A New Universalitity Class for Dynamic Processes.

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We discuss a one-dimensional model of a fluctuating interface with a dynamic exponent $z = 1$. The processes that occur are adsorption, which is local, and desorption which is non-local and may take place over regions of the order of the system size. In the thermodynamic limit, the time dependence of the system is given by characters of the $c = 0$ conformal theory of percolation. It is for the first time that one can show in a rigorous way, a connection between CFT and stochastic processes. The finite-size scaling behavior of the average height, interface width and other observables are obtained. The avalanches produced during desorption are analyzed and we show that the probability distribution of the avalanche sizes obeys finite-size scaling with new critical exponents.

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The structure of growing interfaces continues to be a subject of major interest and a characterization of the various universality classes of critical behavior remains an open question [1]. In this letter we present a one-dimensional model of a fluctuating interface which belongs to a new universality class where the dynamical critical exponent $z = 1$. In this model the interface evolves following non-local Markovian dynamics. The relaxation rules are such that one has avalanches and therefore if we are looking at the properties of these avalanches, one can consider our model as belonging also to the sandpile class. Following the work of Bak, Tang and Wiesenfeld [2] many types of sandpile models have been studied [3]. As for interface models, a clear picture of the factors which determine different universality classes for sandpile models did not yet emerge [4]. Studying the finite-size scaling (FSS) properties of the probability distribution function (PDF) of avalanches of different sizes we found new exponents which show that our model belongs to a new universality class also of sandpile models.

As we are going to show, the Hamiltonian which gives the time evolution of our model is given by the well known critical $U_q(sl(2))$ invariant XXZ quantum chain at a special value of the asymmetry parameter. The FSS limit of the spectrum of the Hamiltonian is known exactly using the Bethe Ansatz [5] and is given by characters of the Virasoro algebra of conformal field theory (CFT), this is why $z = 1$. In fact the knowledge of the CFT is what induced us to study this model. Its defining rules are not designed to describe a specific phenomenon, but follow from the equivalence to a known XXZ spin chain. Moreover [6], the PDF describing the stationary state is related to combinatorial aspects of the ice model defined on a rectangle with special boundary conditions. This is probably the reason why we are able to make a number of educated guesses for various quantities connected to the model even for finite lattices.

We consider an interface on a one-dimensional lattice of size $L + 1$ ($L = 2n$ even). The non-negative heights h_i ($i = 0, 1, .., L$) that specify the interface, obey restricted solid-on-solid (RSOS) rules,

$$h_{i+1} - h_i = \pm 1, \quad h_0 = h_L = 0, \quad h_i \geq 0. \tag{1}$$

Alternatively, we can describe the interface using slope variables $s_i = (h_{i+1} - h_{i-1})/2$, ($s_0 = s_L = 0$). The interface evolves through adsorption and desorption according to the following rules. Adsorption, which locally changes the height h_i to $h_i + 2$, takes place with a rate equal to 1 at a local minimum of the interface ($s_i = 0$, $h_i < (h_{i-1}, h_{i+1})$). The non-vanishing rates for desorption can be understood using the notion of active segments. A segment of a configuration with the end-points a and b is defined by the conditions: $h_a = h_b = h$ and $h_i > h$ for $a < i < b$. We call a segment active if at least one of the boundary slopes s_a or s_b is nonzero. In the desorption process, all the $b - a - 1$ heights h_i contained in a segment decrease by two units ($h_i \mapsto h_i - 2$), the other heights remaining unchanged, with a rate

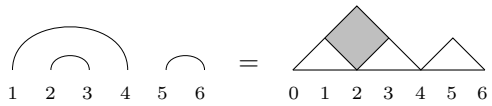
$$\delta(s_a - 1) + \delta(s_b + 1), \tag{2}$$

where δ is the discrete Kronecker symbol. In order to find which desorption processes can take place, for each of the $C_n = (2n)!/((n+1)(n!)^2)$ configurations one first looks at how many active segments one has and then one uses (2). If the two slopes s_a and s_b are zero, the desorption rate is zero. This observation has two consequences. First, in the stationary state we expect to see predominantly configurations with large terraces, i.e. intervals where the slope is zero for all the sites. Next, it is meaningful to consider clusters. Those are segments where the end-points have the heights equal to zero. According to the rules (2) desorption takes place only within a cluster.

The dynamics of the interface can easily be visualized using tiles (tilted squares) which cover the area between the interface and the substrate ($h_{2i} = 0$, $h_{2i+1} = 1$). For a given configuration c, the number of tiles is $u(c) =$

$\frac{1}{2}\sum_{i=2}^{L-2} h_i + 1 - L/4$. According to the rules given above, through adsorption one adds one tile, through desorption one loses $b-a-1$ tiles (this is an odd number). If one looks at our model from the point of view of self-organized critical phenomena, the number of tiles u lost in a desorption process defines the size of an avalanche.

Before describing the physical properties of the model we will give another formulation which enables us to obtain the FSS spectrum of the Hamiltonian and therefore to find the dynamic exponent. To each RSOS configuration with $L+1$ sites one can associate a configuration of n non-intersecting half-loops and $L = 2n$ sites. The height h_i is the number of half-loops above the midpoint above the sites i and $i+1$ in the half-loop configuration. For example, the following picture illustrates the association between a configuration with three half-loops and an $L = 6$ RSOS configuration with two clusters and a tile,



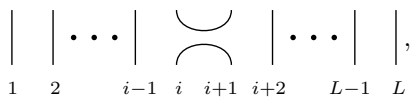
(3)

The half-loop configurations on the other hand, can be associated with the left ideal of the Temperley Lieb (TL) algebra at $Q = 1$ [7]. The $L-1$ generators of the algebra satisfy the relations

$$e_i^2 = e_i,\ e_i e_{i\pm1} e_i = e_i,\ [e_i, e_j] = 0,\ |i-j| > 1, \tag{4}$$

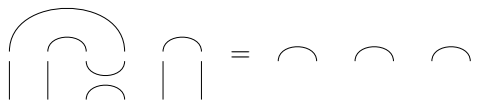
and have the graphical representation

e_i =



(5)

The left ideal is generated by the action of TL generators on $I_0 = \prod_{i=1}^{n} e_{2i-1}$. For example the action of e_3 on the half-loop configuration (3) is



(6)

The Hamiltonian gives the time evolution in the vector space of the half-loop (RSOS) configurations,

$$\frac{\mathrm{d}}{\mathrm{d}t} P_c(t) = -\sum_d H_{cd} P_d(t), \tag{7}$$

and is given by

$$H = \sum_{j=1}^{L-1} (1 - e_j)\,. \tag{8}$$

$P_c(t)$ is the (unnormalized) probability to find the system in the configuration c at the time t and $r_{cd} = -H_{cd}$ give the rates for the transitions $d \to c$. Since H is an intensity matrix ($\sum_c H_{cd} = 0$) [6], it has a zero eigenvalue with a trivial bra and a nontrivial ket which gives the probabilities in the stationary state,

$$\begin{aligned} &\langle 0|\, H = 0, \quad \langle 0| = (1, 1, \ldots, 1), \\ &H|0\rangle = 0, \quad |0\rangle = \sum_c P_c |c\rangle, \quad P_c = \lim_{t\to\infty} P_c(t). \end{aligned} \tag{9}$$

Let us compute the dynamic critical exponent z. We use the following representation of the TL algebra

$$\begin{aligned} e_i \;=\; &\frac{1}{4} - \frac{1}{2}\Big[\Big(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y - \Delta \sigma_i^z \sigma_{i+1}^z\Big) \\ &+ h\Big(\sigma_i^z - \sigma_{i+1}^z\Big)\Big], \end{aligned} \tag{10}$$

where $\sigma^x, \sigma^y, \sigma^z$ are Pauli matrices, $-2\Delta = q + q^{-1} = 1$, $2h = q - q^{-1} = \mathrm{i}\sqrt{3}$ and $q = \mathrm{e}^{\pi \mathrm{i}/3}$. In this representation H given by (8) becomes the Hamiltonian of the XXZ quantum chain [8] with L sites and the energy gaps E_k (the ground-state energy is zero for any L) scale like

$$E_k = \pi v_\mathrm{s} x_k L^{-z}, \tag{11}$$

with $z = 1$, $v_\mathrm{s} = 3\sqrt{3}/2$ is the sound velocity [5] and x_k are related to surface exponents of a $c = 0$ CFT of percolation [9]. In the continuum limit, the spectrum of the Hamiltonian (8) in the full algebra is given by the Virasoro characters [9]

$$\chi_s(q) = q^{s(2s-1)/3}(1 - q^{2s+1}) \prod_{k=1}^{\infty} (1 - q^k)^{-1}. \tag{12}$$

Here $q = \exp(\pi T v_\mathrm{s}/L)$ has a different meaning than in (10) and parametrizes the temporal (T) and spatial (L) extent of the stochastic process. In the subspace of the RSOS configurations the spectrum is given by $\chi_0(q)$. To our knowledge, for the first time, a connection is made between stochastic processes and CFT. This implies among other things that the space and time correlation lengths are the same and that, in the continuum limit, the forms of the space-time correlation functions in the stationary state are known [10].

As discussed in detail in [6] the stationary state of our model with L sites is related to the two-dimensional ice model [11] defined on a rectangle of dimension $L \times (L-1)/2$ with special boundary conditions [12, 13]. This model is equivalent to a fully packed loop (FPL) model on the rectangle, all configurations being equally probable [14]. Let us briefly explain how one can better understand the stationary state $|0\rangle$ of our model. If we choose the (unnormalized) probability of the "pyramid" configuration ($s_1 = ... = s_{n-1} = 1$, $s_n = 0$, $L = 2n$) to be equal to one (this configuration has the smallest probability), then the normalization factor of the stationary state is equal to [15]

$$\langle 0|0\rangle = \prod_{j=0}^{n-1} (3j+2) \frac{(2j+1)!(6j+3)!}{(4j+2)!(4j+3)!}, \tag{13}$$

This is precisely $A_{\rm V}(2n+1)$, the number of configurations of the FPL model on the rectangle. Moreover, the (unnormalized) probabilities for the other half-loop configurations are integer numbers which are equal to the numbers of configurations of the FPL model with the external occupied edges connected in the same way as in the configuration of half-loops [6, 16]. This implies that the PDF describing the stationary distribution in the RSOS configurations space corresponds to a uniform probability distribution in the space of the FPL model, or equivalently the ice model. The RSOS configuration with the largest (unnormalized) probability is the one with the longest terrace ($s_1 = .. = s_L = 0$) in which the interface coincides with the substrate, its value is [15]

$$N_8(2n) = \prod_{j=1}^{n-1} (3j+1)\, \frac{(2j)!(6j)!}{(4j)!(4j+1)!}. \qquad (14)$$

The normalized PDF is obviously $p_c = P_c/\langle 0|0\rangle$.

We now describe some properties of the model. They were obtained either numerically for lattice sizes up to $L = 18$ or "guessed" and checked up to $L = 18$. We start with the properties of the stationary state. The fraction of the interface covered by terraces is [17]

$$\tau(L) = \frac{1}{L-1}\sum_{j=1}^{L-1}\langle 1-|s_j|\rangle = \frac{3L^2-2L-6}{(L-1)(4L+2)}. \qquad (15)$$

This implies that for large L, three quarters of the surface is covered by terraces. We now consider the number of clusters per configuration $C = \sum_{j=1}^{L}\delta(h_j)$. Its average and the average of its square have the following expressions

$$\langle C\rangle = \frac{1}{3}\prod_{j=0}^{n-1}\frac{(2j+1)(3j+4)}{(j+1)(6j+1)} - \frac{1}{3} \approx 0.738L^{2/3}, \qquad (16)$$

$$\langle C^2\rangle \approx (1.29 - 0.49L^{-0.63})\langle C\rangle^2. \qquad (17)$$

From our results we deduce that the compressibility [18] of the gas of clusters diverges like the size of the system,

$$\kappa = L\frac{\langle C^2\rangle - \langle C\rangle^2}{\langle C\rangle^2}. \qquad (18)$$

The average height $\langle\overline{h}\rangle$ and interface width w_∞, which characterizes the roughness of the surface in the stationary state,

$$\overline{h^m} = \frac{1}{L}\sum_{i=1}^{L} h_i^m, \quad w_\infty = \sqrt{\langle\overline{h^2} - \overline{h}^2\rangle}, \qquad (19)$$

are compatible with the following behavior,

$$\langle\overline{h}\rangle \approx 0.14\ln(L/2), \quad w_\infty \approx 0.34\,(\ln L/2)^{1/3}. \qquad (20)$$

This implies that the PDF of the heights has a very small dispersion.

As in any fit involving logs, the formulae given in (20) are probably not the last word. What is certain is that the width grows slowly with the size of the system implying that the surface is only marginally rough. It is interesting to mention that marginally rough surfaces (with $z = 1.581$ corresponding to the directed percolation universality class) were also encountered [19, 20] at a critical point dividing a moving rough KPZ phase from a smooth, massive phase. In these models a factor 3 between the powers of the logs (like in (20)), which implies a very narrow distribution of the heights, was also seen [21].

To complete the interface growth picture, we calculated the ratio of the time dependent interface width $w(t, L)$ to w_∞ for various lattice sizes. The curves were obtained by solving on a computer the differential equations (7) taking the substrate as the initial configuration. If $z = 1$ and if the Family-Vicsek scaling applies, this ratio should converge to a function of t/L. We have checked that this is indeed the case.

We now consider the avalanches in our model. If in the stationary state one considers the configuration d with $u(d)$ tiles, it changes with rate r_{cd} into the configuration c with $u(c)$ tiles. The rate of changes in which u tiles are lost, can be written

$$R(u, L) = \sum_{c\neq d}\delta(u(d) - u(c) - u)r_{cd}p_d, \qquad (21)$$

We have found the following values for the rate of adsorption of one tile, $R(u = -1, L)$, and the rate of all possible avalanches, $R(u > 0, L)$,

$$R(-1, L) = \frac{3L(L-2)}{4(2L+1)}, \qquad (22)$$

$$R(u > 0, L) = L - 2 - 2R(-1, L). \qquad (23)$$

Notice that for large L the rate per unit length to have an avalanche ($u > 0$) is only 1/4.

Since through desorption one loses an odd number of tiles, it is convenient to write $u = 2v - 1$ and to consider v as the size of an avalanche. Given the occurrence of an avalanche, its size v is distributed according to the PDF

$$S(v, L) = \frac{R(2v-1, L)}{R(u > 0, L)}. \qquad (24)$$

FSS [3] predicts the following form for this PDF,

$$S(v, L) = v^{-\tau}F(v/L^D). \qquad (25)$$

One way to get the exponents of the FSS function is to consider the moments [22],

$$\langle v^k\rangle = \sum_{v=1} v^k S(v, L) \sim L^{\sigma(k)}, \qquad (26)$$

and one expects

$$\sigma(k) = \begin{cases} 0 & k < \tau - 1 \\ D(k+1-\tau) & k > \tau - 1 \end{cases} \qquad (27)$$

From equation (23) one gets $\langle v \rangle = (5L+4)/4(L+2)$, which implies that for large L the average size of an avalanche is 3/2 tiles. A numerical investigation of the other moments indicates $\sigma(k) = k-2$. This implies $D = 1$ and $\tau = 3$. These exponents have been checked directly looking at the FSS scaling function as shown in Fig 1. Assuming $D = 1$, a best fit gives $\tau = 3.3$ compatible with the result obtained from the momenta. Together with $z = 1$, the observed exponents define a new universality class of sandpile models.

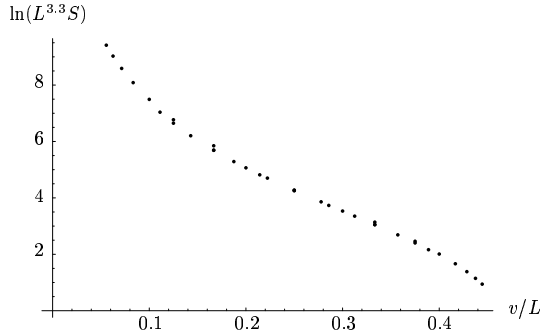


FIG. 1: Avalanche scaling function. The data are obtained for $L = 2, \ldots, 18$.

To conclude, we have presented a model of a critical fluctuating interface which can also be seen as a sandpile model. We have also shown the connection between the model and a CFT with $c = 0$, which among other things implies that the dynamic exponent $z = 1$. Several FSS exponents of expectation values have been computed in the stationary state but their identification with the scaling dimensions of the CFT was not yet done. The FSS exponents of the avalanche PDF have also been determined. Much is still to be understood in this model, such as correlation functions and its off-critical behavior. We hope to come back to these topics in a future publication.

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