

# Critical $Q = 1$ Potts Model and Temperley-Lieb Stochastic Processes

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We consider the groundstate wave function and spectra of the  $L$ -site XXZ  $U_q[sl(2)]$  invariant quantum spin chain with anisotropy parameter  $\Delta = 1/2$ . This chain is related to the critical  $Q = 1$  Potts model and exhibits  $c = 0$  conformal invariance. We show that the problem is related to Hamiltonians describing one-dimensional stochastic processes defined on a Temperley-Lieb algebra. The bra groundstate wave function is trivial and the ket groundstate wave function gives the probability distribution of the stationary state. The stochastic processes can be understood as interface RSOS growth models with nonlocal rates. Allowing defects which can hop on the interface one obtains stochastic models having the same stationary state and spectra (but not degeneracies) as the XXZ chain.

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## I. INTRODUCTION

It is known that the  $Q = 1$  Potts model [1] is related to other models in statistical mechanics such as percolation [2] and the  $O(1)$  loop model [3]. It is only recently realized, however, that at criticality it is related to other topics in physics and mathematics. Consider the ferromagnetic quantum  $U_q[sl(2)]$  invariant XXZ Hamiltonian [4]

$$H_{\text{XXZ}} = \sum_{j=1}^{L-1} (1 - e_j) \quad (1.1)$$

$$e_j = \frac{1}{4} - \frac{1}{2} \left[ \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right) + h \left( \sigma_j^z - \sigma_{j+1}^z \right) \right] \quad (1.2)$$

where  $\sigma^x, \sigma^y, \sigma^z$  are Pauli matrices,  $q = e^{\pi i/3}$  and

$$2\Delta = q + q^{-1} = \sqrt{Q} = 1, \quad h = \frac{1}{2}(q - q^{-1}) = i\frac{\sqrt{3}}{2} \quad (1.3)$$

The value  $\Delta = 1/2$  relates to the critical  $Q = 1$  Potts model. The  $e_j$  are the generators of a Temperley-Lieb (TL) algebra

$$e_j^2 = e_j, \quad e_j e_{j\pm 1} e_j = e_j, \quad [e_i, e_j] = 0, \quad |i - j| > 1. \quad (1.4)$$

The groundstate energy is zero, that is  $H_{\text{XXZ}}|0\rangle = 0|0\rangle$ , for any number of sites as can be seen by taking the quotient  $e_j = 1$  of the TL algebra.

As pointed out by Read and Saleur [5] the spectrum of the Hamiltonian (1.1) is also related, in the continuum

limit, to the spectra of nonlinear sigma models defined on coset supermanifolds. Such models are relevant in the integer quantum Hall effect [6] and problems with quenched disorder [7]. The spectra of all of these models are given by a generalization of the  $c = 0$  Virasoro algebras related to logarithmic conformal field theories (LCFTs) [8] (for reviews of the subject see [9]). What is common to all these theories is the Jordan cell structure of the representations which occur when studying their spectra. This is due to the fact that, at  $q$  a root of unity, indecomposable representations of  $U_q[sl(2)]$  appear and the same is true for models with global supersymmetry (indecomposable representations are common place in superalgebras). The Jordan cell structure also appears in the definition of LCFTs.

Another field where  $c = 0$  theories are relevant are stochastic models if the dynamical critical exponent  $z = 1$ . This possibility was suggested by numerical calculations in a 3-state model at the spinoidal point [10]. In stochastic models the groundstate is again zero for any number of sites and, this time, the groundstate wave function has a direct physical meaning: it describes the probability distribution of the stationary state.

In unrelated developments, study [11] of the XXZ chain has lead to various classes of alternating sign matrices (ASM) [12] — a subject of deep interest in combinatorics. Initially the ASM numbers were found in the periodic chain ( $L$  odd) [13] and have subsequently also been found for open [14] and twisted boundaries [14,15]. They also appear in the ice model with domain wall boundaries [16] as was shown in [17,18]. This model is also related to the combinatorial problem of domino tiling [19].

A relevant question is — why should number theoret-

ical properties of wave functions in quantum mechanics be significant? In this letter we answer this question by showing that these ASM numbers are related to directly measurable quantities. Specifically, we bring under one roof some of the above topics and relate the Hamiltonian (1.1) to stochastic processes on TL algebras where the stationary state probability distribution is given by the groundstate wavefunction  $|0\rangle$ . A much more detailed version of this work will be published elsewhere [20].

## II. TEMPERLEY-LIEB ALGEBRA

The generators  $\{e_j, j = 1, 2, \dots, L-1\}$  of the TL algebra can be represented graphically using monoids [21]

$$e_j = \left| \begin{array}{c} | \\ 1 \end{array} \right| \left| \begin{array}{c} | \\ 2 \end{array} \right| \cdots \left| \begin{array}{c} | \\ j-1 \end{array} \right| \left| \begin{array}{c} \cup \\ j \end{array} \right| \left| \begin{array}{c} \cup \\ j+1 \end{array} \right| \left| \begin{array}{c} | \\ j+2 \end{array} \right| \cdots \left| \begin{array}{c} | \\ L-1 \end{array} \right| \left| \begin{array}{c} | \\ L \end{array} \right| \quad (2.1)$$

The number of independent words  $C_L$  in the TL algebra with  $L-1$  generators is given by the Catalan numbers

$$C_n = \frac{1}{n+1} \binom{2n}{n} = 1, 2, 5, 14, \dots \quad n = 1, 2, 3, 4, \dots \quad (2.2)$$

We write  $e_j = v_{j,j+1} v_{j,j+1}^T$  and move to the left ideal by ignoring the upper half of the monoid diagrams

$$\begin{array}{ccc} \left| \begin{array}{c} \cup \\ 1 \end{array} \right| \left| \begin{array}{c} \cup \\ 2 \end{array} \right| \left| \begin{array}{c} \cup \\ 3 \end{array} \right| \left| \begin{array}{c} \cup \\ 4 \end{array} \right| & \mapsto & \left| \begin{array}{c} \cup \\ 1 \end{array} \right| \left| \begin{array}{c} \cup \\ 2 \end{array} \right| \left| \begin{array}{c} \cup \\ 3 \end{array} \right| \left| \begin{array}{c} \cup \\ 4 \end{array} \right| = v_{1,2} v_{3,4} = \left| \begin{array}{c} \diagup \\ 0 \end{array} \right| \left| \begin{array}{c} \diagdown \\ 1 \end{array} \right| \left| \begin{array}{c} \diagup \\ 2 \end{array} \right| \left| \begin{array}{c} \diagdown \\ 3 \end{array} \right| \left| \begin{array}{c} \diagup \\ 4 \end{array} \right| \\ \left| \begin{array}{c} \cup \\ 1 \end{array} \right| \left| \begin{array}{c} \cup \\ 2 \end{array} \right| \left| \begin{array}{c} \cup \\ 3 \end{array} \right| \left| \begin{array}{c} \cup \\ 4 \end{array} \right| & \mapsto & \left| \begin{array}{c} \cup \\ 1 \end{array} \right| \left| \begin{array}{c} \cup \\ 2 \end{array} \right| \left| \begin{array}{c} \cup \\ 3 \end{array} \right| \left| \begin{array}{c} \cup \\ 4 \end{array} \right| = v_{1,4} v_{2,3} = \left| \begin{array}{c} \diagup \\ 0 \end{array} \right| \left| \begin{array}{c} \diagdown \\ 1 \end{array} \right| \left| \begin{array}{c} \diagup \\ 2 \end{array} \right| \left| \begin{array}{c} \diagdown \\ 3 \end{array} \right| \left| \begin{array}{c} \diagup \\ 4 \end{array} \right| \end{array} \quad (2.3)$$

This ideal is equivalent to the link states in the  $O(1)$  loop interpretation [14]. Let us assume that  $L$  is even. Then the number of independent words in the left ideal is  $C_{L/2}$ . The diagrams give the action of the TL generators on the ideal

$$\begin{aligned} e_j(v_{j-1,j+2} v_{j,j+1}) &= \left| \begin{array}{c} \cup \\ j-1 \end{array} \right| \left| \begin{array}{c} \cup \\ j \end{array} \right| \left| \begin{array}{c} \cup \\ j+1 \end{array} \right| \left| \begin{array}{c} \cup \\ j+2 \end{array} \right| = \left| \begin{array}{c} \cup \\ j-1 \end{array} \right| \left| \begin{array}{c} \cup \\ j \end{array} \right| \left| \begin{array}{c} \cup \\ j+1 \end{array} \right| \left| \begin{array}{c} \cup \\ j+2 \end{array} \right| \\ &= v_{j-1,j+2} v_{j,j+1} \end{aligned} \quad (2.4)$$

$$\begin{aligned} e_{j-1}(v_{j-1,j+2} v_{j,j+1}) &= \left| \begin{array}{c} \cup \\ j-1 \end{array} \right| \left| \begin{array}{c} \cup \\ j \end{array} \right| \left| \begin{array}{c} \cup \\ j+1 \end{array} \right| \left| \begin{array}{c} \cup \\ j+2 \end{array} \right| = \left| \begin{array}{c} \cup \\ j-1 \end{array} \right| \left| \begin{array}{c} \cup \\ j \end{array} \right| \left| \begin{array}{c} \cup \\ j+1 \end{array} \right| \left| \begin{array}{c} \cup \\ j+2 \end{array} \right| \\ &= v_{j-1,j} v_{j,j+1} \end{aligned} \quad (2.5)$$

It is convenient to encode the words in the left ideal by restricted solid-on-solid (RSOS or Dyck) paths  $|a\rangle = (a_0, a_1, \dots, a_{L/2})$  where  $a_j$  is the number of half-loops above the midpoint between sites  $j$  and  $j+1$  and  $a_{j+1} - a_j = \pm 1$  for each  $j$ . For  $L = 6$  the RSOS paths are

$$L = 6 : \quad \{(0, 1, 2, 3, 2, 1, 0), (0, 1, 2, 1, 2, 1, 0), (0, 1, 0, 1, 2, 1, 0), (0, 1, 2, 1, 0, 1, 0), (0, 1, 0, 1, 0, 1, 0)\} \quad (2.6)$$

In this basis for the TL ideal, the  $C_{L/2} \times C_{L/2}$  matrix representative of  $H$  for  $L = 6$  is

$$H = -H_{\text{XXZ}} = - \sum_{j=1}^{L-1} (1 - e_j) = \begin{pmatrix} -4 & 1 & 0 & 0 & 0 \\ 2 & -3 & 1 & 1 & 0 \\ 0 & 1 & -3 & 0 & 1 \\ 0 & 1 & 0 & -3 & 1 \\ 2 & 0 & 2 & 2 & -2 \end{pmatrix} \quad (2.7)$$

Since  $(1 - e_j) = (1 - e_j)^2$  are projectors we see that  $H$  is nonpositive definite. The matrix  $H$  is representation independent. It is straightforward to reconstruct the corresponding eigenvectors of  $H_{\text{XXZ}}$  using

$$v_{j,k} = \overset{j}{\frown} \underset{k}{\smile} = q^{-1/2} |\uparrow\rangle_j \otimes |\downarrow\rangle_k - q^{1/2} |\downarrow\rangle_j \otimes |\uparrow\rangle_k \quad (2.8)$$

where  $q = e^{\pi i/3}$  and  $|\uparrow\rangle = (1, 0)$ ,  $|\downarrow\rangle = (0, 1)$  are the spin basis vectors. Moving to the left ideal changes the degeneracies in the spectra but not the groundstate wave function.

## III. TEMPERLEY-LIEB STOCHASTIC PROCESSES

It is easy to show that for even  $L$  the Hamiltonians  $H$  are intensity matrices (see [22] for some properties) satisfying  $H_{ab} > 0$  and  $H_{aa} = -\sum_{b \neq a} H_{ab}$ . The occurrence of intensity matrices reveals a novel connection with a stochastic process with time evolution given by the Euclidean Schrödinger equation [23]

$$\frac{d}{dt} P_a(t) = \sum_b H_{ab} P_b(t) \quad (3.1)$$

where  $P_a(t)$  is the (unnormalized) probability of finding the system in the state (RSOS path)  $|a\rangle$  at time  $t$ . In [20] we will exhibit the rules giving the transition rates  $H_{ab}$  for the general case which define RSOS growth models (for a review of growth models see [24]). These rates  $H_{ab} = 0, 1, 2$  for  $a \neq b$  are simple. At each time step adsorption occurs with rate  $H_{ab} = 1$  and desorption with rate  $H_{ab} = 1, 2$ . These transition processes correspond to the addition of a single (diamond shape) tile to the RSOS path or the removal of a partial layer of tiles and are indicated above and below the main diagonal respectively in (2.7). Notice that the desorption process is nonlocal.

Since  $H$  is an intensity matrix it has a zero eigenvalue with a trivial bra

$$\langle 0| H = 0, \quad \langle 0| = (1, 1, \dots, 1) \quad (3.2)$$

and nontrivial ket wave functions giving the unnormalized probabilities of the unique stationary state

$$H|0\rangle = 0, \quad |0\rangle = \sum_a P_a |a\rangle, \quad P_a = \lim_{t \rightarrow \infty} P_a(t). \quad (3.3)$$

The wavefunction  $|0\rangle$  describes a critical statistical system of weighted RSOS paths. The normalized probabilities are  $p_a = P_a/S_1(L)$  where  $S_1(L) = \langle 0|0\rangle$ . The conjectured [14,18] magic properties of the groundstate wave functions of  $H_{\text{XXZ}}$  now assume a direct physical significance.

We have calculated the wavefunctions exactly up to  $L = 18$ . If we now normalize the wave functions  $|0\rangle$  to have smallest entry 1

$$L = 6 : \quad |0\rangle = (1, 4, 5, 5, 11), \quad S_1(6) = 26 \quad (3.4)$$

we confirm [14] that the normalization factors  $S_1(L) = \langle 0|0\rangle$  satisfy

$$S_1(2n) = A_V(2n+1) = 1, 3, 26, 646, \dots \quad (3.5)$$

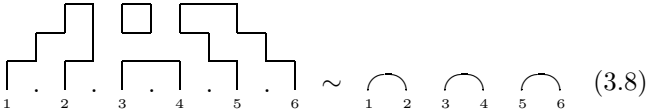
where  $A_V$  is the number of vertically symmetric  $(2n+1) \times (2n+1)$  ASMs [17,12]

$$A_V(2n+1) = (-3)^{n^2} \prod_{i=1}^{2n+1} \prod_{j=1}^n \frac{3(2j-i)+1}{2j-i+2n+1} \quad (3.6)$$

We also confirm that for the highest path  $P_a = 1$  and for the lowest path  $P_a = N_8(L/2)$  where  $N_8$  is the number of cyclically symmetric transpose complement plane partitions [17,12]

$$N_8(n) = \prod_{j=1}^{n-1} (3j+1) \frac{(2j)!(6j)!}{(4j)!(4j+1)!} \quad (3.7)$$

As for the combinatorial significance we conjecture [20], following [18], that the integers appearing in the groundstate are given by the number of configurations of the fully packed loop model on a  $(2L-1) \times (\lfloor L/2 \rfloor + 1)$  pyramid grid domain with specified boundary conditions and links determined by the diagrams in the left ideal. For example, for  $L = 6$  and  $v_{1,2}v_{3,4}v_{5,6}$  there are 11 configurations of the type



$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \sim \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad (3.8)$$

More generally, and to accommodate odd  $L$ , we can allow  $2s$  or fewer defects represented by unpaired (vertical) lines where  $s = 0, \frac{1}{2}, 1, \dots, \lfloor L/2 \rfloor$  is the spin. The number of independent words in the left ideal is then  $C_{(L+2s)/2}$



$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \rightarrow \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$$

Under the action of the TL generators, the defects can hop and adjacent defects can be annihilated

$$e_j v_{j-1,j} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = v_{j,j+1} \quad (3.9)$$

The RSOS paths for  $L = 5, 2s = 1$  and  $L = 4, 2s = 2$  are

$$L = 5, 2s = 1 : \quad \{(|0, 1, 2, 1, 0\rangle, (0, 1, 2, 1, 0, 1|), \\ (|0, 1, 0, 1, 0\rangle, (0, 1, 0, 1, 0|), (0, 1, 0|0, 1, 0\rangle)\} \quad (3.10)$$

$$L = 4, 2s = 2 : \quad \{(|0, 1, 0\rangle, (|0, 1, 0|), (0, 1, 0|), \\ (0, 1, 2, 1, 0\rangle, (0, 1, 0, 1, 0|)\} \quad (3.11)$$

The matrix representatives of  $H$  and wave functions are

$$H = \begin{pmatrix} -3 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 \\ 2 & 1 & -2 & 0 & 1 \\ 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & -2 \end{pmatrix}, \quad \begin{pmatrix} -2 & 1 & 0 & | & 0 & 0 \\ 1 & -2 & 1 & | & 0 & 0 \\ 0 & 1 & -2 & | & 0 & 0 \\ \hline 0 & 0 & 0 & | & -2 & 1 \\ 1 & 0 & 1 & | & 2 & -1 \end{pmatrix} \quad (3.12)$$

$$|0\rangle = \begin{cases} (1, 1, 3, 3, 3), & S_1(5) = 11, L = 5, 2s = 1 \\ (0, 0, 0, 1, 2), & S_1(4) = 3, L = 4, 2s = 2 \end{cases} \quad (3.13)$$

The matrices are again nonpositive definite intensity matrices. For two or more defects the Hamiltonian is (Jordan cell) block triangular and the spectrum decomposes according to the  $sl(2)$  fusion rules which for 2 defects is  $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$ .

For  $L \leq 17$  odd and  $2s = 1$  we have verified [14] that

$$S_1(2n+1) = N_8(n+1) = 2, 11, 170, 7429, \dots \quad (3.14)$$

It is sometimes convenient to remove the defects by adding  $2s$  sites on the right, joining the defects to these sites by half-loops and working with extended RSOS paths on  $L + 2s$  sites without defects. In this scheme the defects are incorporated as boundary conditions.

#### IV. PROPERTIES OF STATIONARY STATES

The stationary states are independent of the initial condition and Jordan cell structure but the time evolution does depend on which Jordan blocks are selected out by the initial  $t = 0$  probabilities. A stochastic process can also be defined by the action of  $H$  on the whole TL algebra ( $C_L$  states) giving a different Jordan cell structure and (non-unique) stationary states.

Many properties of the stationary states can be studied by finite-size scaling. We give preliminary results on the average perimeter  $\langle \ell(a) \rangle$ , average area  $\langle N(a) \rangle$  and average number of clusters  $\langle C(a) \rangle$  where we define

$$\ell(a) = \sum_{j=1}^{2\lfloor L/2 \rfloor - 1} \frac{1}{2} |a_{j+1} - a_{j-1}|, \quad N(a) = \sum_{j=1}^{2\lfloor L/2 \rfloor} [a_j/2] \quad (4.1)$$

$$C(a) = \sum_{j=1}^L \delta(a_j, 0) - 2s, \quad \langle \dots \rangle = \sum_a \dots p_a \quad (4.2)$$

Here  $N(a)$  is the number of tiles and we have ignored the fixed horizontal contribution  $2\lfloor L/2 \rfloor$  to the perimeter.

By extrapolating results for  $L \leq 18$  we find numerically the following scaling behaviour for large even  $L$

$$\langle \ell(a) \rangle \sim .249(1) L, \quad \langle N(a) \rangle \sim .065(1) L^{1+\nu}, \quad \nu = .50(3) \quad (4.3)$$

$$\langle C(a) \rangle \sim 1.17(1) L^x, \quad x = .667(3) \quad (4.4)$$

The same exponents but different amplitudes are obtained for  $\langle \ell(a) \rangle$  and  $\langle N(a) \rangle$  with odd  $L$ . The finite-size behaviour of other properties describing the critical weighted RSOS paths model and its connection to conformal invariance will be discussed in [20].

## V. CONFORMALLY INVARIANT SPECTRA

We assert that the spectra of the intensity matrices  $H$  are described by a conformal field theory [5] with central charge  $c = 0$ , conformal weights

$$\Delta_s = \frac{s(2s-1)}{3} = 0, 0, \frac{1}{3}, 1, \dots \quad s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \quad (5.1)$$

and Virasoro characters with  $q$  the modular parameter

$$\chi_s(q) = q^{\Delta_s} (1 - q^{2s+1}) \prod_{n=1}^{\infty} (1 - q^n)^{-1} \quad (5.2)$$

Finite-size corrections [25] to the energy levels  $E_n$ ,  $H|n\rangle = -E_n|n\rangle$ , for large  $L$  with  $n = 0, 1, 2, \dots$  are

$$\frac{LE_n}{\pi v} = \Delta_s + k_n + o(1), \quad k_n \in \mathbb{N} \quad (5.3)$$

where  $k_n$  labels descendents and  $v$  is the sound velocity. We have calculated numerically the finite-size spectra up to size  $L = 16$  and obtain the following estimates using Van den Broeck-Schwartz [26] approximants with sound velocity  $v = 2.597(1)$

$n$	$s = 0$		$s = \frac{1}{2}$		$s = 1$	
	$\Delta_s + k_n$	exact	$\Delta_s + k_n$	exact	$\Delta_s + k_n$	exact
0	0	0	0	0	0.333(1)	1/3
1	1.999(1)	2	1.0	1	1.34(2)	4/3
2	3.003(4)	3	1.999(3)	2	2.4(2)	7/3
3	4.01(2)	4	3.003(6)	3	2.3(2)	7/3
4	3.99(8)	4	2.999(6)	3	3.4(3)	10/3
5	4.7(5)	5	3.8(5)	4	3.5(4)	10/3

TABLE I. Table of energy level estimates for  $2s = 0, 1, 2$  defects giving the characters  $\chi_s(q)$  for  $s = 0, \frac{1}{2}, 1$ .

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