Integrable Lattice Realizations of $N = 1$ Superconformal Boundary Conditions

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Abstract

We construct integrable boundary conditions for $\widehat{s\ell}(2)$ coset models with central charges $c = \frac{3}{2} - \frac{12}{m(m+2)}$ and $m = 3, 4, \ldots$ The associated cylinder partition functions are generating functions for the branching functions but these boundary conditions manifestly break the superconformal symmetry. We show that there are additional integrable boundary conditions, satisfying the boundary Yang-Baxter equation, which respect the superconformal symmetry and lead to generating functions for the superconformal characters in both Ramond and Neveu-Schwarz sectors. We also present general formulas for the cylinder partition functions. This involves an alternative derivation of the superconformal Verlinde formula recently proposed by Nepomechie.

1 Introduction

It is known that for certain families of rational conformal field theories (CFTs) [4] it is possible to construct complete sets of integrable and conformal boundary conditions. More specifically, if the associated critical Yang-Baxter integrable lattice model is known, then fusion techniques can be used to construct integrable boundary conditions which satisfy the boundary Yang-Baxter equation and give rise to all of the conformal boundary conditions in the continuum scaling limit. This program has been carried out in particular for $\widehat{s\ell}(2)$ minimal [3] and $\mathbb{Z}_k$ parafermion models [13]. In these cases the Virasoro characters and parafermionic string functions are dictated by the relevant chiral algebra. In some cases, however, there exists an extended chiral symmetry and in such situations, at least from the viewpoint of CFT, the actual chiral algebra which is used is a matter of choice depending on the symmetries which are to be preserved. A relevant question is then whether integrable and conformal boundary conditions can be obtained which are compatible with the extended chiral symmetry. If the answer is yes, as we expect is generally the case, then this observation necessarily implies the existence of new solutions to the boundary Yang-Baxter equations for the underlying critical lattice model.

In this paper we consider the level two $\widehat{s\ell}(2)$ coset models which can alternatively be viewed as $N=1$ superconformal theories. We review these theories from the different CFT viewpoints in Section 2. We give a generalized Verlinde formula for the fusion coefficients of the superconformal theories, being valid for all values of the central charge. It coincides with a formula recently proposed by Nepomechie [15] for the theories without fixed point, using a different approach. In Section 3 we define lattice realizations and use a generalized fusion procedure to construct integrable boundary conditions. In the case corresponding to the usual fusion procedure, this leads to integrable and conformal boundary conditions for the coset models. The generalized fusion leads to additional integrable boundary conditions which we posit to be compatible with the superconformal symmetry. This is explained in Section 4. In Section 5 we confirm numerically that these solutions of the boundary Yang-Baxter equations indeed lead to the branching functions and to the superconformal characters in the continuum scaling limit. The integrable superconformal boundary conditions can be extended off-criticality and are highly relevant to the study of superconformal bulk and boundary flows via TBA [16].

2 Superconformal theories

In this section we review the properties of $N=1$ superconformal theories, focusing on the A-type of the A-D-E classification [5] of torus partition functions. We give an alternative description using the coset construction and relate both approaches. We give explicit expressions for the S matrices and derive the fusion rules. We derive a generalized Verlinde formula, which describes the fusion of superconformal boundary conditions, following the general framework of Behrend, Pearce, Petkova and Zuber [4].

2.1 Coset description

The coset description of these models is given by the coset [10, 14]

$$\frac{\widehat{s\ell}(2)_{m-2}\otimes\widehat{s\ell}(2)_2}{\widehat{s\ell}(2)_m}. \tag{2.1}$$

The unitary heighest weight representations have central charges

$$c=\frac{3}{2}\left[1-\frac{8}{m(m+2)}\right]\qquad m=3,4,\ldots \tag{2.2}$$

Its branching functions $b^{(l)}_{r,s}(q)$ satisfy

$$\chi_{r-1,m-2}(q)\chi_{l,2}(q)=\sum_{s=1}^{m+1}b^{(l)}_{r,s}(q)\chi_{s-1,m}(q), \tag{2.3}$$

where the ranges of the indices are $1 \le r \le m-1$, $1 \le s \le m+1$, $0 \le l \le 2$. The $\chi_{r,s}(q)$ are the characters of the affine Lie algebra $\widehat{s\ell}(2)$ at levels $m-2$, 2 and m, respectively. The branching functions have the symmetries

$$\begin{aligned} b_{r,s}^{(l)}(q) &= b_{m-r,m-s+2}^{(2-l)}(q), \\ b_{r,s}^{(l)}(q) &= 0, \qquad r+s+l = 0 \bmod 2. \end{aligned} \tag{2.4}$$

The conformal weights of the non-vanishing branching functions are given by

$$\begin{aligned} \Delta_{r,s}^{(l)} =& \frac{[(m+2)r-ms]^2-4}{8m(m+2)} + \frac{1}{8}\left(\frac{3}{4} - (-1)^{(l+s-r)/2}\right)\left(1+(-1)^{r-s}\right) \\ &+ \frac{2}{32} + \delta_{l,0}\delta_{r,m-1}\delta_{s,m+1} + \delta_{l,2}\delta_{r,1}\delta_{s,1}. \end{aligned} \tag{2.5}$$

For $m=3$, we obtain the Kac table of conformal weights as shown below.

r ($l=0$), s horizontal:

	$\frac{1}{10}$		$\frac{3}{2}$
0		$\frac{3}{5}$	

r ($l=1$), s horizontal:

$\frac{7}{16}$		$\frac{3}{80}$	
	$\frac{3}{80}$		$\frac{7}{16}$

r ($l=2$), s horizontal:

	$\frac{3}{5}$		0
$\frac{3}{2}$		$\frac{1}{10}$	

For $m=4$, we obtain the following set of weights:

r ($l=0$), s horizontal:

$\frac{3}{2}$		$\frac{1}{6}$		$\frac{3}{2}$
	$\frac{1}{16}$		$\frac{9}{16}$	
0		$\frac{2}{3}$		1

r ($l=1$), s horizontal:

	$\frac{9}{16}$		$\frac{1}{16}$	
$\frac{3}{8}$		$\frac{1}{24}$		$\frac{3}{8}$
	$\frac{1}{16}$		$\frac{9}{16}$	

r ($l=2$), s horizontal:

1		$\frac{2}{3}$		0
	$\frac{9}{16}$		$\frac{1}{16}$	
$\frac{3}{2}$		$\frac{1}{6}$		$\frac{3}{2}$

The branching functions can be expressed in terms of the branching coefficients $d_{j_1 j_2 j_3}(q)$ defined [7, 8] by

$$\begin{aligned} d_{j_1 j_2 j_3}(q) =& q^{j_1^2/4m_1 + j_2^2/4m_2 - j_3^2/4m_3 - 1/8} Q(q)^{-3} \sum_{\epsilon_1,\epsilon_2=\pm 1} \left(\sum_{k,n_1,n_2}^{(1)} - \sum_{k,n_1,n_2}^{(2)} \right) \\ & \times (-1)^{k+(\epsilon_1+\epsilon_2)/2} q^{k(k-1)/2 + k(j_3+1)/2 + \sum_{i=1}^{2}[k\epsilon_i(m_i n_i + j_i/2) + m_i n_i^2 + j_i n_i]}, \end{aligned} \tag{2.6}$$

where

$$Q(q) = \prod_{n=1}^{\infty}(1-q^n), \qquad m_1 = m, \qquad m_2 = 4, \qquad m_3 = m+2, \tag{2.7}$$

and the two sums are restricted to values of k, n_1, n_2 satisfying

$$\sum^{(1)} \;:\; k \geq \xi + 1, \qquad \eta \leq \frac{j_3+1}{2} + \sum_{i=1}^{2} \epsilon_i \left(m_i n_i + \frac{j_i}{2} \right) \in \mathbb{Z}, \tag{2.8}$$

$$\sum^{(2)} \;:\; k \leq \xi, \qquad \eta - 1 \geq \frac{j_3+1}{2} + \sum_{i=1}^{2} \epsilon_i \left(m_i n_i + \frac{j_i}{2} \right) \in \mathbb{Z}.$$

The integers $\xi = \xi(\epsilon_i, n_i)$ and $\eta = \eta(\epsilon_i, n_i)$ can be chosen arbitrarily for fixed ϵ_i and n_i. The nonvanishing branching functions are given in terms of these as

$$b^{(l)}_{r,s}(q) = d_{r,l+1,s}(q). \tag{2.9}$$

Under modular transformations, the branching functions transform as

$$b^{(l)}_{r,s}\left(e^{2\pi i\tau}\right) = \sum_{r'=1}^{m-1} \sum_{s'=1}^{m+1} \sum_{l'=0}^{2} S^{(r',s',l')}_{(r,s,l)} \; b^{(l')}_{r',s'}\left(e^{-2\pi i/\tau}\right), \tag{2.10}$$

where the modular matrix S satisfies

$$S^T = S^{-1}, \qquad S^2 = \mathbb{I}. \tag{2.11}$$

The entries of the modular matrix are given explicitly by

$$S^{(r',s',l')}_{(r,s,l)} = \frac{\sqrt{2}}{\sqrt{m(m+2)}} \sin\frac{\pi r r'}{m} \sin\frac{\pi s s'}{m+2} \sin\frac{\pi (l+1)(l'+1)}{4}. \tag{2.12}$$

The modular invariant partition functions of the coset models (2.1) have been classified by Cappelli [5] in terms of a pair of graphs (G', G) where G' is of A-type or D-type, and G is of A-D-E type. Throughout the paper, we restrict ourselves to A-type models, whose allowed spin values are given by the adjacency matrix of the graph A. According to [5], the modular invariant torus partition function for these (A_{m-1}, A_{m+1}) models is given by

$$Z(q) = \sum_{r=1}^{m-1} \sum_{s=1}^{m+1} \sum_{l=0}^{2} |b^{(l)}_{rs}(q)|^2. \tag{2.13}$$

In analogy to the reasoning in [4], we claim that the cylinder partition functions are given by

$$Z_{(r_1,s_1,l_1)|(r_2,s_2,l_2)} = \sum_{r=1}^{m-1} \sum_{s=1}^{m+1} \sum_{l=0}^{2} n^{(r,s,l)}_{(r_1,s_1,l_1),(r_2,s_2,l_2)} b^{(l)}_{rs}(q), \tag{2.14}$$

where the fusion coefficients $n^{(r_2,s_2,l_2)}_{(r,s,l),(r_1,s_1,l_1)}$ are given by the Verlinde formula

$$n^{(r_2,s_2,l_2)}_{(r,s,l),(r_1,s_1,l_1)} = \sum_{r'=1}^{m-1} \sum_{s'=1}^{m+1} \sum_{l'=0}^{2} \frac{S^{(r',s',l')}_{(r,s,l)} S^{(r',s',l')}_{(r_1,s_1,l_1)} (S^{-1})^{(r',s',l')}_{(r_2,s_2,l_2)}}{S^{(r',s',l')}_{(1,1,0)}} \in \mathbb{N}. \tag{2.15}$$

Due to the coset construction, the fusion coefficients for the branching coefficents can be written in tensor product form

$$n_{(r,s,l),(r',s',l')}^{(r'',s'',l'')} = n_{rr'}^{(m)\,r''} n_{ss'}^{(m+2)\,s''} n_{l+1,l'+1}^{(4)\,l''+1}, \tag{2.16}$$

where $n_{ij}^{(g)\,k}$ are the fusion coefficients of the affine Lie algebra $\widehat{s\ell}(2)$ at level $g-2$. The fusion coefficients $n_{ij}^{(g)\,k}$ can be expressed in terms of the matrix elements of the fused adjacency matrices $F^{(g)\,r}$ of the graph A_{g-1} as

$$n_{ij}^{(g)\,k} = F_{i,j}^{(g)\,k}, \tag{2.17}$$

where $F^{(g)\,r}$ are given recursively in terms of the adjacency matrix of the graph A_{g-1} by the $s\ell(2)$ fusion rules

$$F^{(g)\,r} = A_{g-1} F^{(g)\,r-1} - F^{(g)\,r-2}, \qquad r = 3, \ldots, g-1 \tag{2.18}$$

with initial conditions

$$F^{(g)\,1} = \mathbb{I}_{g-1}, \qquad F^{(g)\,2} = A_{g-1}. \tag{2.19}$$

2.2 Restriction to fundamental domain

The S matrix definition above includes all branching functions. In particular, equal branching functions with different (r,s,l) labels are distinguished, and vanishing branching functions have to be accounted for. It is more desirable to describe the theory by identifying equal branching functions and disregarding vanishing branching functions. Moreover, this restriction makes it possible to compare the coset construction approach to the usual description of superconformal theories as outlined in the following section.

The restriction is made by a change of basis of branching functions

$$\widetilde{b} = M\, b, \qquad \widetilde{S} = M\, S\, M^{-1}, \tag{2.20}$$

such that the new matrix $\widetilde{S}$ is block diagonal. We can then restrict to one of the blocks as fundamental domain. This affects the fusion coefficients, however. Under the basis change (2.20), fusion coefficients are given by a modified Verlinde formula

$$\widetilde{n}_{ij}^{k} = \sum_n \frac{(\widetilde{S}M)_{in}(\widetilde{S}M)_{jn}(\widetilde{S}M)^{-1}_{nk}}{(M^{-1}\widetilde{S}M)_{1n}}. \tag{2.21}$$

This means that the fusion coefficients are no longer integers, if the new basis is not suitably chosen. A general method to resolve this problem has been given in [17, 9] in the setup of diagonal coset theories. We will follow a different approach, which is more suited to our special case.

We first define a fundamantal domain $E = E_0^+ \cup E_1^- \cup E_f$ of (r,s,l) values. This we do by taking the labels corresponding to nonzero branching functions in the first Kac Table, and half of the corresponding values in the middle Kac Table, including the fixed point. Explicitly, we define

$$\begin{aligned} E_0^\pm &= \left\{(r,s,0)\,|\,(-1)^{r-s} = \pm 1\right\}, \\ E_1^\pm &= \left\{(r,s,1)\,|\,(-1)^{r-s} = \pm 1, s \le (m+1)/2; \right. \\ &\quad \left.(-1)^{r-s} = \pm 1, s = m/2+1, r < m/2\right\}, \\ E_f &= \{(m/2, m/2+1, 1), m \text{ even}\}. \end{aligned} \tag{2.22}$$

Next, we perform a change of basis, where the matrix $\widetilde{S}$ takes block diagonal form, by

$$\begin{aligned} \widetilde{b}^{(l)}_{r,s} &= \frac{1}{2}\left[b^{(l)}_{r,s} + b^{(2-l)}_{m-r,m-s+2}\right] \qquad (r,s,l) \in E_0^\pm \cup E_1^\pm \cup E_f, \\ \widetilde{b}^{(l)}_{r,s} &= \frac{1}{2}\left[b^{(l)}_{r,s} - b^{(2-l)}_{m-r,m-s+2}\right] \qquad \text{otherwise.} \end{aligned} \tag{2.23}$$

From now on, we restrict labels (r,s,l) to the block E. The matrix $\widetilde{S}$ is given by

$$\widetilde{S} = \begin{pmatrix} 2S_{a,b} & S_{a,f} \\ 2S_{f,b} & 0 \end{pmatrix}, \tag{2.24}$$

where $a, b \in E_0^+ \cup E_1^-$ and $f \in E_f$. It can be checked that all fusion coefficients $\widetilde{n}_{ij}^k$ are integral, and that they can be computed using the (usual) Verlinde formula for $\widetilde{S}$, restricted to the fundamental domain E.

2.3 Superconformal data

The unitary heighest weight representations of the $N = 1$ superconformal algebra have central charge (2.2) and conformal dimensions

$$\Delta_{r,s} = \frac{[(m+2)r - ms]^2 - 4}{8m(m+2)} + \frac{1}{32}\left[1 - (-1)^{r-s}\right], \tag{2.25}$$

where $1 \le r \le m-1$ and $1 \le s \le m+1$. The cases $r - s$ even or odd correspond to the Neveu-Schwarz and to the Ramond sector, respectively. For $m = 3$ and $m = 4$, the Kac table of conformal dimensions are

$m = 3$ (rows r, columns s):

$\frac{7}{16}$	$\frac{1}{10}$	$\frac{3}{80}$	0
0	$\frac{3}{80}$	$\frac{1}{10}$	$\frac{7}{16}$

$m = 4$ (rows r, columns s):

1	$\frac{9}{16}$	$\frac{1}{6}$	$\frac{1}{16}$	0
$\frac{3}{8}$	$\frac{1}{16}$	$\frac{1}{24}$	$\frac{1}{16}$	$\frac{3}{8}$
0	$\frac{1}{16}$	$\frac{1}{6}$	$\frac{9}{16}$	1

Note that these tables may be obtained by combining the appropriate coset Kac tables. In the Neveu Schwarz sector $r-s$ even, this amounts to identifying fields corresponding to superpartners.
The superconformal characters are given by [12]

$$\begin{aligned}
\chi^{NS}_{r,s}(q) &= q^{-c/24}\prod_{n=1}^{\infty}\frac{1+q^{n-1/2}}{1-q^n}\sum_{n=-\infty}^{\infty}\left(q^{\gamma_{r,s}(n)}-q^{\gamma_{-r,s}(n)}\right) \\
\chi^{R}_{r,s}(q) &= q^{-c/24+1/16}\prod_{n=1}^{\infty}\frac{1+q^{n}}{1-q^n}\sum_{n=-\infty}^{\infty}\left(q^{\gamma_{r,s}(n)}-q^{\gamma_{-r,s}(n)}\right) \\
\widetilde{\chi}^{NS}_{r,s}(q) &= q^{-c/24}\prod_{n=1}^{\infty}\frac{1-q^{n-1/2}}{1-q^n}\sum_{n=-\infty}^{\infty}(-1)^{mn}\left[q^{\gamma_{r,s}(n)}-(-1)^{rs}q^{\gamma_{-r,s}(n)}\right],
\end{aligned} \tag{2.26}$$

where

$$\gamma_{r,s}(n)=\frac{\left[2m(m+2)n-r(m+2)+sm\right]^2-4}{8m(m+2)}. \tag{2.27}$$

The branching coefficients can be expressed in terms of the superconformal characters as

$$\begin{aligned}
b^{(0)}_{r,s}(q) &= \frac{1}{2}\left[\chi^{NS}_{r,s}(q)+(-1)^{\frac{r-s}{2}}\widetilde{\chi}^{NS}_{r,s}(q)\right], \qquad r-s \text{ even}, \\
b^{(1)}_{r,s}(q) &= \chi^{R}_{r,s}(q), \qquad r-s \text{ odd}, \\
b^{(2)}_{r,s}(q) &= \frac{1}{2}\left[\chi^{NS}_{r,s}(q)-(-1)^{\frac{r-s}{2}}\widetilde{\chi}^{NS}_{r,s}(q)\right], \qquad r-s \text{ even}.
\end{aligned} \tag{2.28}$$

This relation is invertible and gives the superconformal characters in terms of the branching functions as

$$\begin{aligned}
\chi^{NS}_{r,s}(q) &= b^{(0)}_{r,s}(q)+b^{(2)}_{r,s}(q), \qquad r-s \text{ even}, \\
\chi^{R}_{r,s}(q) &= b^{(1)}_{r,s}(q), \qquad r-s \text{ odd}, \\
\widetilde{\chi}^{NS}_{r,s}(q) &= (-1)^{(r-s)/2}\left(b^{(0)}_{r,s}(q)-b^{(2)}_{r,s}(q)\right), \qquad r-s \text{ even}.
\end{aligned} \tag{2.29}$$

For m odd, the S matrix is given by [12]

$$S=\begin{pmatrix} S^{[NS,NS]} & 0 & 0 \\ 0 & 0 & S^{[\widetilde{NS},R]}\sqrt{2} \\ 0 & S^{[R,\widetilde{NS}]}/\sqrt{2} & 0 \end{pmatrix}, \tag{2.30}$$

with matrix elements

$$\begin{aligned}
S^{[NS,NS]}_{(rs),(r's')} &= \frac{4}{\sqrt{m(m+2)}}\sin\frac{\pi rr'}{m}\sin\frac{\pi ss'}{m+2}, \\
S^{[\widetilde{NS},R]}_{(rs),(r's')} &= (-1)^{(r-s)/2}\frac{4}{\sqrt{m(m+2)}}\sin\frac{\pi rr'}{m}\sin\frac{\pi ss'}{m+2}, \\
S^{[R,\widetilde{NS}]}_{(rs),(r's')} &= (-1)^{(r'-s')/2}\frac{4}{\sqrt{m(m+2)}}\sin\frac{\pi rr'}{m}\sin\frac{\pi ss'}{m+2},
\end{aligned} \tag{2.31}$$

For m even, we have to divide $S^{[\widetilde{NS},R]}_{(rs),(r's')}$ by a factor of two if $(r's') = (m/2, m/2+1)$. For the definition of the superconformal S-matrix, we restrict the values of the conformal labels (r,s) to the following fundamental domain

$$\begin{aligned} E_{\widetilde{NS}} = E_{NS} &= \{(r,s)\,|\, r-s \mod 4 = 0\}\,, \\ E_R &= \{(r,s)\,|\, r-s \text{ odd}, s \leq (m+1)/2; \\ & \qquad r-s \text{ odd}, s = m/2+1, r \leq m/2\}\,. \end{aligned} \tag{2.32}$$

Here, we do not adopt the choice of [12], since it is incomplete if m is even. For m odd, both choices are equivalent. The S matrix is real and satisfies $S^2 = \mathbb{I}$. The above expression can be obtained from the S matrix corresponding to the coset construction by performing the basis transformation (2.28).

The fusion coefficients are given by the generalized Verlinde formula [15]

$$\begin{aligned} n^{NS_k}_{NS_i,NS_j} &= \sum_{l\in E_{NS}} \frac{S^{[NS,NS]}_{il} S^{[NS,NS]}_{jl} (S^{[NS,NS]})^{-1}_{kl}}{S^{[NS,NS]}_{0l}} \\ n^{\widetilde{NS}_k}_{\widetilde{NS}_i,\widetilde{NS}_j} &= \sum_{l\in E_R} \frac{S^{[\widetilde{NS},R]}_{il} S^{[\widetilde{NS},R]}_{jl} (S^{[\widetilde{NS},R]})^{-1}_{kl}}{S^{[\widetilde{NS},R]}_{0l}} \\ n^{NS_k}_{R_i,R_j} &= 2\sum_{l\in E_{NS}} \frac{S^{[R,\widetilde{NS}]}_{il} S^{[R,\widetilde{NS}]}_{jl} (S^{[NS,NS]})^{-1}_{kl}}{S^{[NS,NS]}_{0l}} \\ n^{R_k}_{NS_i,R_j} &= \sum_{l\in E_{NS}} \frac{S^{[NS,NS]}_{il} S^{[R,\widetilde{NS}]}_{jl} (S^{[R,\widetilde{NS}]})^{-1}_{kl}}{S^{[NS,NS]}_{0l}} \end{aligned} \tag{2.33}$$

and likewise for the other allowed combinations of sectors. It can be checked that the fusion coefficients are integers. We emphasize that these formulae can be obtained from the fusion coefficients of the coset construction by performing a change of basis according to (2.21). The transformation has to be performed on the heighest weight coset fields and superconformal fields, who are related via

$$\begin{aligned} \Phi^{NS}_{r,s} &= (\Phi^{(0)}_{r,s} + \Phi^{(2)}_{r,s})/2, \qquad r-s \text{ even}, \\ \Phi^{R}_{r,s} &= \Phi^{(1)}_{r,s}, \qquad r-s \text{ odd}, \\ \Phi^{\widetilde{NS}}_{r,s} &= (-1)^{(r-s)/2}\left(\Phi^{(0)}_{r,s} - \Phi^{(2)}_{r,s}\right)/2, \qquad r-s \text{ even}. \end{aligned} \tag{2.34}$$

Note that the matrix of basis change is different from the corresponding one for the branching functions and superconformal characters (2.29).

According to [5], the modular invariant torus partition function for the (A_{m-1}, A_{m+1}) models is given by

$$Z(q) = \sum_{r-s\, even} \left(|\chi^{NS}_{r,s}(q)|^2 + |\widetilde{\chi}^{NS}_{r,s}(q)|^2\right) + \sum_{r-s\, odd} |\chi^R_{r,s}(q)|^2. \tag{2.35}$$

This expression coincides with (2.13), which is readily seen from the relation (2.28) between the branching functions and the superconformal characters. We claim that the cylinder partition functions are given by the generalized Verlinde formula

$$Z_{i|j} = \sum_k n_{i,j}^k \chi_k(q). \tag{2.36}$$

The fusion coefficients $n_{i,j}^k$ are given by the generalized Verlinde formula as discussed above. Here, the indices $i, j, k = (r, s)$ range over the fundamental domain (2.32).

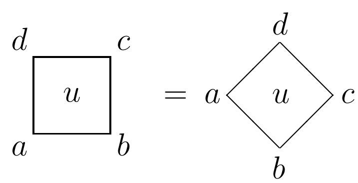
3 Lattice realization

In this section we discuss lattice realizations of the coset and superconformal theories on a cylinder. We cite explicit expressions for the face weights of A-type lattice models [6] at arbitrary fusion level (p, q). We explain how to construct integrable boundary weights using the fusion principle and define double-row transfer matrices, generalizing the methods in [1, 3]. This is then specified to the case $p = q = 2$, which corresponds to the superconformal theories. We explain how the conformal data connect to the eigenvalues of the double-row transfer matrices.

3.1 Face weights and boundary weights

We consider the critical A-type lattice models at fusion level (p, q). We denote the face weights by

$$W^{p,q}\left(\begin{array}{cc} d & c \\ a & b \end{array}\bigg|\, u\right) = \tag{3.1}$$



The values of adjacent spins are constrained by the fused adjacency conditions. Specifically, nonzero weights only occur for spins a, b, c, d satisfying the adjacency condition $F^p_{ab}F^q_{bc}F^p_{cd}F^q_{da} = 1$, where the adjacency matrices F^r are defined in (2.17). We introduce the notation

$$[u]_m = \prod_{k=0}^{m-1} \sin(u - k\lambda), \qquad \left[\begin{array}{c} u \\ m \end{array}\right] = \prod_{k=0}^{m-1} \frac{\sin(u - k\lambda)}{\sin(m\lambda - k\lambda)}. \tag{3.2}$$

The non-vanishing weights have been given in explicit form in [6]. In either of the four cases $|a-b|=p$ or $|b-c|=q$ or $|c-d|=p$ or $|d-a|=q$ they have the factorized form

$$W^{p,q}\left(\begin{matrix} a & a+2s-p \\ a+2r-q & a+2s-p+q \end{matrix}\middle| u\right) = \frac{\begin{bmatrix}(p-s)\lambda \\ q-r\end{bmatrix}\begin{bmatrix}(a-s+r-p-1)\lambda \\ r\end{bmatrix}\begin{bmatrix}s\lambda-u \\ r\end{bmatrix}\begin{bmatrix}(a+s)\lambda-u \\ q-r\end{bmatrix}}{\begin{bmatrix}(a+r)\lambda \\ q-r\end{bmatrix}\begin{bmatrix}(a+2r-q-1)\lambda \\ r\end{bmatrix}} \tag{3.3}$$

$$W^{p,q}\left(\begin{matrix} a & a+2s-p \\ a+2r-q & a+2s-p-q \end{matrix}\middle| u\right) = \frac{\begin{bmatrix}s\lambda \\ r\end{bmatrix}\begin{bmatrix}(a+s)\lambda \\ q-r\end{bmatrix}\begin{bmatrix}(p-s)\lambda-u \\ q-r\end{bmatrix}\begin{bmatrix}(a+s-p+r-1)\lambda+u \\ r\end{bmatrix}}{\begin{bmatrix}(a+r)\lambda \\ q-r\end{bmatrix}\begin{bmatrix}(a+2r-q-1)\lambda \\ r\end{bmatrix}} \tag{3.4}$$

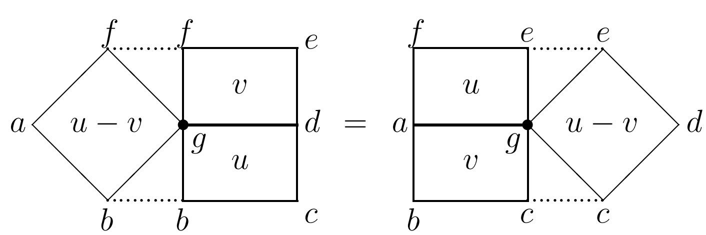
Here, $\lambda=\pi/g$ is the spectral parameter, and g is the Coxeter number of A_{g-1}. Using these weights, the remaining ones are given by

$$W^{p,q}\left(\begin{matrix} a & b \\ a+2r-q & b+2s-q \end{matrix}\middle| u\right)\begin{bmatrix}q\lambda \\ s\end{bmatrix} = \sum_{j=\max(0,r+s-q)}^{\min(r,s)} W^{p,s}\left(\begin{matrix} a & b \\ a+2j-s & b+s \end{matrix}\middle| u+(q-s)\lambda\right) W^{p,q-s}\left(\begin{matrix} a+2j-s & b+s \\ a+2r-q & b+2s-q \end{matrix}\middle| u\right) \tag{3.5}$$

These face weights satisfy the fused Yang-Baxter equation

$$\begin{aligned}\sum_g W^{r,q}\left(\begin{matrix} f & g \\ a & b \end{matrix}\middle| u-v\right) W^{p,q}\left(\begin{matrix} g & d \\ b & c \end{matrix}\middle| u\right) W^{p,r}\left(\begin{matrix} f & e \\ g & d \end{matrix}\middle| v\right) \\ =\sum_g W^{p,r}\left(\begin{matrix} a & g \\ b & c \end{matrix}\middle| v\right) W^{p,q}\left(\begin{matrix} f & e \\ a & g \end{matrix}\middle| u\right) W^{r,q}\left(\begin{matrix} e & d \\ g & c \end{matrix}\middle| u-v\right)\end{aligned} \tag{3.6}$$

This can be expressed graphically by

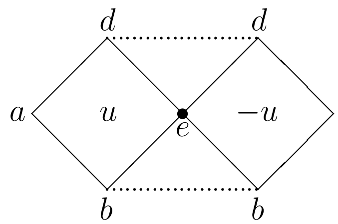


(3.7)

The face weights satisfy the fused inversion relation

$$\sum_e W^{q,r}\left(\begin{array}{cc} d & e \\ a & b \end{array}\middle|\, u\right) W^{r,q}\left(\begin{array}{cc} d & c \\ e & b \end{array}\middle|\, -u\right) = \rho^{qr}(u)\rho^{rq}(-u)\delta_{ac}F^q_{ab}F^r_{ad}, \tag{3.8}$$

where $\rho^{qr}(u)$ are model dependent functions. For us, only the case $p = q = 2$ will be of interest. In this case, the functions can be disregarded because they are common factors. This will simplify the reflection equations, see below. This can be graphically described by



$$= \rho^{qr}(u)\rho^{rq}(-u)\delta_{ac}F^q_{ab}F^r_{ad}. \tag{3.9}$$

For the definition of boundary weights we will need the braid limit of the above bulk weights. The braid limit of the (p, q) weights is defined as

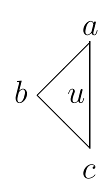
$$W^{pq}\left(\begin{array}{cc} d & c \\ a & b \end{array}\right) = \lim_{u\to -i\infty} \frac{1}{\sin^q u} W^{p,q}\left(\begin{array}{cc} d & c \\ a & b \end{array}\middle|\, u\right). \tag{3.10}$$

The braid limit may be obtained from the above weights by the substitution

$$\lim_{u\to -i\infty} \frac{1}{\sin^m u}\left[\begin{array}{c} a \pm u \\ m \end{array}\right] = \frac{(\pm 1)^m}{[m\lambda]_m} e^{\pm ima \mp im(m-1)\lambda/2}. \tag{3.11}$$

We now explain how to define boundary weights, which will realize the different types of boundary conditions corresponding to the coset description and to the superconformal description. We explain the general method to obtain integrable boundary weights from known boundary weights, using fused face weights, as discussed in [2]. We start with a simple initial boundary condition

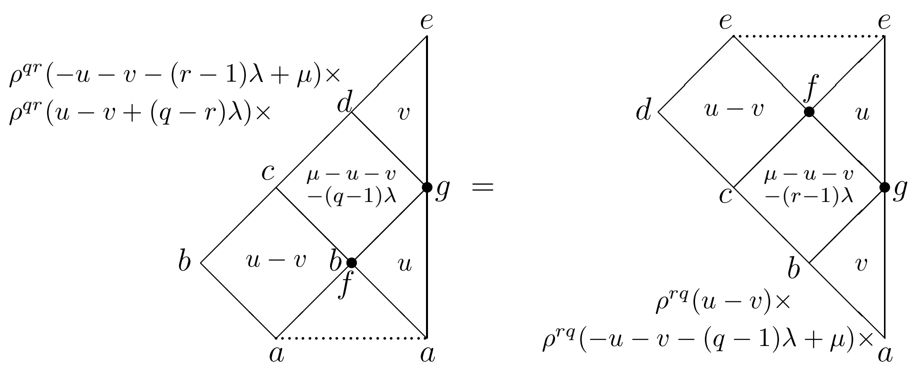
$$B^q_0\left(b \begin{array}{c} a \\ c \end{array}\middle|\, u\right) = \tag{3.12}$$



with weights satisfying the right reflection equation

$$
\begin{aligned}
&\rho^{qr}(u-v+(q-r)\lambda)\rho^{qr}(-u-v-(r-1)\lambda+\mu) \\
&\times \sum_{fg} W^{r,q}\left(\begin{array}{cc} c & f \\ b & a \end{array}\middle|\, u-v\right) W^{r,q}\left(\begin{array}{cc} d & g \\ c & f \end{array}\middle|\, -u-v-(q-1)\lambda+\mu\right) \\
&\times B_0^q\left(f \begin{array}{c} g \\ a \end{array}\middle|\, u\right) B_0^r\left(d \begin{array}{c} e \\ g \end{array}\middle|\, u\right) \\
&= \rho^{rq}(u-v)\rho^{rq}(-u-v-(q-1)\lambda+\mu) \\
&\times \sum_{fg} W^{q,r}\left(\begin{array}{cc} e & f \\ d & c \end{array}\middle|\, u-v+(q-r)\lambda\right) W^{q,r}\left(\begin{array}{cc} f & g \\ c & b \end{array}\middle|\, -u-v-(r-1)\lambda+\mu\right) \\
&\times B_0^q\left(f \begin{array}{c} e \\ g \end{array}\middle|\, u\right) B_0^r\left(b \begin{array}{c} g \\ a \end{array}\middle|\, u\right),
\end{aligned} \tag{3.13}
$$

where $\rho^{qr}(u)$ are model dependent functions. This is depicted grahically below.

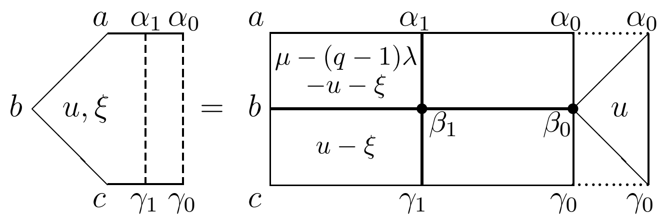


(3.14)

We obtain further integrable boundary weights satisfying the boundary Yang-Baxter equation by applying s-type fusion with the braid bulk weights and r-type fusion with the face weights [3]. In order to be able to perform the fusion construction, we introduce dangling variables for each type of fusion. This leads to boundary weights of the form

$$
\begin{aligned}
&B^{q,(rs)}\left(b \begin{array}{ccc} a & \alpha_1 & \alpha_0 \\ c & \gamma_1 & \gamma_0 \end{array}\middle|\, u,\xi\right) = \\
&\sum_{\beta_0,\beta_1} W^{r-1,q}\left(\begin{array}{cc} b & \beta_1 \\ c & \gamma_1 \end{array}\middle|\, u-\xi\right) W^{r-1,q}\left(\begin{array}{cc} a & \alpha_1 \\ b & \beta_1 \end{array}\middle|\, \mu-(q-1)\lambda-u-\xi\right) \\
&W^{s-1,q}\left(\begin{array}{cc} \beta_1 & \beta_0 \\ \gamma_1 & \gamma_0 \end{array}\right) W^{s-1,q}\left(\begin{array}{cc} \alpha_1 & \alpha_0 \\ \beta_1 & \beta_0 \end{array}\right) B_0^q\left(\beta_0 \begin{array}{c} \alpha_0 \\ \gamma_0 \end{array}\middle|\, u\right),
\end{aligned} \tag{3.15}
$$

graphically depicted by



(3.16)

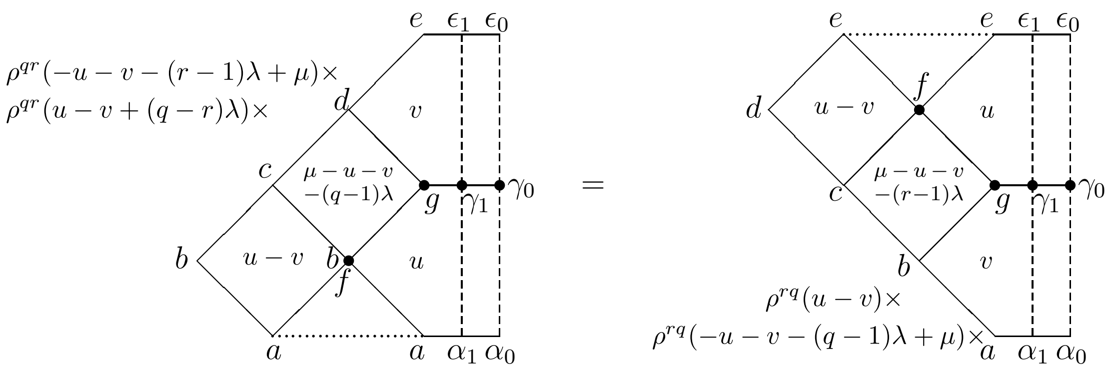
Here, μ and ξ are arbitrary fixed parameters. The parameter μ is the crossing parameter (see (3.28)), which we fix to be $\mu = 2\lambda$ in our numerical calculations. The value of the inhomgeneity parameter ξ will later be chosen such that the corresponding boundary weights take simple form and are conformally invariant.

The above construction generally introduces dangling variables for the boundary weight. In some cases, however, the dependence on these variables disappears. This is, for example, the case for boundary conditions corresponding to the unitary minimal models. Here, the (r,s)-type boundary conditions corresponding to the Virasoro characters of type (r,s) are obtained by starting with a simple "vacuum" solution with boundary spins $a = c = 1$. Applying fusion $s-1$ times with face weights in the braid limit gives integrable, $(1,s)$-type boundary weights. Since the spin variable of the vacuum weight has only one value, the new boundary weight is not dependent on the value of this internal spin. Again, the new $(1,s)$-type weights are diagonal and have only one spin value $a = c = s$. Repeated $r-1$ times fusion with the full weights leads to the (r,s)-type boundary conditions. These are again, by construction, independent of the dangling variable s.

As we will discuss in the next paragraph, above construction yields, for suitable choices of the starting weights, boundary weights corresponding to each branching function and each superconformal character. In contrast to the boundary weights for the unitary minimal models, these boundary weights will however generally depend on internal dangling spins. The weights satisfy a generalized fused right reflection equation

$$\begin{aligned}
&\rho^{qr}(u-v+(q-r)\lambda)\rho^{qr}(-u-v-(r-1)\lambda+\mu) \\
&\times \sum_{fg\gamma_0\gamma_1} W^{r,q}\left(\begin{array}{cc} c & f \\ b & a \end{array}\middle| u-v\right) W^{r,q}\left(\begin{array}{cc} d & g \\ c & f \end{array}\middle| -u-v-(q-1)\lambda+\mu\right) \\
&\times B^{q}\left(f\begin{array}{ccc} g & \gamma_1 & \gamma_0 \\ a & \alpha_1 & \alpha_0 \end{array}\middle| u\right) B^{r}\left(d\begin{array}{ccc} e & \epsilon_1 & \epsilon_0 \\ g & \gamma_1 & \gamma_0 \end{array}\middle| u\right) \\
&= \rho^{rq}(u-v)\rho^{rq}(-u-v-(q-1)\lambda+\mu) \\
&\times \sum_{fg\gamma_0\gamma_1} W^{q,r}\left(\begin{array}{cc} e & f \\ d & c \end{array}\middle| u-v+(q-r)\lambda\right) W^{q,r}\left(\begin{array}{cc} f & g \\ c & b \end{array}\middle| -u-v-(r-1)\lambda+\mu\right) \\
&\times B^{q}\left(f\begin{array}{ccc} e & \epsilon_1 & \epsilon_0 \\ g & \gamma_1 & \gamma_0 \end{array}\middle| u\right) B^{r}\left(b\begin{array}{ccc} g & \gamma_1 & \gamma_0 \\ a & \alpha_1 & \alpha_0 \end{array}\middle| u\right),
\end{aligned} \tag{3.17}$$

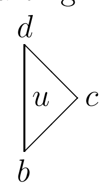
where $\rho^{qr}(u)$ are model dependent functions.



(3.18)

Similarly, we define left boundary weights, starting from a boundary weight

$$B_0^q\left(\begin{array}{cc} a & \\ & b \\ c & \end{array}\middle| u\right) =$$

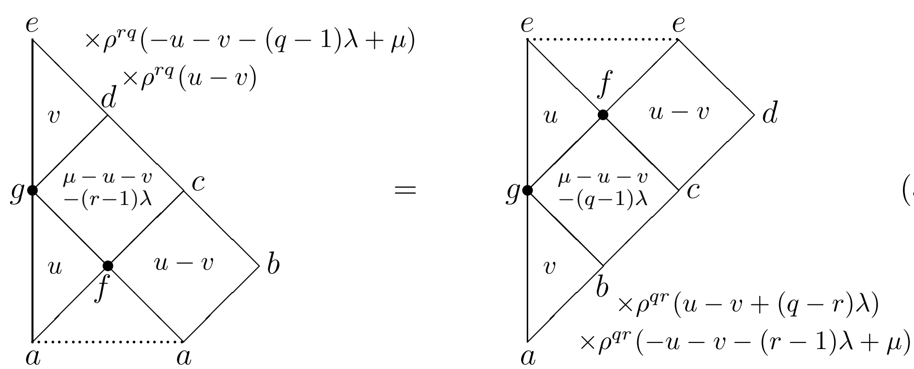


(3.19)

satisfying the left reflection equation

$$\begin{aligned}
&\rho^{rq}(u-v)\rho^{rq}(-u-v-(q-1)\lambda+\mu)\\
&\times\sum_{fg} W^{q,r}\left(\begin{array}{cc} c & b \\ f & a \end{array}\middle| u-v+(q-r)\lambda\right) W^{q,r}\left(\begin{array}{cc} d & c \\ g & f \end{array}\middle| -u-v-(r-1)\lambda+\mu\right)\\
&\times B_0^q\left(\begin{array}{cc} g & \\ & f \\ a & \end{array}\middle| u\right) B_0^r\left(\begin{array}{cc} e & \\ & d \\ g & \end{array}\middle| u\right)\\
&=\rho^{qr}(u-v+(q-r)\lambda)\rho^{qr}(-u-v-(r-1)\lambda+\mu)\\
&\times\sum_{fg} W^{r,q}\left(\begin{array}{cc} e & d \\ f & c \end{array}\middle| u-v\right) W^{r,q}\left(\begin{array}{cc} f & c \\ g & b \end{array}\middle| -u-v-(q-1)\lambda+\mu\right)\\
&\times B_0^q\left(\begin{array}{cc} e & \\ & f \\ g & \end{array}\middle| u\right) B_0^r\left(\begin{array}{cc} g & \\ & b \\ a & \end{array}\middle| u\right).
\end{aligned} \tag{3.20}$$

This is depicted graphically below.

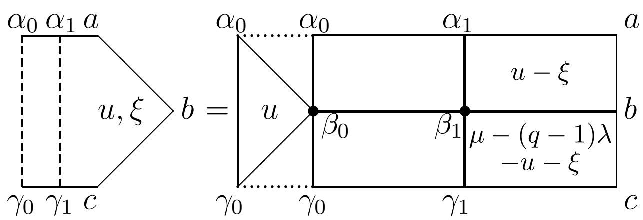


(3.21)

We obtain integrable boundary weights by subsequently applying s-type fusion and r-type fusion. The new left boundary weights are given explicitly by

$$
\begin{aligned}
&B^{q,(rs)}\left(\begin{array}{ccc|c} \alpha_0 & \alpha_1 & a & \\ \gamma_0 & \gamma_1 & c & \end{array}\!\!\! b \,\middle|\, u,\xi\right) = \\
&\sum_{\beta_0,\beta_1} B_0^q\left(\begin{array}{c} \alpha_0 \\ \gamma_0 \end{array} \beta_0 \,\middle|\, u\right) W^{s-1,q}\left(\begin{array}{cc} \beta_0 & \beta_1 \\ \gamma_0 & \gamma_1 \end{array}\right) W^{s-1,q}\left(\begin{array}{cc} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \end{array}\right) \\
&W^{r-1,q}\left(\begin{array}{cc} \beta_1 & b \\ \gamma_1 & c \end{array} \,\middle|\, \mu-(q-1)\lambda-u-\xi\right) W^{r-1,q}\left(\begin{array}{cc} \alpha_1 & a \\ \beta_1 & b \end{array} \,\middle|\, u-\xi\right),
\end{aligned}
\tag{3.22}
$$

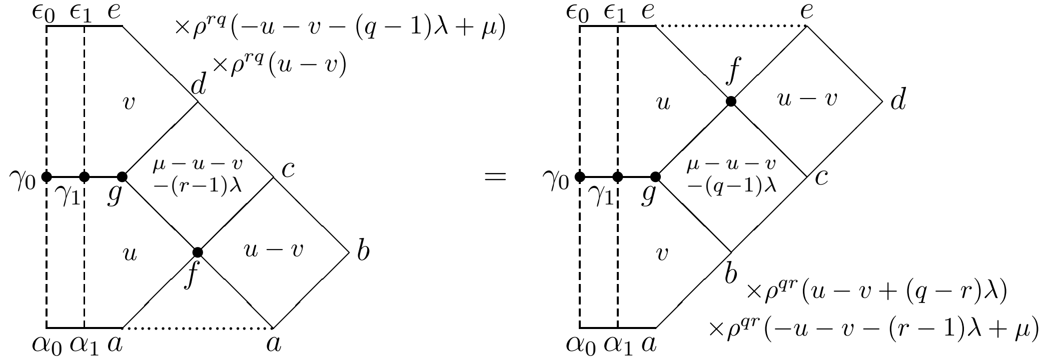
depicted graphically as



(3.23)

These boundary weights satisfy a generalized fused left reflection equation

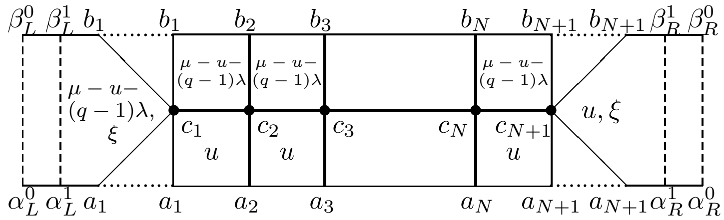
$$
\begin{aligned}
&\rho^{rq}(u-v)\rho^{rq}(-u-v-(q-1)\lambda+\mu) \\
&\times \sum_{fg\gamma_0\gamma_1} W^{q,r}\left(\begin{array}{cc} c & b \\ f & a \end{array} \,\middle|\, u-v+(q-r)\lambda\right) W^{q,r}\left(\begin{array}{cc} d & c \\ g & f \end{array} \,\middle|\, -u-v-(r-1)\lambda+\mu\right) \\
&\times B^q\left(\begin{array}{ccc} \gamma_0 & \gamma_1 & g \\ \alpha_0 & \alpha_1 & a \end{array} f \,\middle|\, u\right) B^r\left(\begin{array}{ccc} \epsilon_0 & \epsilon_1 & e \\ \gamma_0 & \gamma_1 & g \end{array} d \,\middle|\, u\right) \\
&= \rho^{qr}(u-v+(q-r)\lambda)\rho^{qr}(-u-v-(r-1)\lambda+\mu) \\
&\times \sum_{fg\gamma_0\gamma_1} W^{r,q}\left(\begin{array}{cc} e & d \\ f & c \end{array} \,\middle|\, u-v\right) W^{r,q}\left(\begin{array}{cc} f & c \\ g & b \end{array} \,\middle|\, -u-v-(q-1)\lambda+\mu\right) \\
&\times B^q\left(\begin{array}{ccc} \epsilon_1 & \epsilon_0 & e \\ \gamma_1 & \gamma_0 & g \end{array} f \,\middle|\, u\right) B^r\left(\begin{array}{ccc} \gamma_1 & \gamma_0 & g \\ \alpha_1 & \alpha_0 & a \end{array} b \,\middle|\, u\right).
\end{aligned}
\tag{3.24}
$$



(3.25)

The fused double-row transfer matrices are defined by

$$
\begin{aligned}
&< \alpha_L^0, \alpha_L^1, a_1, \ldots, a_{N+1}, \alpha_R^1, \alpha_R^0 | \mathbf{D}^{pq}(u, \xi_1, \xi_2) | \beta_L^0, \beta_L^1, b_1, \ldots, b_{N+1}, \beta_R^1, \beta_R^0 > \\
&= \sum_{c_1 \ldots c_{N+1}} B^q \left(\begin{array}{ccc} \beta_L^0 & \beta_L^1 & b_1 \\ \alpha_L^0 & \alpha_L^1 & a_1 \end{array} c_1 \middle| -u-(q-1)\lambda+\mu, \xi_1 \right) \\
&\times \left[\prod_{j=1}^{N} W^{p,q} \left(\begin{array}{cc} c_j & c_{j+1} \\ a_j & a_{j+1} \end{array} \middle| u \right) W^{p,q} \left(\begin{array}{cc} b_j & b_{j+1} \\ c_j & c_{j+1} \end{array} \middle| -u-(q-1)\lambda+\mu \right) \right] \\
&\times B^q \left(c_{N+1} \begin{array}{ccc} b_{N+1} & \beta_R^1 & \beta_R^0 \\ a_{N+1} & \alpha_R^1 & \alpha_R^0 \end{array} \middle| u, \xi_2 \right) \qquad (3.26)
\end{aligned}
$$

$=$ 

The fused double-row transfer matrices form a commuting family

$$\mathbf{D}^{pq}(u)\mathbf{D}^{pq}(v) = \mathbf{D}^{pq}(v)\mathbf{D}^{pq}(u). \qquad (3.27)$$

This can be shown by using the fused Yang-Baxter equation (3.6), inversion relation (3.8), and the generalized reflection equations (3.17) and (3.24) in the diagram proof given in [1]. It can also be shown by similar arguments involving boundary crossing equations, which we do not give here, that the fused double-row transfer matrices satisfy crossing symmetry

$$\mathbf{D}^{pq}(u) = \mathbf{D}^{pq}(-u-(q-1)\lambda+\mu). \qquad (3.28)$$

3.2 Finite-size corrections

The properties of the lattice models connect to the data of the associated conformal field theories through the finite-size corrections to the eigenvalues of the double-row transfer matrices. Let us denote the double-row transfer matrix with boundary coset or supersymmetric labels i on the left and j on the right by $\mathbf{D}_{i|j}$. If we write the eigenvalues of $\mathbf{D}_{i|j}$ as

$$D_n(u) = \exp(-E_n(u)), \quad n = 0, 1, 2, \ldots \qquad (3.29)$$

then the finite size corrections take the form

$$E_n(u) = 2Nf(u) + f_{i|j}(u) + \frac{2\pi \sin\theta}{N}\left(-\frac{c}{24} + \Delta_n + k_n\right) + o\left(\frac{1}{N}\right), \qquad k_n \in \mathbb{N}, \quad (3.30)$$

where $f(u)$ is the bulk free energy, $f_{i|j}(u)$ is the boundary free energy, c is the central charge, Δ_n is a conformal weight and the anisotropy angle is given by

$$\theta = gu, \qquad (3.31)$$

where g is the Coxter number of the graph A_{g-1}.

The bulk and boundary free energies can be computed using inversion relations [1, 13]. This we do not do since we are interested only in the conformal partition functions. Removing the bulk and boundary contributions to the partition function on a cylinder leads to the conformal partition function $Z_{i|j}(q)$ with left and right boundaries i and j. This can be expressed as a linear form in characters

$$Z_{i|j} = \sum_k n_{i,j}^k \chi_k(q). \tag{3.32}$$

where the fusion coefficients $n_{ij}^{\ k} \in \mathbb{Z}$ give the operator content, and k has to be summed over an appropriate domain. For the coset models, this is given by (2.14), whereas for the superconformal models this is given by (2.36). With the introduction of two dangling variables per boundary weight, we are effectively dealing with four boundary conditions, such that each eigenvector is fourfold degenerate. For M double rows the modular parameter is

$$q = \exp(2\pi i\tau), \qquad \tau = i\frac{M}{N}\sin\theta, \tag{3.33}$$

where M/N is the aspect ratio of the cylinder.

4 Coset and superconformal boundary weights

In this section, we define the integrable coset boundary weights and integrable superconformal boundary weights. Since it is not obvious from the construction of the weights how to identify the (r,s) labels of the fusion construction with the (r,s) labels in the Kac tables, we have to make this identification from numerical data. In the sequel we focus on right boundary weights. Since the left boundary weights are defined in the same manner, we do not give the corresponding expressions here.

We first give the boundary weights corresponding to the branching functions $b_{r,s}^{(l)}(q)$, which we will denote by $B^{(r,s,l)}\left(b \begin{array}{ccc} a & \alpha_1 & \alpha_0 \\ c & \gamma_1 & \gamma_0 \end{array} \middle| u\right)$. The starting weight is given by

$$B_0^2\left(b \begin{array}{c} a \\ c \end{array} \middle| u\right) = \delta_{a,1}\delta_{c,1}\delta_{b,3}. \tag{4.1}$$

This weight gives the vacuum character of the above models and generalizes the vacuum boundary condition of the unfused A models [3] to fusion level 2. It is the coset vacuum. We use this boundary weight on the left of the double-row transfer matrix. Since the cylinder partition function reduces to a single branching function, it is easy to identify labels of boundary weights on the right with their corresponding Kac labels. It can be checked that the $(r,1)$ weights obtained from the above starting weight correspond to the weights (6.32) in [1].

As it turns out, the different boundary weights for different sectors l correspond to different choices of the inhomogeneity parameter ξ. For the crossing parameter fixed to $\mu = 2\lambda$, we have explicitly

$$\begin{aligned} B^{(r,s,0)}\left(b\ \begin{matrix} a & \alpha_1 & \alpha_0 \\ c & \gamma_1 & \gamma_0 \end{matrix}\ \middle|\ u\right) &= B^{2,(r,s)}\left(b\ \begin{matrix} a & \alpha_1 & \alpha_0 \\ c & \gamma_1 & \gamma_0 \end{matrix}\ \middle|\ u, \xi = 5\lambda/2\right) \\ B^{(r,s,1)}\left(b\ \begin{matrix} a & \alpha_1 & \alpha_0 \\ c & \gamma_1 & \gamma_0 \end{matrix}\ \middle|\ u\right) &= B^{2,(r+1,s)}\left(b\ \begin{matrix} a & \alpha_1 & \alpha_0 \\ c & \gamma_1 & \gamma_0 \end{matrix}\ \middle|\ u, \xi = 3\lambda/2\right) \\ B^{(r,s,2)}\left(b\ \begin{matrix} a & \alpha_1 & \alpha_0 \\ c & \gamma_1 & \gamma_0 \end{matrix}\ \middle|\ u\right) &= B^{2,(r+2,s)}\left(b\ \begin{matrix} a & \alpha_1 & \alpha_0 \\ c & \gamma_1 & \gamma_0 \end{matrix}\ \middle|\ u, \xi = -\lambda\right) \end{aligned} \tag{4.2}$$

As explained above, these weights are do not in fact depend on the dangling variables. This is no longer the case for the boundary weights realizing superconformal boundary conditions, which we now define.

We will denote them by

$$B^{X(r,s)}\left(b\ \begin{matrix} a & \alpha_1 & \alpha_0 \\ c & \gamma_1 & \gamma_0 \end{matrix}\ \middle|\ u\right), \tag{4.3}$$

where $X \in \{NS, \widetilde{NS}, R\}$ stands for the Neveu-Schwarz sector, Neveu-Schwarz tilda sector or for the Ramond sector, respectively. The starting weight in the Neveu-Schwarz sector is given by

$$B_0^{NS}\left(b\ \begin{matrix} a \\ c \end{matrix}\ \middle|\ u\right) = B^{(110)}\left(b\ \begin{matrix} a & 1 & 1 \\ c & 1 & 1 \end{matrix}\ \middle|\ u, -\lambda\right) + B^{(310)}\left(b\ \begin{matrix} c & 1 & 1 \\ a & 1 & 1 \end{matrix}\ \middle|\ u, -\lambda\right) \tag{4.4}$$

This weight is the superconformal vacuum. It satisfies the right reflection equation. This is due to the fact that each summand satisfies the reflection equation by construction, and they are both diagonal with different nonzero spin values. Therefore, the sum in the reflection equation decouples into the two separate reflection equations. At the isotropic point $u = \lambda/2$, the above weight simplifies to

$$B_0^{NS}\left(b\ \begin{matrix} a \\ c \end{matrix}\ \middle|\ \lambda/2\right) = h(\delta_{a,1}\delta_{c,1}\delta_{b,3} + \delta_{a,3}\delta_{c,3}\delta_{b,1}), \tag{4.5}$$

where h is a constant. The starting weight in the Neveu-Schwarz tilda sector is given by

$$B_0^{\widetilde{NS}}\left(b\ \begin{matrix} a \\ c \end{matrix}\ \middle|\ u\right) = B^{(110)}\left(b\ \begin{matrix} a & 1 & 1 \\ c & 1 & 1 \end{matrix}\ \middle|\ u, -\lambda\right) - B^{(31),0}\left(b\ \begin{matrix} a & 1 & 1 \\ c & 1 & 1 \end{matrix}\ \middle|\ u, -\lambda\right) \tag{4.6}$$

The superconformal boundary weights in the Ramond sector are given by

$$B^{R(rs)}\left(b\ \begin{matrix} a & \alpha_1 & \alpha_0 \\ c & \gamma_1 & \gamma_0 \end{matrix}\ \middle|\ u, \xi\right) = B^{NS(r+1s)}\left(b\ \begin{matrix} a & \alpha_1 & \alpha_0 \\ c & \gamma_1 & \gamma_0 \end{matrix}\ \middle|\ u, \xi\right) \tag{4.7}$$

These choices of the superconformal boundary weights correspond precisely to the relation between the branching functions and superconformal characters (2.28). The labels (r, s) which appear are the superconformal labels in the Kac Table.

5 Numerical spectra

Here, we describe our numerical procedure which led to the identification of boundary conditions presented in the previous chapter. We have tested our predictions for the models A_4 and A_5, separately for the coset boundary weights and superconformal boundary weights.

For the coset boundary weights, which do not depend on dangling variables, we were able to compute double row transfer matrices up to 16 faces for A_4 and up to 11 faces for A_5. Due to the introduction of dangling variables, double-row transfer matrices of superconformal boundary weights generally can only be computed for much smaller lattice sizes, typically up to 5 faces for A_5. For $(r, 1)$ or $(1, s)$ type superconformal boundary weights, however, the situation can be improved, since the dependence on one dangling variable is trivial and may be disregarded.

The A_4 model, which has central charge $c = 7/10$, can be related to the tricritical hard square and tricritical Ising model. It can be alternatively realized as a unitary minimal model from the (unfused) *A*-*D*-*E* lattice model A_4. The corresponding conformal boundary conditions have been given previously in [3]. The coset boundary conditions agree with the conformal boundary conditions. This is related to the fact that, for this model, the branching functions are just the Virasoro characters of the model $\mathcal{M}(7/10)$.

The predictions from conformal field theory manifest themselves in the level spacings and degeneracies of the double-row transfer matrix eigenvalues in the large N limit, cf. (3.30). We have chosen $u = \lambda/2$ such that the sine factor reduces to unity. For the double-row transfer matrix at fusion level $(2, 2)$, which is the case of interest for our numerics, we achieved this by choosing the isotropic point $u_c = (\mu - \lambda)/2$, in which case $\mathbf{D}^{22}(u) = \mathbf{D}^{22}(\mu - \lambda - u)$, and setting $\mu = 2\lambda$.

First, we have computed double-row transfer matrices with the vacuum weight on the left and a general boundary weight on the right. In this case, the cylinder partition function reduces to a single character, according to the fusion rules (3.32). In order to check for conformal dimension from given transfer matrix data, we computed reduced energies by subtracting the contributions from the bulk free energy, from the boundary free energy and from the central charge according to (3.30). We then plotted the largest reduced eigenvalue of the transfer matrix against $1/N$ and extrapolated the sequence of numbers to $N = \infty$. In all cases, we obtained agreement with the theoretical value of Δ within numerical accuracy.

The same method has been applied in order to test the exponents and degeneracies of the eigenvalues of the double-row transfer matrix, which are given by the expansion of the characters in powers of q in the large N limit. As example, we extract the superconformal vacuum character for A_4. It has a series expansion

$$\chi_{1,1}^{NS}(q) = q^{-7/240}(1 + q^{\frac{3}{2}} + q^2 + q^{\frac{5}{2}} + q^3 + 2q^{\frac{7}{2}} + 2q^4 + 2q^{\frac{9}{2}} + \mathcal{O}(q^5)) \qquad (5.1)$$

We have computed the double-row transfer matrix with superconformal vacuum weights on the left and on the right up to 15 faces. (Note that the dependence of the boundary weight on dangling variables is trivial.) A polynomial extrapolation of the first ten reduced eigenvalues from lattice sizes 10 to 15 to $N=\infty$ yields the exponents shown in the table.

energy	1	2	3	4	5	6	7	8	9	10
data	0.000	1.495	1.994	2.492	2.993	3.489	3.504	3.990	4.077	4.501
theory	0	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	$\frac{7}{2}$	4	4	$\frac{9}{2}$

In order to test the predictions for fusion rules, we put different boundary weights to the right and to the left and tested for the correct cylinder partition function by examining the first ten eigenvectors of the double-row transfer matrix. In each case tested, we find agreement between theory and prediction within numerical accuracy. We discuss a typical example of the coset theory: The cylinder partition function of the A_5 model with left and right boundary weights of type $(2,3,1)$ is given by (2.14)

$$
\begin{aligned}
Z_{(2,3,1)|(2,3,1)} &= 2\left(b^{(0)}_{1,1}+b^{(0)}_{1,3}+b^{(0)}_{1,5}+b^{(0)}_{3,1}+b^{(0)}_{3,3}+b^{(0)}_{3,5}\right) &(5.2)\\
&= q^{-1/24}(2+2q^{\frac{1}{6}}+2q^{\frac{2}{3}}+2q+2q^{\frac{7}{6}}+4q^{\frac{3}{2}}+\mathcal{O}(q^{\frac{5}{3}})). &(5.3)
\end{aligned}
$$

We have computed the double-row transfer matrix of this model up to 9 faces. A polynomial extrapolation of the first ten reduced eigenvalues from lattice sizes 4 to 9 to $N=\infty$ yields the exponents shown in the table.

energy	1	2	3	4	5	6	7	8	9	10
data	0.000	0.000	0.168	0.168	0.675	0.675	1.012	1.012	1.183	1.183
theory	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$	1	1	$\frac{7}{6}$	$\frac{7}{6}$

6 Conclusion

We have discussed $N=1$ superconformal theories on the torus and on the cylinder and derived a generalized Verlinde formula for the fusion coefficients. For the diagonal theories classified by (A,A) graphs, we have given a lattice realization of the corresponding superconformal boundary conditions. This can be used to study superconformal bulk and boundary flows via TBA [16].

Using the methods introduced here, the non-diagonal theories can be investigated as well. The corresponding (A,G)-type theories, where G is of A-D-E -type, may be obtained by constructiong the integrable A-D-E -lattice models at fusion level $(2,2)$, together with their superconformal boundary conditions. Whereas this is a straightforward generalization of the methods presented here (see also [3, 13]), it is not obvious how to obtain lattice realizations of the (D,A) and (D,E) theories.

Focusing on the coset construction, we have given a complete set of coset boundary conditions in the simplest case of the level two $\widehat{s\ell}(2)$ coset models corresponding to the $N=1$ superconformal theories. The above methods can be used to obtain integrable and conformal boundary conditions for the coset models at fusion level higher than two [11] by an obvious generalization. Our claim is that the corresponding coset boundary weights give a complete realization of coset boundary conditions.

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