

# On the Classification of Bulk and Boundary Conformal Field Theories

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## Abstract

The classification of rational conformal field theories is reconsidered from the standpoint of boundary conditions. Solving Cardy's equation expressing the consistency condition on a cylinder is equivalent to finding integer valued representations of the fusion algebra. A complete solution not only yields the admissible boundary conditions but also gives valuable information on the bulk properties.

The classification of conformal field theories (CFTs) remains an important issue, both in the study of bulk and boundary critical phenomena [1] and in string theory [2]. The guiding principle is that of consistency of the theory on an arbitrary  $2D$  surface, with or without boundaries.

To define a rational conformal field theory, one first specifies a chiral algebra  $\mathcal{A}$ , *e.g.* the Virasoro algebra or one of its extensions, at a certain level. Rationality means that at this level,  $\mathcal{A}$  has only a finite set  $\mathcal{I}$  of admissible irreducible representations  $\mathcal{V}_i$ ,  $i \in \mathcal{I}$ . We denote by  $\mathcal{V}_i^*$  the representation conjugate to  $\mathcal{V}_i$  and  $i = 1$  refers to the vacuum representation.

We suppose that the characters  $\chi_i(q)$  of these representations, the symmetric matrix  $S_i^j$  of modular transformations of the  $\chi$ 's and the fusion coefficients  $N_{ij}^k$  of the  $\mathcal{V}$ 's, assumed to be given in terms of  $S$  by the Verlinde formula [3], are all known.

A physical theory is then fully defined by the collection of bulk and boundary fields and their 3-point couplings (“structure constants”). In particular, the spectrum of bulk fields is described by the finite set  $\text{Spec}$  of pairs  $(j, \bar{j})$  of representations (with possible multiplicities) of the left and right copies of  $\mathcal{A}$ , such that the Hilbert space of the theory on an infinitely long cylinder reads  $\mathcal{H} = \bigoplus_{\text{Spec}} \mathcal{V}_j \otimes \mathcal{V}_{\bar{j}}$ . We denote by  $\mathcal{E}$  the finite set of labels of the left-right symmetric elements (up to conjugation) of the spectrum:  $\mathcal{E} = \{j | (j, \bar{j} = j^*) \in \text{Spec}\}$ , with these same multiplicities. In terms of all these data one is in principle able to compute exactly all correlation functions of the CFT on an arbitrary  $2D$  surface with or without boundaries [4].

These data, however, are subject to consistency constraints. Much emphasis was originally put on bulk properties, namely on the consistency of the 4-point functions on the sphere [5] and the zero-point function (the modular invariant partition function) on a torus [6]. In this letter, we want to show that the consistency of the partition function on a finite cylinder is equivalent to a well-posed algebraic problem. Once solved, this not only determines the possible boundary conditions but also yields substantial information on the bulk properties, by determining the diagonal part  $\mathcal{E}$  of the set  $\text{Spec}$ . The consistency condition on a cylinder is the well known equation of Cardy [7], but it seems that its consequences have never been fully exploited.

We recall Cardy's discussion. Let  $W_n$  and  $\bar{W}_n$  denote the spin  $s_W$  generators of the  $\mathcal{A}$  chiral algebra acting on the left and right sectors. Then the  $\mathcal{A}$ -invariant boundary states satisfy the conditions  $(W_n - (-1)^{s_W} \bar{W}_{-n})|\phi\rangle = 0$ . (Here we assume that the “gluing automorphism” [8, 2] is trivial.) Solutions to this system of equations are spanned by special states  $|j\rangle\rangle$  (called Ishibashi states [9]) labelled by the finite set  $\mathcal{E}$ . Let  $Z_{AB}$  be the partition function of the CFT on a cylinder of perimeter  $T$  and length  $L$  with boundary conditions  $A$  and  $B$ . Regarded as resulting from the periodic “time”  $T$  evolution of the system with prescribed boundary conditions, it is a linear form in the characters with integer coefficients:  $Z_{AB} = \sum_{i \in \mathcal{I}} n_{iA}^B \chi_i(q)$ ,  $q = e^{-\pi T/L}$ . It also results from the “time”  $L$  evolution between states  $|A\rangle$  and  $|B\rangle$ : it is then a sesquilinear form in the components of these boundary states on Ishibashi states. If we write

$$|A\rangle = \sum_{j \in \mathcal{E}} \frac{\psi_A^j}{(S_1^j)^{\frac{1}{2}}} |j\rangle\rangle, \quad (1)$$

then  $Z_{AB} = \sum_{j \in \mathcal{E}} \psi_A^j (\psi_B^j)^* \chi_j(\tilde{q})/S_1^j$ ,  $\tilde{q} = e^{-4\pi L/T}$ , and Cardy's equation results from the identification of  $\chi_i$  in these two alternative expressions of  $Z_{AB}$ :

$$n_{iA}^B = \sum_{j \in \mathcal{E}} \frac{S_i^j}{S_1^j} \psi_A^j (\psi_B^j)^* \quad \text{for all } i \in \mathcal{I}. \quad (2)$$

We have assumed that there is some quantity, for example a  $Z_N$  charge, that discriminates representations with the same character, *e.g.*  $\chi_i = \chi_{i^*}$ , and enables one to write equation (2) unambiguously. With the norm appropriate for boundary states [4, 7], orthogonality of  $|A\rangle$  and  $|B\rangle$  amounts to  $n_{1A}^B = \sum_{j \in \mathcal{E}} \psi_A^j (\psi_B^j)^* = \delta_{AB}$ . Reality of  $Z_{AB}$  implies that  $n_{iA}^B = n_{i^*B}^A$ . It is also natural to introduce conjugate states  $|A^*\rangle$  such that  $\psi_{A^*}^j = \psi_A^{j^*} = (\psi_A^j)^*$ . Suppose we have found a complete set of boundary states, *i.e.* such that  $\sum_B \psi_B^j (\psi_B^{j'})^* = \delta_{jj'}$ . Then, using the fact that the ratios  $\frac{S_i^j}{S_1^j}$  for a given  $j$  form a representation of the fusion algebra, as a consequence of the Verlinde formula:  $\frac{S_{i_1}^j}{S_1^j} \frac{S_{i_2}^j}{S_1^j} = \sum_{i_3} N_{i_1 i_2}^{i_3} \frac{S_{i_3}^j}{S_1^j}$ , it follows that the matrices  $n_i = (n_{iA}^B)$  also form a representation of the fusion algebra

$$\sum_B n_{i_1 A}^B n_{i_2 B}^C = \sum_{i_3} N_{i_1 i_2}^{i_3} n_{i_3 A}^C . \quad (3)$$

Equation (2) expresses  $n_i$  in terms of its eigenvalues  $\frac{S_i^j}{S_1^j}$  and its eigenvectors  $\psi^j$ . The matrices commute and are normal ( $n_i$  commutes with  $n_i^T = n_{i^*}$ ).

Thus the problem of finding all the orthonormal and complete solutions to Cardy's equation (2) is equivalent to that of finding all matrix representations of the fusion algebra with nonnegative integer coefficients that satisfy  $n_i^T = n_{i^*}$ . The fact that classes of boundary partition functions are associated with representations of the fusion algebra has been recognized before [7, 10, 11]. To the best of our knowledge, however, the general and simple argument above has not been given.

The search for representations of the fusion algebra is a well posed problem that may be approached by algebraic or combinatorial methods. As the fusion algebra has a finite number of generators, one first determines the representations of these generators. The latter are matrices whose possible eigenvalues  $\frac{S_i^j}{S_1^j}$  are known. They may be regarded as the adjacency matrices of graphs that characterize the representation. For theories with the affine (current) algebra  $\widehat{\mathfrak{sl}}(2)$  as a chiral algebra, this problem has been solved long ago [10]. The representations of  $\widehat{\mathfrak{sl}}(2)$  at level  $k$  are labelled by an integer  $1 \leq j \leq k+1$ , and Cardy's equation says that the generator  $n_2 = n_{2^*}$  has eigenvalues  $\frac{S_2^j}{S_1^j} = 2 \cos \frac{\pi j}{k+2}$ . The only symmetric indecomposable matrices with nonnegative integer entries and eigenvalues less than 2 are the adjacency matrices of *A-D-E* Dynkin diagrams and of the "tadpole" graphs  $A_{2n}/\mathbb{Z}_2$  [12]. Only the former solutions are retained as their spectrum matches the spectrum of  $\widehat{\mathfrak{sl}}(2)$  theories, known by their modular invariant partition functions [13]. For a theory classified by a Dynkin diagram  $G$  of *A-D-E* type, the set  $\mathcal{E}$  is the set of Coxeter exponents of  $G$ . The matrices  $n_i$  are then the "fused adjacency matrices" or "intertwiners" defined recursively by equation (3):  $n_{i+1} = n_2 n_i - n_{i-1}$ ,  $i = 2, 3, \dots, k$ ;  $n_1 = I$  [10, 14], and one verifies that all their entries are non negative integers. This set of complete orthonormal solutions of Cardy's equation for  $\widehat{\mathfrak{sl}}(2)$  theories is unique up to a relabelling of the states  $|A\rangle$ .

For minimal  $c < 1$  theories, if  $c = 1 - \frac{6(g-h)^2}{gh}$  and the theory is classified by a pair  $(A_{h-1}, G)$ ,  $h$  odd,  $G$  a Dynkin diagram of *A-D-E* type with Coxeter number  $g$ , a class of solutions is obtained by tensor products of the solutions of the  $\widehat{\mathfrak{sl}}(2)$  case:  $n_{(rs)} = N_r \otimes n_s$

where  $r = 1, 3, \dots, h - 2$ ,  $s = 1, \dots, g - 1$ ,  $N_r$  are the fusion matrices of  $\widehat{sl}(2)$  at level  $h - 2$  and  $n_s$  are the intertwiners just mentioned pertaining to  $G$  [15]. It is likely that all (orthonormal and complete) solutions of Cardy's equation are of this type. We have checked it explicitly in the cases of  $G = D_4, D_6, E_7$ .

For more complicated CFTs, like those based on higher rank affine algebras, no general result on the representations of the fusion algebra is known, although a few steps have been taken [10, 16]. See also some recent more abstract work in this direction [17].

The considerations of the present paper put on a firmer ground the general program of classification of CFTs through the classification of  $\mathbb{N}$ -valued matrix representations of the fusion algebra proposed in [10]. For a given chiral algebra  $\mathcal{A}$  and an  $\mathbb{N}$ -valued matrix representation  $\{n_i\}$  of its fusion algebra, the commuting and normal matrices  $n_i$  may be diagonalized in an orthonormal basis  $\psi^j$  labelled by representations  $\mathcal{V}_j$  of  $\mathcal{A}$ , with possible multiplicities, which according to Cardy's equation must give the diagonal part  $\mathcal{E}$  of the spectrum of the theory in the bulk. It then must be a relatively easy task to decide if this diagonal subset may be supplemented to make a fully consistent theory in the bulk, namely if

$$\sum_{j \in \mathcal{E}} |\chi_j(q)|^2 + \text{off-diagonal terms} \quad (4)$$

may be made modular invariant. For example, in  $\widehat{sl}(2)$  theories, as discussed above, from the possible solutions to Cardy's equation, one recovers the classification of modular invariants, once solutions of type  $A_{2n}/\mathbb{Z}_2$  have been discarded. For higher rank,  $\widehat{sl}(3)$  for example, it is also known that some representations of the fusion algebra do not give rise to a modular invariant [10]. In some cases [10, 16], there are explicit expressions of the modular invariant partition function in terms of the matrices  $n_i$ . In block diagonal cases,  $Z = \sum_{B \in T} |\sum_i n_i^B \chi_i|^2$ , for a special subset of boundary conditions  $T$  and a special boundary state denoted  $1 \in T$ .

The eigenvectors  $\psi$  carry also physical information on boundary and bulk properties. The  $g$ -factors introduced by Affleck and Ludwig [18] giving the groundstate degeneracies are easily seen to be (in a unitary theory)  $g_A = \psi_A^1 / \sqrt{S_1^1}$ . The bulk-boundary reflection coefficients [4] are expressed in our notation as

$$\sqrt{\frac{S_1^j}{S_1^1}} C_{(j,j^*),1}^A = \frac{\psi_A^j}{\psi_A^1} \quad j \in \mathcal{E} . \quad (5)$$

These ratios realize the Pasquier algebra [19] with structure constants

$$M_{ij}{}^k = \sum_A \frac{\psi_A^i \psi_A^j (\psi_A^k)^*}{\psi_A^1} . \quad (6)$$

In  $\widehat{sl}(2)$  and minimal models, these  $M_{ij}{}^k$  are, in a suitable normalization, the relative OPE coefficients in the bulk  $C_{(i^*)(j^*)}^{(kk^*)}$ , as seen by the independent study of bulk [16] and boundary [4] locality equations.

It is thus suggested that the classification of  $N$ -valued representations of the fusion algebra is a profitable route to the classification of CFTs and determination of a large part of their data.

## Acknowledgements

REB and PAP are supported by the Australian Research Council. VBP acknowledges the support and hospitality of ICTP, Trieste, and the partial support of the Bulgarian National Research Foundation (contract  $\Phi$ -643). J-BZ acknowledges partial support of the EU Training and Mobility of Researchers Program (Contract FMRX-CT96-0012), which made possible an extremely profitable stay in SISSA, Trieste. VBP and JBZ would like to thank J. Fuchs, V. Schomerus and C. Schweigert for inviting them to a very stimulating meeting on “Conformal field theory of D-branes” at DESY.

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