Excited TBA Equations
for Tricritical Ising Model

I: Massive (hep-th/0012223)
II: Massless Flow (in preparation)

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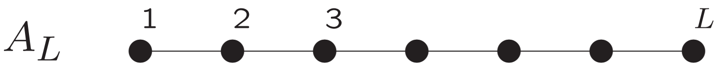
ABF Models

Height Adjacency:



Face Weights:

$$W\left(\begin{matrix} a\pm 1 & a \\ a & a\mp 1\end{matrix}\,\middle|\, u\right) = \frac{\vartheta_1(\lambda-u)}{\vartheta_1(\lambda)}$$

$$W\left(\begin{matrix} a & a\pm 1 \\ a\mp 1 & a\end{matrix}\,\middle|\, u\right) = \left(\frac{\vartheta_1((a-1)\lambda)\vartheta_1((a+1)\lambda)}{\vartheta_1^2(a\lambda)}\right)^{1/2} \frac{\vartheta_1(u)}{\vartheta_1(\lambda)}$$

$$W\left(\begin{matrix} a & a\pm 1 \\ a\pm 1 & a\end{matrix}\,\middle|\, u\right) = \frac{\vartheta_1(a\lambda\pm u)}{\vartheta_1(a\lambda)}$$

Boundary Weights: $(r,s) =$ Kac labels

$$K\left(\begin{matrix} r & \\ r & r\pm 1\end{matrix}\,\middle|\, u,\xi_L\right) = \left(\frac{\vartheta_1((r\pm 1)\lambda)}{\vartheta_1(r\lambda)}\right)^{\frac{1}{2}} \frac{\vartheta_1(u\pm\xi_L)\vartheta_1(u\mp r\lambda\mp\xi_L)}{\vartheta_1^2(\lambda)}$$

$$K\left(\begin{matrix} & s \\ s\pm 1 & s\end{matrix}\,\middle|\, u,\xi_R\right) = \left(\frac{\vartheta_1((s\pm 1)\lambda)}{\vartheta_1(s\lambda)}\right)^{\frac{1}{2}} \frac{\vartheta_4(u\pm\xi_R)\vartheta_4(u\mp s\lambda\mp\xi_R)}{\vartheta_4^2(\lambda)}$$

$$\vartheta_1(u) = \vartheta_1(u,p), \qquad \vartheta_4(u) = \vartheta_4(u,p), \qquad \lambda = \frac{\pi}{L+1}$$

$$\text{Temperature} = t = p^2 = \pm\exp(-2\pi\varepsilon)$$

$t = 0$ is a critical point

Regimes:

Regime I:	$-\pi/2+\lambda \le u \le 0,$	$-1 < t < 0$
Regime II:	$-\pi/2+\lambda \le u \le 0,$	$0 < t < 1$
Regime III:	$0 \le u \le \lambda,$	$0 < t < 1$
Regime IV:	$0 \le u \le \lambda,$	$-1 < t < 0$

Critical Points & CFT

Regime III/IV: $c = 1 - \dfrac{6}{L(L+1)}$ Unitary Minimal

Regime I/II: $c = \dfrac{2(L-2)}{L+1}$ $\mathbb{Z}_{L-1}$

Scaling Limit & QFT

- An integrable QFT is obtained by thermal perturbation from the CFT in the continuum scaling limit of an $N \times M$ lattice:

$$t \to 0, \qquad N, M \to \infty, \qquad a \to 0$$

$$\mu = N|t|^\nu, \qquad R = Na, \qquad R' = Ma, \qquad 4\mu = mR$$

$$\nu = \text{correlation length exponent} = \begin{cases} \dfrac{L+1}{4}, & \text{III/IV} \\ \dfrac{L+1}{2(L-1)}, & \text{I/II} \end{cases}$$

$$\mu = \text{perturbation parameter}$$

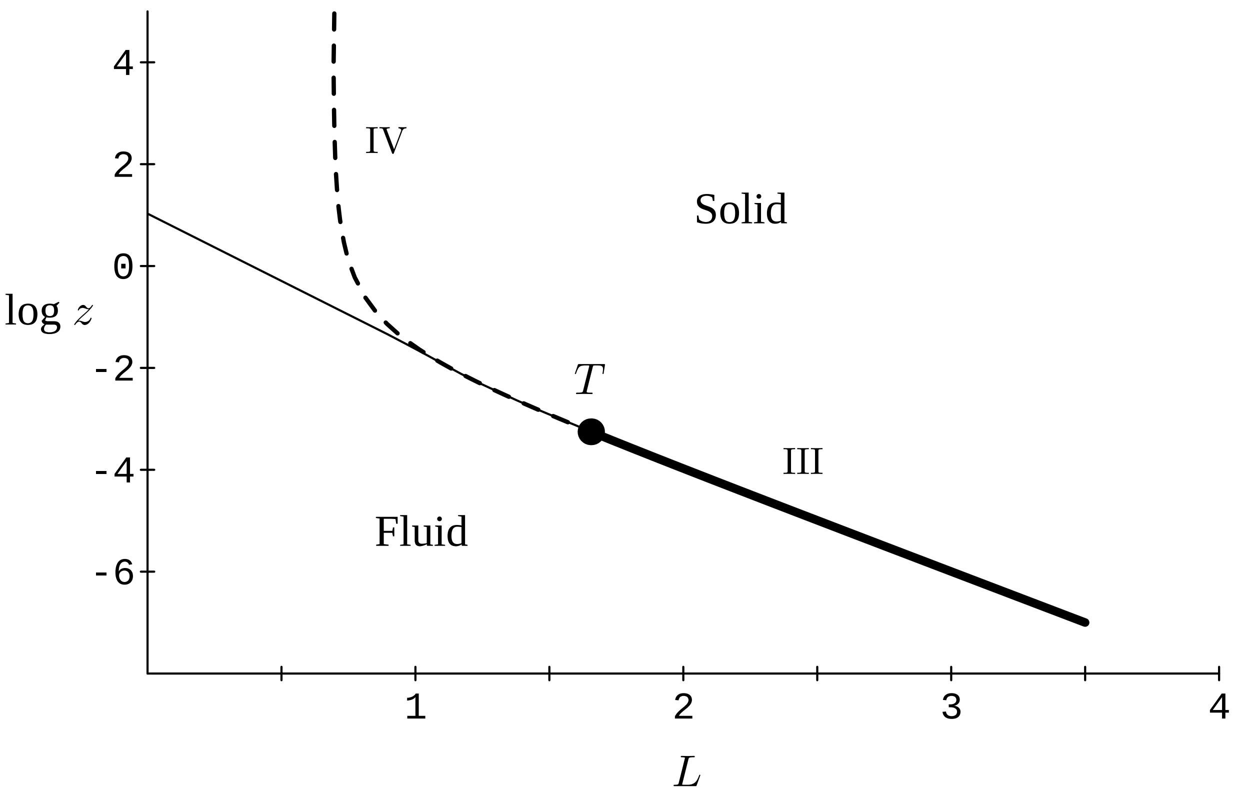
- Crude finite-size scaling gives

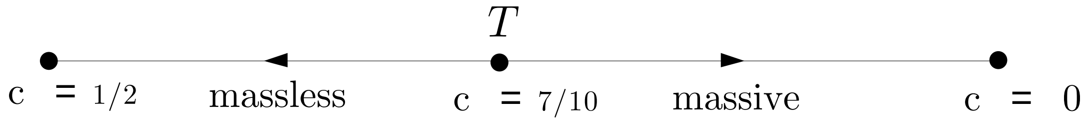
$$\xi \sim |t|^{-\nu} \sim N, \qquad \xi^{-1} \sim m$$

A_4 Phase Diagram: Interacting Hard Squares

$$L = 4, \qquad \lambda = \frac{\pi}{5}$$

$T =$ Tricritical Ising Point

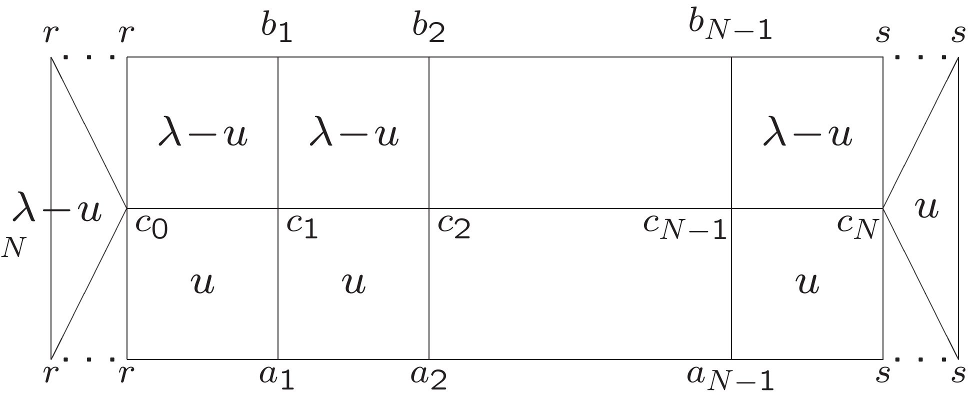




RG Flow Diagram

Commuting Transfer Matrices

$$\boldsymbol{D}(u)_{a,b} = \sum_{c_0,\dots,c_N}$$



YBE + Inversion + BYBE ⇒ Integrability

Normalized Double-Row Transfer Matrices:

$$\boldsymbol{t}(u) = S_{r,s}(u)S(u)[i\,\frac{\vartheta_1(u+2\lambda,p)\vartheta_1(\lambda,p)}{\vartheta_1(u+3\lambda,p)\vartheta_1(u+\lambda,p)}]^{2N}\boldsymbol{D}(u)$$

where

$$S(u) = \frac{\vartheta_1(2u-\lambda,p)^2}{\vartheta_1(2u-3\lambda,p)\vartheta_1(2u+\lambda,p)}$$

$$S_{r,s}(u) = (-1)^s h_r(u-\xi_L)h_{-r}(u+\xi_L)\bar{h}_s(u-\xi_R)\bar{h}_{-s}(u+\xi_R)$$

$$\begin{aligned} h_r(u) &= \frac{\vartheta_1(\lambda,p)\vartheta_1(u+(3-r)\lambda,p)\vartheta_1(u+(1-r)\lambda,p)}{\vartheta_1(u,p)\vartheta_1(u-\lambda,p)\vartheta_1(u+2\lambda,p)} \\ \bar{h}_s(u) &= \frac{\vartheta_4(\lambda,p)\vartheta_4(u+(3-s)\lambda,p)\vartheta_4(u+(1-s)\lambda,p)}{\vartheta_4(u,p)\vartheta_4(u-\lambda,p)\vartheta_4(u+2\lambda,p)} \end{aligned}$$

Universal TBA Functional Equation:

$$\boxed{\boldsymbol{t}(u)\boldsymbol{t}(u+\lambda) = \boldsymbol{I} + \boldsymbol{t}(u+3\lambda)}$$

Finite Size Corrections

$$\boxed{-\tfrac{1}{2}\log D(u) = N f_{\mathsf{bulk}}(u) + b_{r,s}(u) + \frac{R\sin\vartheta}{N}E(R) + o(\frac{1}{N})}$$

$$\begin{aligned}
f_{\mathsf{bulk}}(u) &= \text{bulk free energy}\\
b_{r,s}(u) &= \text{boundary free energy}\\
E(R) &= \text{scaling energy}
\end{aligned}$$

$$\vartheta = \text{anisotropy angle} = \begin{cases} (L+1)u, & \text{III, IV}\\ -\dfrac{2(L+1)u}{L-1}, & \text{I, II}\end{cases}$$

Conformal point: (eg. $R \to 0$, $R \to \infty$)

$$\boxed{\frac{RE(R)}{2\pi} \to -\frac{c}{24} + \Delta_{r,s} + n, \qquad n \in \mathbb{N}}$$

$$\begin{aligned}
c &= \text{central charge}\\
\Delta_{r,s} &= \text{conformal weights}
\end{aligned}$$

$$\boxed{c_{\mathsf{eff}} = -\frac{12RE(R)}{\pi}}$$

Solution of TBA Equations

Combine Methods:

- Klümper & Pearce 91/92:	Conformal weights
- O'Brien, Pearce & Warnaar 97:	All conformal excitations
- Pearce & Nienhuis 98:	Off-critical

$$\boxed{\boldsymbol{t}(u)\boldsymbol{t}(u+\lambda) = \boldsymbol{I} + \boldsymbol{t}(u+3\lambda)}$$

Solve for $\log t(u)$ by Fourier Series:

$$\boxed{\begin{array}{c}\text{Periodicity} + \text{Analyticity/Zeros}\\ \Rightarrow\ \text{Non-Linear Integral Equation}\end{array}}$$

Periodicity: $t = \pm \exp(-2\pi\varepsilon)$

$$\text{period rectangle} = \begin{cases} (-\frac{\lambda}{2}, \frac{9\lambda}{2}) \times (-\frac{\pi i\varepsilon}{2}, \frac{\pi i\varepsilon}{2}), & \text{III} \\ \\ (-\frac{\lambda}{2}, \frac{9\lambda}{2}) \times (-\pi i\varepsilon, \pi i\varepsilon), & \text{IV} \end{cases}$$

Additional symmetry in Regime IV:

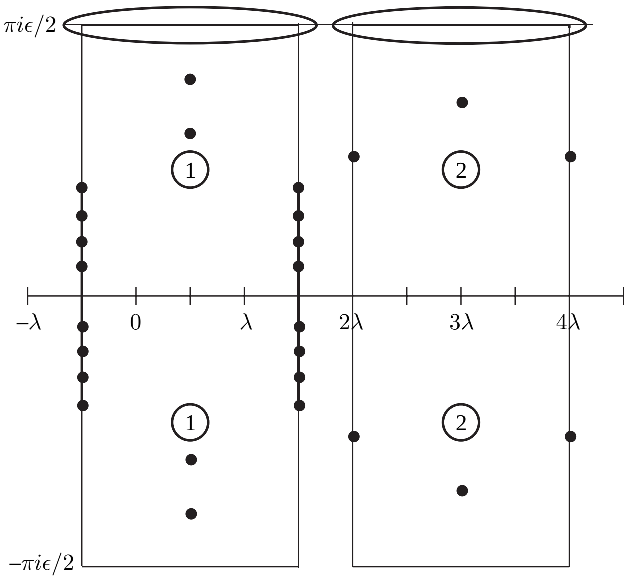
$$t(u \pm \pi/2 + \pi i\varepsilon) = t(u), \qquad \text{IV}$$

Classification of Solutions by Location of Zeros:

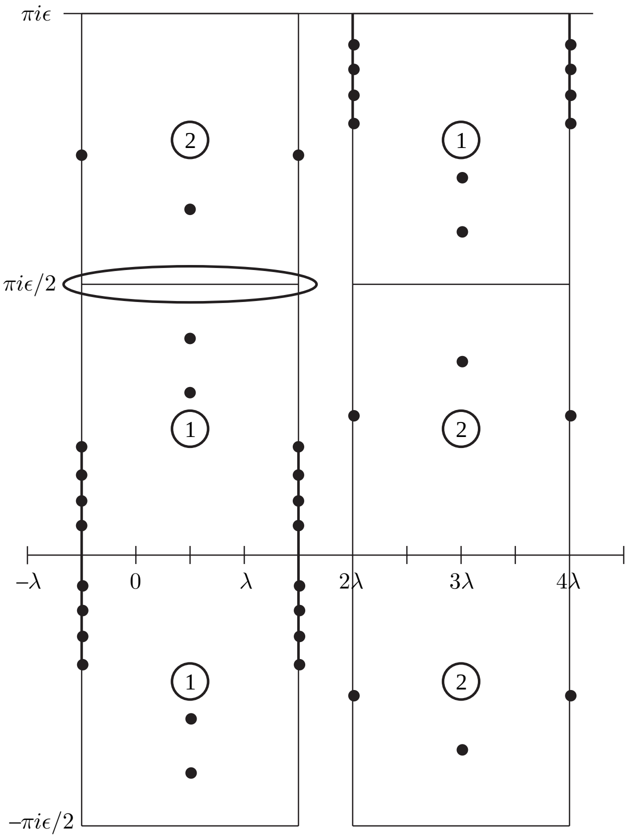
$$\boxed{\begin{array}{c}(\boldsymbol{m},\boldsymbol{n})\ \text{Systems} + \text{Quantum Numbers}\\ =\ \text{String Content} + \text{Excitations}\end{array}}$$

Period Rectangles/Zeros in u-Plane

Regime III: Two Strips



Regime IV: Two Strips or One Extended Strip



Symmetry: $t(u \pm \pi/2 + \pi i\varepsilon) = t(u) = t(\overline{u})$, IV

(m,n) Systems

Vacuum Sector $(r,s)=(1,1)$:

$$\boxed{\boldsymbol{m}+\boldsymbol{n}=\frac{1}{2}(N\boldsymbol{e}_1+\mathcal{I}\boldsymbol{m})}$$

$$\boldsymbol{m}=(m_1,m_2),\quad \boldsymbol{n}=(n_1,n_2),\quad \boldsymbol{e}_1=(1,0)$$

$\mathcal{I}=A_2$ adjacency matrix, $\quad N=\#$ columns

$$\boxed{\begin{aligned} m_1+n_1&=\tfrac{N+m_2}{2}\\ m_2+n_2&=\tfrac{m_1}{2}\end{aligned}}$$

$$N,m_1,m_2=\text{even},\quad m_2\le\frac{m_1}{2}$$

$N\to\infty$ Limit:

$$n_1=O(\frac{N}{2}),\quad m_1,m_2,n_2=O(1)$$

Other Sectors (r,s):

$$\boxed{\boldsymbol{m}+\boldsymbol{n}=\frac{1}{2}(N\boldsymbol{e}_1+\mathcal{I}\boldsymbol{m}+\boldsymbol{e}_{s-1}+\boldsymbol{e}_{L-r})}$$

String Contents:

$m_i=$ number of 1-strings in strip $i=1,2$

$n_i=$ number of 2-strings in strip $i=1,2$

Finitized Characters

Vacuum Sector $(r,s)=(1,1)$:

The generator of conformal spectra in UV limit ($R\to 0$) is

$$\boxed{Z^{(N)}_{(1,1)}(q)=\sum_E q^E=q^{-c/24}\sum_{(\boldsymbol{m},\boldsymbol{n})} q^{\frac{1}{4}\boldsymbol{m}C\boldsymbol{m}}\begin{bmatrix} m_1+n_1 \\ m_1 \end{bmatrix}\begin{bmatrix} m_2+n_2 \\ m_2 \end{bmatrix}=\chi^{(N)}_{1,1}(q)}$$

$$\begin{aligned} q &= \text{modular parameter} = e^{-2\pi\frac{M}{N}\sin\vartheta} \\ C &= A_2 \text{ Cartan matrix} = 2-\mathcal{I} \\ \begin{bmatrix} m+n \\ m \end{bmatrix} &= \text{Gaussian polynomial} = \sum_{I_1=0}^{n}\sum_{I_2=0}^{I_1}\cdots\sum_{I_m=0}^{I_{m-1}} q^{I_1+\ldots+I_m} \end{aligned}$$

Completeness: $A=A_4$ adjacency matrix

$$\lim_{q\to 1}\chi^{(N)}_{1,1}(q)=\sum_{(\boldsymbol{m},\boldsymbol{n})}\binom{m_1+n_1}{m_1}\binom{m_2+n_2}{m_2}=[A^N]_{1,1}$$

$N\to\infty$ Limit:

$$\boxed{\lim_{N\to\infty}\chi^{(N)}_{1,1}(q)=q^{-c/24}\sum_{m_1,\,m_2\,\text{even}}\frac{q^{\frac{1}{4}\boldsymbol{m}C\boldsymbol{m}}}{(q)_{m_1}}\begin{bmatrix} \frac{m_1}{2} \\ m_2 \end{bmatrix}=\chi_{1,1}(q)}$$

Other Sectors (r,s):

$$\chi^{(N)}_{(r,s)}(q) = q^{-\frac{c}{24}+\Delta_{r,s}-\frac{1}{4}(s-r)(s-r-1)}\sum_{(\boldsymbol{m},\boldsymbol{n})} q^{\frac{1}{4}\boldsymbol{m}C\boldsymbol{m}-\frac{1}{2}m_{s-1}}\begin{bmatrix} m_1+n_1 \\ m_1 \end{bmatrix}\begin{bmatrix} m_2+n_2 \\ m_2 \end{bmatrix}$$

Quantum Numbers

- The quantum numbers *uniquely* label energy levels

$$\boxed{I = (I_1^{(1)}, \ldots, I_{m_1}^{(1)} | I_1^{(2)}, \ldots, I_{m_2}^{(2)})}$$

- The associated conformal energy for $(r,s) = (1,1)$ is

$$\boxed{E = -\frac{c}{24} + \boldsymbol{m}C\boldsymbol{m} + \sum_{j=1}^{m_1} I_j^{(1)} + \sum_{k=1}^{m_2} I_k^{(2)}}$$

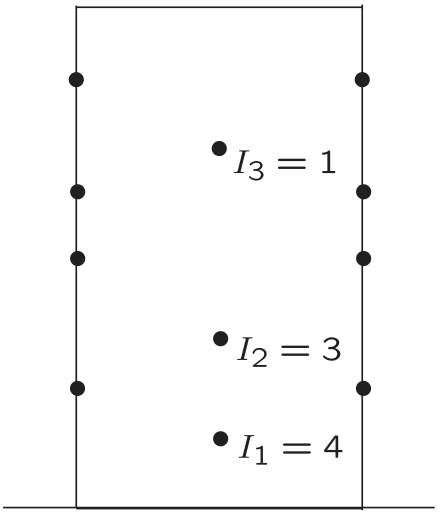
$$\boldsymbol{m}C\boldsymbol{m} = \frac{1}{2}(m_1^2 - m_1 m_2 + m_2^2)$$

- In each strip $i = 1, 2$

$I_j^{(i)}$ = {number of 2-strings with imaginary parts greater than given 1-string labelled j}

- Clearly,

$$n_i \geq I_1^{(i)} \geq I_2^{(i)} \geq \ldots \geq I_{m_i}^{(i)} \geq 0, \qquad i = 1, 2$$



Massive TBA Equations: $(r,s)=(1,1)$

Quantum Number: $I=(I_1^{(1)},\ldots,I_{m_1}^{(1)}|I_1^{(2)},\ldots,I_{m_2}^{(2)})$

$$RE(R)=2mR\sum_{j=1}^{m_1}\cosh\beta_j^{(1)}-\frac{mR}{2\pi}\int_{-\infty}^{\infty}d\vartheta\,\cosh\vartheta\log\left(1+e^{-\epsilon_2(\vartheta)}\right)$$

$$\begin{aligned}\epsilon_1(\vartheta)\;&=\;-\log\tanh^2\frac{\vartheta}{2}-\frac{1}{2\pi}\int_{-\infty}^{\infty}d\vartheta'\,\frac{\log(1+e^{-\epsilon_2(\vartheta')})}{\cosh(\vartheta-\vartheta')}\\&\quad-\sum_{j=1}^{m_1}\log[\tanh(\frac{\vartheta}{2}+\frac{\beta_j^{(1)}}{2})\tanh(\frac{\vartheta}{2}-\frac{\beta_j^{(1)}}{2})]\end{aligned}$$

$$\begin{aligned}\epsilon_2(\vartheta)\;&=\;2mR\cosh\vartheta-\log\tanh^2\frac{\vartheta}{2}-\frac{1}{2\pi}\int_{-\infty}^{\infty}d\vartheta'\,\frac{\log(1+e^{-\epsilon_1(\vartheta')})}{\cosh(\vartheta-\vartheta')}\\&\quad-\sum_{k=1}^{m_2}\log[\tanh(\frac{\vartheta}{2}+\frac{\beta_k^{(2)}}{2})\tanh(\frac{\vartheta}{2}-\frac{\beta_k^{(2)}}{2})]\end{aligned}$$

$$\begin{aligned}&-2mR\sinh\beta_j^{(1)}=n_j^{(1)}\pi+\int\frac{d\vartheta}{2\pi}\,\frac{\log(1+e^{-\epsilon_1(\vartheta)})}{\sinh(\beta_j^{(1)}-\vartheta)}-i\log[\tanh^2(\frac{\pi i}{4}-\frac{\beta_j^{(1)}}{2})]\\&-i\sum_{k=1}^{m_2}\log[\tanh(\frac{\pi i}{4}-\frac{\beta_k^{(2)}+\beta_j^{(1)}}{2})]-i\sum_{k=1}^{m_2}\log[\tanh(\frac{\pi i}{4}+\frac{\beta_k^{(2)}-\beta_j^{(1)}}{2})]\end{aligned}$$

$$\begin{aligned}0\;&=\;n_k^{(2)}\pi+\int\frac{d\vartheta}{2\pi}\,\frac{\log(1+e^{-\epsilon_2(\vartheta)})}{\sinh(\beta_k^{(2)}-\vartheta)}-i\log[\tanh^2(\frac{\pi i}{4}-\frac{\beta_k^{(2)}}{2})]\\&-i\sum_{j=1}^{m_1}\log[\tanh(\frac{\pi i}{4}-\frac{\beta_k^{(2)}+\beta_j^{(1)}}{2})]-i\sum_{j=1}^{m_1}\log[\tanh(\frac{\pi i}{4}+\frac{\beta_j^{(1)}-\beta_k^{(2)}}{2})]\end{aligned}$$

$$\begin{aligned}n_j^{(1)}\;&=\;2(m_1-j)-m_2+1+2I_j^{(1)},\qquad j=1,2,\ldots,m_1\\n_k^{(2)}\;&=\;2(m_2-k)-m_1+1+2I_k^{(2)},\qquad k=1,2,\ldots,m_2\end{aligned}$$

Massless Flow

Thermal Bulk Flow:

$$\boxed{\mathcal{M}(4,5) + \phi_{1,3} \to \mathcal{M}(3,4)}$$

Flow of Primary Fields:

$$\boxed{\chi^{4,N+2}_{r,s}(q) \;\mapsto\; \chi^{3,N}_{r',s'}(q)}$$

$$
\Delta^{(4,5)}_{r,s} \qquad\qquad\qquad\qquad \Delta^{(3,4)}_{r',s'}
$$

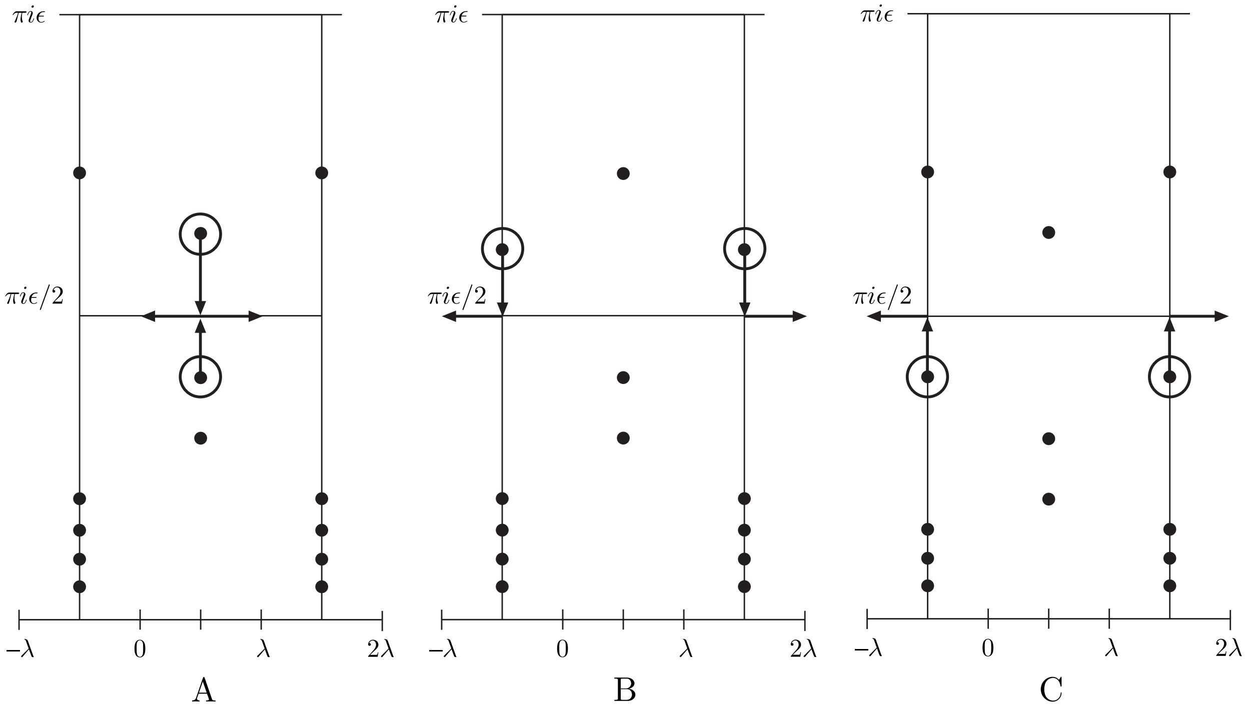
$$
\begin{array}{c|ccc}
s & & & \\
4 & \frac{3}{2} & \frac{7}{16} & 0 \\
3 & \frac{3}{5} & \frac{3}{80} & \frac{1}{10} \\
2 & \frac{1}{10} & \frac{3}{80} & \frac{3}{5} \\
1 & 0 & \frac{7}{16} & \frac{3}{2} \\
\hline
 & 1 & 2 & 3 \quad r
\end{array}
\qquad \mapsto \qquad
\begin{array}{c|ccc}
s & & & \\
4 & \frac{1}{2} & \frac{1}{16} & 0 \\
3 & 0 & \frac{1}{16} & \frac{1}{2} \\
2 & \frac{1}{2} & \frac{1}{16} & 0 \\
1 & 0 & \frac{1}{16} & \frac{1}{2} \\
\hline
 & 1 & 2 & 3 \quad r
\end{array}
$$

Cylinder: (6 Flows) Primary $\mapsto$ Primary

Torus: (1 Flow) Primary $\mapsto \begin{cases} \text{Primary} \\ \text{Descendant} \end{cases}$

Mechanisms A, B, C

$$\boxed{(I_1^{(1)}, I_2^{(1)}, \ldots, I_{m_1}^{(1)} \mid I_1^{(2)}, I_2^{(2)}, \ldots, I_{m_2}^{(2)}) \mapsto (I'_1, I'_2, \ldots, I'_m)}$$



$$\boxed{I_{m_1}^{(1)} = I_{m_2}^{(2)} = 0} \qquad \boxed{I_{m_1}^{(1)} = 0,\ I_{m_2}^{(2)} > 0} \qquad \boxed{I_{m_1}^{(1)} \geq 0}$$

$(r, s) = (1, 1)$

$$(\boldsymbol{m}, \boldsymbol{n}): \quad \left.\begin{aligned} m_1 + n_1 &= \tfrac{N+2+m_2}{2} \\ m_2 + n_2 &= \tfrac{m_1}{2} \end{aligned}\right\} \mapsto \; m + n = \frac{N+m}{2}$$

$$m = \begin{cases} m_1 + m_2 - 2, & \text{A} \\ m_1 + m_2, & \text{B,C} \end{cases} \qquad n = \begin{cases} n_1 + n_2, & \text{A} \\ n_1 + n_2 - 1, & \text{B,C} \end{cases}$$

$$\text{A:} \quad \begin{cases} I'_j = n_2 + I_j^{(1)}, & j = 1, 2, \ldots, m_1 - 1 \\ I'_{m_1-1+k} = n_2 - I_{m_2-k}^{(2)}, & k = 1, 2, \ldots, m_2 - 1 \end{cases}$$

$$\text{B,C:} \quad \begin{cases} I'_j = n_2 + I_j^{(1)} - 1, & j = 1, 2, \ldots, m_1 \\ I'_{m_1+k} = n_2 - I_{m_2+1-k}^{(2)}, & k = 1, 2, \ldots, m_2 \end{cases}$$

Character Map

$$\chi_{1,1}^{4,N+2}(q) = q^{-\frac{7}{240}} \sum_{(\boldsymbol{m},\boldsymbol{n})_{N+2}} q^{\frac{1}{4}\boldsymbol{m}C\boldsymbol{m}} \begin{bmatrix} m_1+n_1 \\ m_1 \end{bmatrix} \begin{bmatrix} m_2+n_2 \\ m_2 \end{bmatrix}$$

$$\begin{aligned}
= \; & q^{-\frac{7}{240}} \sum_{m_1,m_2 \text{ even}} q^{\frac{1}{2}(m_1^2-m_1m_2+m_2^2)} \left\{ \begin{bmatrix} m_1+n_1-1 \\ m_1-1 \end{bmatrix} \begin{bmatrix} m_2+n_2-1 \\ m_2-1 \end{bmatrix} \right. \\
& \left. + q^{m_2} \begin{bmatrix} m_1+n_1-1 \\ m_1-1 \end{bmatrix} \begin{bmatrix} m_2+n_2-1 \\ m_2 \end{bmatrix} + q^{m_1} \begin{bmatrix} m_1+n_1-1 \\ m_1 \end{bmatrix} \begin{bmatrix} m_2+n_2 \\ m_2 \end{bmatrix} \right\}
\end{aligned}$$

$$\begin{aligned}
\mapsto \; & q^{-\frac{1}{48}} \sum_{m_1,m_2 \text{ even}} q^{\frac{1}{2}m^2} \left\{ q^{n_2(m_1-1)+n_2(m_2-1)} \begin{bmatrix} m_1+n_1-1 \\ m_1-1 \end{bmatrix} \begin{bmatrix} m_2+n_2-1 \\ m_2-1 \end{bmatrix}_{\frac{1}{q}} \right. \\
& + q^{(n_2-1)m_1+(n_2-1)m_2} \begin{bmatrix} m_1+n_1-1 \\ m_1-1 \end{bmatrix} \begin{bmatrix} m_2+n_2-1 \\ m_2 \end{bmatrix}_{\frac{1}{q}} \\
& \left. + q^{n_2m_1+n_2m_2} \begin{bmatrix} m_1+n_1-1 \\ m_1 \end{bmatrix} \begin{bmatrix} m_2+n_2 \\ m_2 \end{bmatrix}_{\frac{1}{q}} \right\} \\
= \; & q^{-\frac{1}{48}} \sum_{m \text{ even}} q^{\frac{1}{2}m^2} \sum_{m_1 \text{ even}} \left\{ q^{n_2(m_1-1)} \begin{bmatrix} m_1+n_1-1 \\ m_1-1 \end{bmatrix} \begin{bmatrix} m_2+n_2-1 \\ m_2-1 \end{bmatrix} \right. \\
& \left. + q^{(n_2-1)m_1} \begin{bmatrix} m_1+n_1-1 \\ m_1-1 \end{bmatrix} \begin{bmatrix} m_2+n_2-1 \\ m_2 \end{bmatrix} + q^{n_2m_1} \begin{bmatrix} m_1+n_1-1 \\ m_1 \end{bmatrix} \begin{bmatrix} m_2+n_2 \\ m_2 \end{bmatrix} \right\}
\end{aligned}$$

$$\begin{aligned}
= \; & q^{-\frac{1}{48}} \sum_{m \text{ even}} q^{\frac{1}{2}m^2} \sum_{m_1 \text{ even}} q^{\frac{3}{2}m_1^2-mm_1} \left\{ q^{m-\frac{7}{2}m_1+2} \begin{bmatrix} \frac{N+m-m_1+2}{2} \\ m_1-1 \end{bmatrix} \begin{bmatrix} \frac{m_1-2}{2} \\ m-m_1+1 \end{bmatrix} \right. \\
& \left. + q^{-m_1} \begin{bmatrix} \frac{N+m-m_1}{2} \\ m_1-1 \end{bmatrix} \begin{bmatrix} \frac{m_1-2}{2} \\ m-m_1 \end{bmatrix} + \begin{bmatrix} \frac{N+m-m_1}{2} \\ m_1 \end{bmatrix} \begin{bmatrix} \frac{m_1}{2} \\ m-m_1 \end{bmatrix} \right\}
\end{aligned}$$

$$= q^{-\frac{1}{48}} \sum_{m \text{ even}} q^{\frac{1}{2}m^2} \begin{bmatrix} \frac{N+m}{2} \\ m \end{bmatrix} = \chi_{1,1}^{3,N}(q)$$

A_4 Counting:

$$\sum_{m \text{ even}} \binom{\frac{N+m}{2}}{m} = [A^{N+2}]_{1,1}$$

UV Massless TBA Equations: $(r,s)=(1,1)$

UV Quantum Number: $I=(I_1^{(1)},\dots,I_{m_1}^{(1)}|I_1^{(2)},\dots,I_{m_2}^{(2)})$

$$RE(R)=mR[\sum_{j=1}^{m_1}e^{-\beta_j^{(1)}}+\sum_{k=1}^{m_2}e^{-\beta_k^{(2)}}]-\frac{mR}{2\pi}\int_{-\infty}^{\infty}d\vartheta\,e^{\vartheta}\,\log(1+e^{-\epsilon_1(\vartheta)})$$

$$\begin{aligned}\epsilon_1(\vartheta)\;=\;& mRe^{\vartheta}-\log\frac{e^{\vartheta}+\sqrt{2}+e^{-\vartheta}}{e^{\vartheta}-\sqrt{2}+e^{-\vartheta}}-\frac{1}{2\pi}\int_{-\infty}^{\infty}d\vartheta'\,\frac{\log(1+e^{-\epsilon_2(\vartheta')})}{\cosh(\vartheta-\vartheta')}\\ &-\sum_{j=1}^{m_1}\log\tanh\,(\frac{\beta_j^{(1)}}{2}-\frac{\vartheta}{2})-\sum_{k=1}^{m_2}\log\tanh\,(\frac{\beta_k^{(2)}}{2}-\frac{\vartheta}{2})\end{aligned}$$

$$\begin{aligned}\epsilon_2(\vartheta)\;=\;& mRe^{-\vartheta}-\log\frac{e^{\vartheta}+\sqrt{2}+e^{-\vartheta}}{e^{\vartheta}-\sqrt{2}+e^{-\vartheta}}-\frac{1}{2\pi}\int_{-\infty}^{\infty}d\vartheta'\,\frac{\log(1+e^{-\epsilon_1(\vartheta')})}{\cosh(\vartheta-\vartheta')}\\ &-\sum_{j=1}^{m_1}\log\tanh\,(\frac{\beta_j^{(1)}}{2}+\frac{\vartheta}{2})-\sum_{k=1}^{m_2}\log\tanh\,(\frac{\beta_k^{(2)}}{2}+\frac{\vartheta}{2})\end{aligned}$$

$$\begin{aligned}\epsilon_2(\beta_j^{(1)}-\frac{\pi i}{2}) &= n_j^{(1)}\pi i, \qquad j=1,2,\dots,m_1\\ \epsilon_1(\beta_k^{(2)}-\frac{\pi i}{2}) &= n_k^{(2)}\pi i, \qquad k=1,2,\dots,m_2\end{aligned}$$

$$\begin{aligned}n_j^{(1)} &= 2(m_1-j)-m_2+1+2I_j^{(1)}, \qquad j=1,2,\dots,m_1\\ n_k^{(2)} &= 2(m_2-k)-m_1+1+2I_k^{(2)}, \qquad k=1,2,\dots,m_2\end{aligned}$$

IR Massless TBA Equations: $(r,s)=(1,1)$

IR Quantum Number: $I'=(I'_1,I'_2,\ldots,I'_m)$

$$RE(R)=mR\sum_{j=1}^{m}e^{-\beta'_j}-\frac{mR}{2\pi}\int_{-\infty}^{\infty}d\vartheta\,e^{-\vartheta}\,\log\left(1+e^{-\epsilon'(\vartheta)}\right)$$

$$\begin{aligned}\epsilon'(\vartheta) &= mRe^{-\vartheta}-\log\frac{e^{\vartheta}+\sqrt{2}+e^{-\vartheta}}{e^{\vartheta}-\sqrt{2}+e^{-\vartheta}}-\frac{1}{2\pi}\int_{-\infty}^{\infty}d\vartheta'\,\frac{\log(1+e^{-\epsilon'(\vartheta')})}{\cosh(\vartheta+\vartheta')}\\ &\quad -\sum_{j=1}^{m}\log\tanh\left(\frac{\beta'_j}{2}+\frac{\vartheta}{2}\right)\end{aligned}$$

$$\epsilon'(\beta'_j-\frac{\pi i}{2}) = n'_j\pi i, \qquad j=1,2,\ldots,m$$

$$n'_j = 2(m-j)+1+2I'_j, \qquad j=1,2,\ldots,m$$

Numerical Agreement

- Where the numerical solution of UV and IR equations both converge they agree to 5 digits in c_{eff} and location of zeros.

Conclusions

- TBA equations can be derived for all excitations in lattice approach in both massive and massless regimes.

- The TBA equations can be solved numerically. In massless regimes this reveals a pattern of RG flows between characters.

- It would be of interest to extend this work to periodic boundary conditions and other models.