigenvalue p.d.f

$$\frac{1}{C} \prod_{l=1}^N e^{-\beta x_l^2/2} \prod_{1 \leq j < k \leq N} |x_k - x_j|^\beta, \quad (4.3)$$

and  $(2N)^{1/2}$  is the leading order location of the largest eigenvalue. In this section (4.1) will be explicitly verified for  $\beta = 2$ , with the quantity  $\nu(N)$  shown to be given by

$$\nu(N) = N^{1/3} \frac{2^{1/3} (\sqrt{1+Y} - 1)^2}{(Y^2 - (2+Y)(\sqrt{1+Y} - 1)^2)^{1/3}}. \quad (4.4)$$

Once  $\nu(N)$  has been determined, the validity of (4.1) for  $n = 1$  can be established by matching the one-body density in the neighbourhood of the smallest eigenvalue implied by (3.24) with the asymptotic behaviour [For93]

$$\rho_{(1)}^{\text{soft}}(x) \underset{x \rightarrow -\infty}{\sim} \frac{\sqrt{|x|}}{\pi} \quad (4.5)$$

(this idea is motivated by a similar procedure used in [Nis96, KF97a]). Now the result (3.24) implies that for all  $\beta$

$$\rho_{(1)} \left( N(1 - \sqrt{1+Y})^2 - \nu(N)x \right) \underset{\substack{N \rightarrow \infty \\ x \rightarrow -\infty}}{\sim} \frac{\sqrt{|x|}}{\pi} \frac{(1+Y)^{1/4}}{(1 - \sqrt{1+Y})^2} \sqrt{\frac{\nu(N)}{N}}, \quad (4.6)$$

valid for  $0 \ll x \ll N/\nu(N)$ . Substituting (4.5) and (4.6) in (4.1) with  $n = 1$ , and using (4.4), shows that (4.1) is satisfied provided

$$(1+Y)^{1/4} \frac{2^{1/2} (\sqrt{1+Y} - 1)}{((2+Y)(Y^2 - \sqrt{1+Y} - 1)^2)^{1/2}} = 1, \quad (4.7)$$

which is readily verified.

In preparation for verifying (4.1) for  $\beta = 2$  and general  $n$ , we first recall some formulas particular to that coupling. For the Laguerre ensemble (1.2) we have [Bro65],

$$\rho_{(n)}^{\text{soft}}(x_1, \dots, x_n) = \det[P_N(x_j, x_k)]_{j,k=1,\dots,n}, \quad (4.8)$$

where with  $L_n^a(x)$  denoting the Laguerre polynomial of degree  $n$ ,

$$\begin{aligned} P_N(x, y) &:= (xy)^{a/2} e^{-(x+y)/2} c_N \frac{L_N^a(x)L_{N-1}^a(y) - L_N^a(y)L_{N-1}^a(x)}{x-y}, \\ &:= (xy)^{a/2} e^{-(x+y)/2} \frac{c_N}{N+a} \frac{L_N^a(x)yL_{N-1}^{a'}(y) - L_N^a(y)xL_{N-1}^{a'}(x)}{x-y}, \quad c_N = \frac{\Gamma(1+N)}{\Gamma(a+N)}. \end{aligned} \quad (4.9)$$

Furthermore, we know that [For93]

$$\rho_{(n)}^{\text{soft}}(x_1, \dots, x_n) = \det[K^{\text{soft}}(x_j, x_k)]_{j,k=1,\dots,n} \quad (4.10)$$

where, with  $\text{Ai}(x)$  denoting the Airy function,

$$K^{\text{soft}}(x, y) := \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}(y)\text{Ai}'(x)}{x-y}. \quad (4.11)$$

Comparison of (4.10) and (4.8), and the fact that (4.1) is valid for  $n = 1$  once (4.4) is established shows that to verify (4.1) it suffices to establish the asymptotic formula

$$x^{NY/2} e^{-x/2} L_N^{YN}(x) \Big|_{x=N(1-\sqrt{1+Y})^2-\nu(N)X} \underset{N \rightarrow \infty}{\sim} k_N(Y) \text{Ai}(X) \quad (4.12)$$

with  $\nu(N)$  given by (4.4) and  $\mu$  fixed. There is no need to specify  $k_N(Y)$ , as its value is uniquely determined by (4.5) and (4.6). Indeed, substituting (4.12) in (4.9), then substituting the resulting expression in (4.8) with  $n = 1$ , setting  $x = y$  and comparing with (4.1) for  $x \rightarrow -\infty$  shows that

$$\left(k_N(Y)\right)^2 = \frac{1}{(1-\sqrt{1+Y})^2} \frac{\nu(N)}{c_N}. \quad (4.13)$$

In table 2 we give the numerical value of the ratio of the l.h.s. to r.h.s. of (4.12) with  $k_N(Y)$  given by (4.13) for various values of  $X, Y, N$ .

	(0, 1)	(1, 2)	(-1, 2)
50	1.0426	1.0567	0.9929
60	1.0402	1.0556	0.9938
70	1.0383	1.0544	0.9944
80	1.0367	1.0533	0.9949
90	1.0353	1.0523	0.9953

**Table 2.** Numerical value of the ratio of the l.h.s. to r.h.s. of (4.12) for various values of  $N$  (column on left) and  $(X, Y)$  (top row).

The asymptotic formula (4.12) can be derived by utilizing the fact that  $y = e^{-x/2} x^{(\alpha+1)/2} L_N^\alpha(x)$  satisfies the second order differential equation

$$y'' + \left( \frac{2N + \alpha + 1}{2x} + \frac{1 - \alpha^2}{4x^2} - \frac{1}{4} \right) y = 0. \quad (4.14)$$

Substituting  $\alpha = YN$ ,  $x = N(1 - \sqrt{1+Y})^2 - \nu(N)X$ , shows that for large  $N$  (4.14) reduces to

$$y'' - \frac{1}{2(1 - \sqrt{1+Y})^6} \left( Y^2 - (2+Y)(1 - \sqrt{1+Y})^2 \right) \frac{(\nu(N))^3}{N} xy = 0. \quad (4.15)$$

With  $\nu(N)$  given by (4.4), this equation reads  $y'' - xy = 0$  and its unique solution which decays as  $X \rightarrow -\infty$  is  $y = k_N(Y)\text{Ai}(X)$ , thus establishing (4.12).

## 4.2 Distribution of the smallest eigenvalue

In principle, knowledge of the  $n$ -point distributions allows the calculation of other statistical quantities such as the distribution of the smallest eigenvalue. This together with the above results implies that, after appropriate change of origin and scale, the p.d.f. of the smallest eigenvalue in the Laguerre ensemble with  $a = YN$  equals, in the  $N \rightarrow \infty$  limit, the p.d.f. for the largest (or equivalently smallest) eigenvalue in the Gaussian ensemble (4.3). For  $\beta = 2$ , this can be explicitly verified from the non-linear equations characterizing the respective p.d.f.'s due to Tracy and Widom [TW94a, TW94b].

Let  $E(x)$  denote the probability that the interval  $(0, x)$  in the Laguerre ensemble (1.2) with  $\beta = 2$  contains no eigenvalues, and let

$$\sigma(x) = x \frac{d}{dx} \log E(x). \quad (4.16)$$

Then it has been derived in [TW94b] that  $\sigma(x)$  satisfies the non-linear equation

$$\begin{aligned} (x\sigma'')^2 &= 4x(\sigma')^3 + \sigma^2 + (2a + 4N - 2x)\sigma\sigma' \\ &+ (a^2 - 2ax - 4Nx + x^2)(\sigma')^2 - 4\sigma(\sigma')^2. \end{aligned} \quad (4.17)$$

Also, let  $\tilde{E}(x)$  denote the probability that there are no eigenvalues between  $(x, \infty)$  in the (infinite dimensional) Gaussian ensemble (4.3) with coordinates as in (4.2). Then it was derived in [TW94a] that the quantity

$$R(x) := \frac{d}{dx} \log \tilde{E}(x) \quad (4.18)$$

satisfies the non-linear equation

$$(R'')^2 + 4R' \left( (R')^2 - xR' + R \right) = 0. \quad (4.19)$$

As the p.d.f. for the smallest (largest) eigenvalue is simply related to  $E(x)$  ( $\tilde{E}(x)$ ) by differentiation, we see that the d.e. (4.17) characterizes the p.d.f. for the smallest eigenvalue in the Laguerre ensemble with  $\beta = 2$ , while (4.19) characterizes the limiting form of the p.d.f. of the eigenvalue at the spectrum edge of the Gaussian ensemble. (Of course boundary conditions must be specified; these follow from the small (large)  $x$  behaviour of the density.) The results of the previous subsection imply that

$$\lim_{N \rightarrow \infty} E \left( N(\sqrt{1+Y} - 1)^2 - \nu(N)x \right) \Big|_{a=YN} = \tilde{E}(x). \quad (4.20)$$

Indeed, a straightforward calculation using the explicit form (4.4) of  $\nu(N)$  shows that after changing variables  $x \mapsto N(\sqrt{1+Y} - 1)^2 - \nu(N)x$  in (4.17) and introducing the function

$$\tilde{\sigma}(x) := \frac{1}{\nu(N)} \sigma\left(N(\sqrt{1+Y} - 1)^2 - \nu(N)x\right), \quad (4.21)$$

the d.e. (4.17) reduces down to (4.19) with  $R = \tilde{\sigma}$ . This is precisely what is required by (4.20).

## Acknowledgement

This work was supported by the Australian Research Council.

## References

- [Ask80] R. Askey. *SIAM J. Math. Analysis*, 11:938–951, 1980.
- [BF97a] T.H. Baker and P.J. Forrester. *Commun. Math. Phys.*, 188:175–216, 1997.
- [BF97b] T.H. Baker and P.J. Forrester. Symmetric Jack polynomials from nonsymmetric theory. preprint, q-alg/9707001, 1997.
- [BFB97] P.W. Brouwer, K. Frahm, and C.W.J. Beenakker. *Phys. Rev. Lett.*, 78:4737–4740, 1997.
- [Bro65] B.V. Bronk. *J. Math. Phys.*, 6:228–237, 1965.
- [Bro95] P.W. Brouwer. *Phys. Rev. B*, 51:16878–16884, 1995.
- [Dys62] F.J. Dyson. *J. Math. Phys.*, 3:1191–1198, 1962.
- [Ede88] A. Edelman. *SIAM J. Matrix Anal. Appl.*, 9:543–560, 1988.
- [Eva94] R. J. Evans. *Contemp. Math.*, 166:341–357, 1994.
- [For93] P.J. Forrester. *Nucl. Phys. B*, 402:709–728, 1993.
- [For94] P.J. Forrester. *J. Math. Phys.*, 35:2539–2551, 1994.
- [For95] P.J. Forrester. *Nucl. Phys. B*, 435:421–429, 1995.
- [Hua63] L.K. Hua. *Analysis of Functions of Several Complex Variables in the Classical Domains*. American Mathematical Society, Providence, RI, 1963.
- [Joh88] K. Johansson. *Bull. Sc. math.*, 2<sup>e</sup> série, 112:257–304, 1988.
- [Joh95] K. Johansson. On fluctuation of eigenvalues of random hermitian matrices. to appear, *Duke Math. J.*
- [Kan93] J. Kaneko. *SIAM J. Math. Anal.*, 24:1086–1110, 1993.

- [KF97a] E. Kanzieper and V. Freilikher. *Phys. Rev. Lett.*, 78:3806–3809, 1997.
- [KF97b] E. Kanzieper and V. Freilikher. *Phys. Rev. E*, 55:3712–3715, 1997.
- [Mac95] I.G. Macdonald. *Hall polynomials and symmetric functions*. Oxford University Press, Oxford, 2nd edition, 1995.
- [Mar88] Ph. A. Martin. *Rev. Mod. Phys.*, 60:1075–1127, 1988.
- [Meh91] M.L. Mehta. *Random Matrices*. Academic Press, New York, 2nd edition, 1991.
- [Mor82] W. G. Morris. *Constant term identities for finite and affine root systems, conjectures and theorems*. PhD thesis, University of Wisconsin, 1982.
- [MPS85] P.A. Mello, P. Pereyra, and T.H. Seligman. *Ann. Phys. (N.Y.)*, 161:254–275, 1985.
- [NF95] T. Nagao and P.J. Forrester. *Nucl. Phys. B*, 435:401–420, 1995.
- [Nis96] S. Nishigaki. *Phys. Lett. B*, 387:139–144, 1996.
- [NW91] T. Nagao and M. Wadati. *J. Phys. Soc. Japan*, 60:3298–3322, 1991.
- [Sta89] R. P. Stanley. *Advances in Math.*, 77:76–115, 1989.
- [TW94a] C.A. Tracy and H. Widom. *Commun. Math. Phys.*, 159:151–174, 1994.
- [TW94b] C.A. Tracy and H. Widom. *Commun. Math. Phys.*, 161:289–309, 1994.